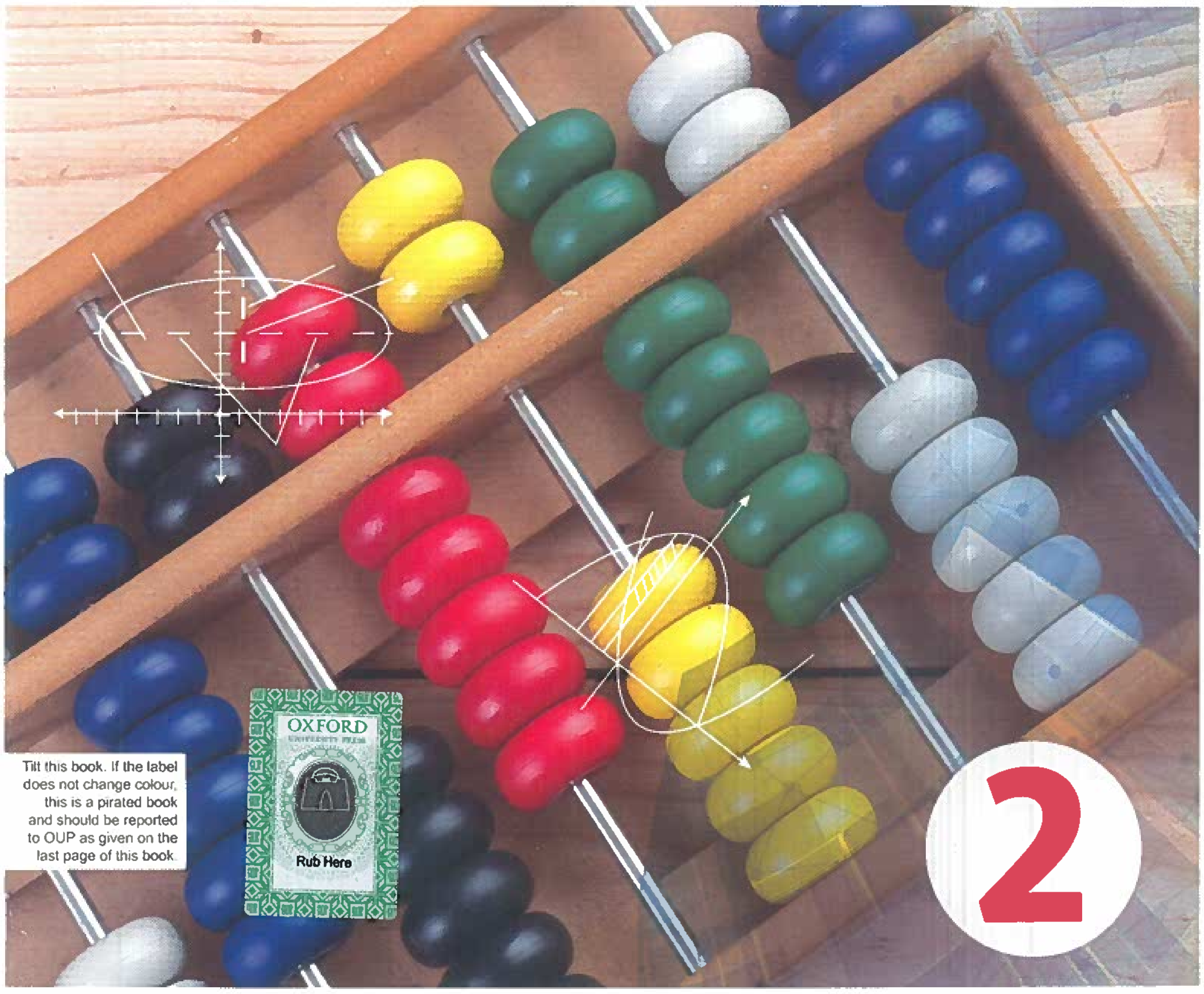
OXFORD

7th EDITION

NEW SYLLABUS MATHEMATICS



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PREFACE

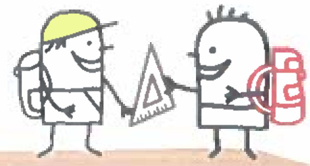
New Syllabus Mathematics (NSM)

is a series of textbooks specially designed to provide valuable learning experiences to engage the hearts and minds of students sitting for the GCE O level examination in Mathematics. Included in the textbooks are **Investigation**, **Class Discussion**, **Thinking Time**, **Journal Writing**, **Performance Task** and **Problems in Real-World Contexts** to support the teaching and learning of Mathematics.

Every chapter begins with a chapter opener which motivates students in learning the topic. Interesting stories about Mathematicians, real-life examples and applications are used to arouse students' interest and curiosity so that they can appreciate the beauty of Mathematics in their surroundings.

The use of ICT helps students to visualise and manipulate mathematical objects more easily, thus making the learning of Mathematics more interactive. Ready-to-use interactive ICT templates are available at http://www.shinglee.com.sg/StudentResources/

KEY FEATURES



CHAPTER OPENER

Each chapter begins with a chapter opener to arouse students' interest and curiosity in learning the topic.

LEARNING OBJECTIVES

Learning objectives help students to be more aware of what they are about to study so that they can monitor their own progress.

RECAP

Relevant prerequisites will be revisited at the beginning of the chapter or at appropriate junctures so that students can build upon their prior knowledge, thus creating meaningful links to their existing schema.

WORKED EXAMPLE

This shows students how to apply what they have learnt to solve related problems and how to present their working clearly. A suitable heading is included in brackets to distinguish between the different Worked Examples.

PRACTISE NOW

At the end of each Worked Example, a similar question will be provided for immediate practice. Where appropriate, this includes further questions of progressive difficulty.

SIMILAR QUESTIONS

A list of similar questions in the Exercise is given here to help teachers choose questions that their students can do on their own.

EXERCISE

The questions are classified into three levels of difficulty – Basic, Intermediate and Advanced.

SUMMARY

At the end of each chapter, a succinct summary of the key concepts is provided to help students consolidate what they have learnt.

REVIEW EXERCISE

This is included at the end of each chapter for the consolidation of learning of concepts.

CHALLENGE YOURSELF

Optional problems are included at the end of each chapter to challenge and stretch high-ability students to their fullest potential.

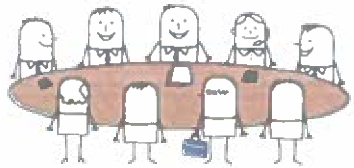
REVISION EXERCISE

This is included after every few chapters to help students assess their learning.

:arning experiences have been infused into Investigation, Class Discussion, Thinking Time, ·urnal Writing and Performance Task.

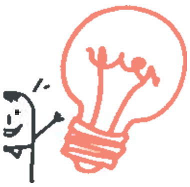
Investigation

Activities are included to guide students to investigate and discover important mathematical concepts so that they can construct their own knowledge meaningfully.



Class Discussion

Questions are provided for students to discuss in class, with the teacher acting as the facilitator. The questions will assist students to learn new knowledge, think mathematically, and enhance their reasoning and oral communication skills.

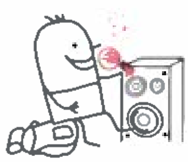


Thinking Time

Key questions are also included at appropriate junctures to check if students have grasped various concepts and to create opportunities for them to further develop their thinking.

Journal Writing

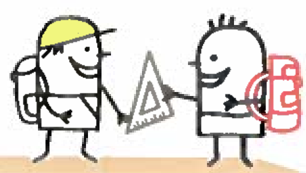
Opportunities are provided for students to reflect on their learning and to communicate mathematically. It can also be used as a formative assessment to provide feedback to students to improve on their learning.



Performance Task

Mini projects are designed to develop research and presentation skills in the students.

MARGINAL NOTES



ATTENTION

This contains important information that students should know.

Problem Solving Tip

This guides students on how to approach a problem.

INFORMATION

This includes information that may be of interest to students.

RECALL

This contains certain mathematical concepts or rules that students have learnt previously.

Just For Fun

This contains puzzles, fascinating facts and interesting stories about Mathematics as enrichment for students.

Internet Resources

This guides students to search on the Internet for valuable information or interesting online games for their independent and self-directed learning.

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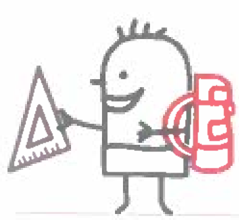

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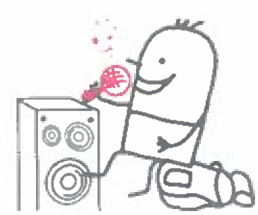

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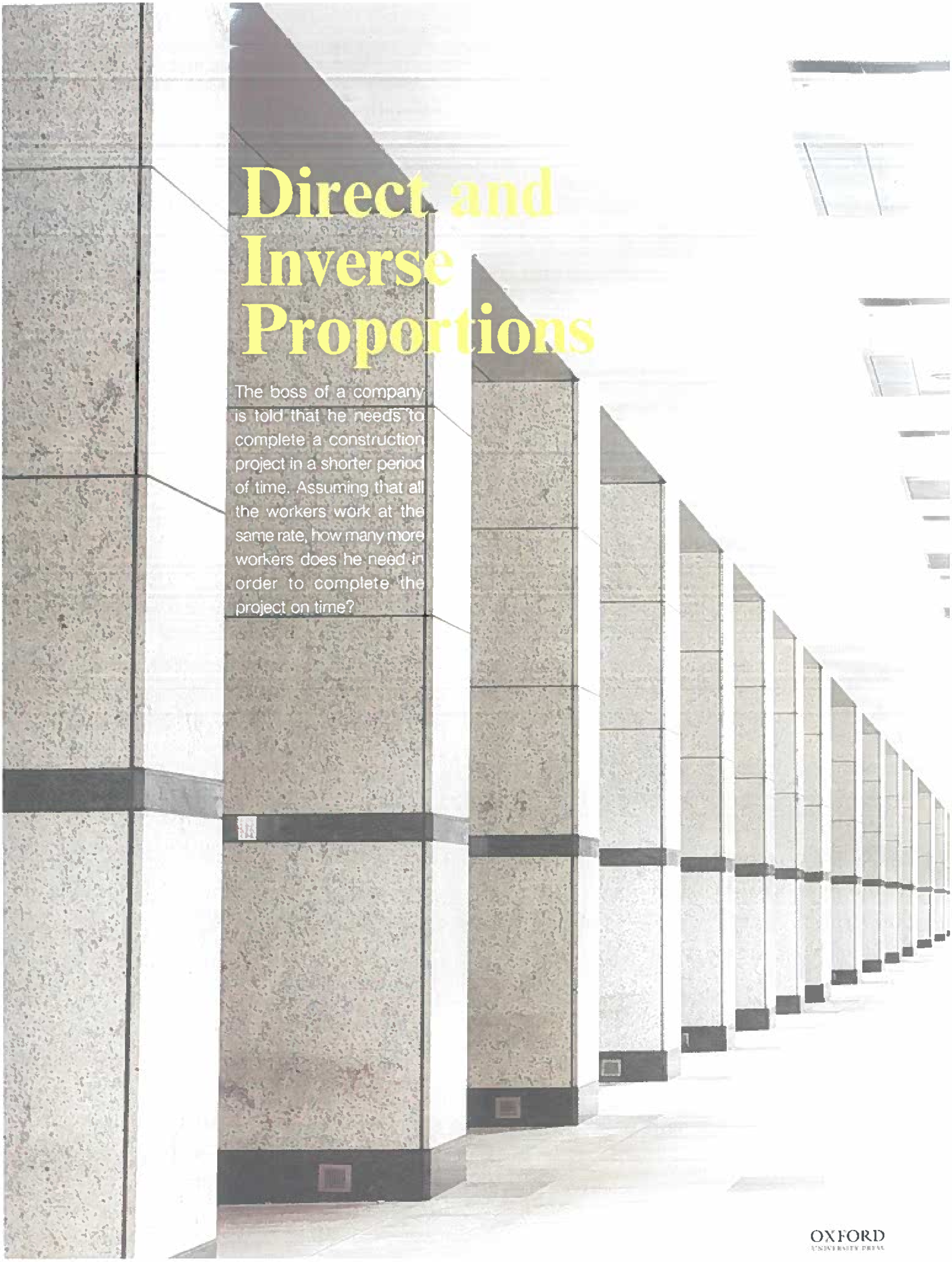

Direct and Inverse Proportions

The boss of a company is told that he needs to complete a construction project in a shorter period of time. Assuming that all the workers work at the same rate, how many more workers does he need in order to complete the project on time?

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chapter One

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- explain the concept of direct proportion using tables, equations and graphs,
- explain the concept of inverse proportion using tables, equations and graphs,
- solve problems involving direct and inverse proportions.

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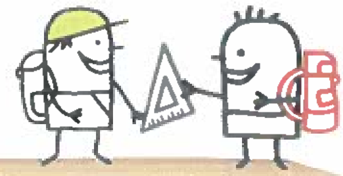
1.1 Direct Proportion



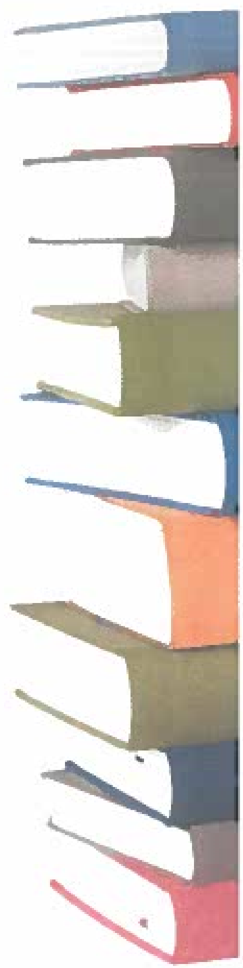
Investigation

Direct Proportion

In Singapore, if we borrow books from a public library and are late in returning the books, we will be fined 15 cents per day for each overdue book. Table 1.1 shows the fines for an overdue book.

Number of days (x)	1	2	3	4	5	6	7	8	9	10
Fine (y cents)	15	30	45	60	75	90	105	120	135	150

Table 1.1



1. If the number of days a book is overdue increases, will the fine increase or decrease?
2. If the number of days a book is overdue is doubled, how will the fine change?
   Hint: Compare the fines when a book is overdue for 3 days and for 6 days.
3. If the number of days a book is overdue is tripled, what will happen to the fine?
4. If the number of days a book is overdue is halved, how will the fine change?
   Hint: Compare the fines when a book is overdue for 10 days and for 5 days.
5. If the number of days a book is overdue is reduced to $\frac{1}{3}$ of the original number, what will happen to the fine?

From the investigation, we notice that as the number of days, x, a book is overdue increases, the fine, y cents, increases *proportionally*, i.e. if x is doubled, then y will be doubled; if x is tripled, then y will be tripled.

Similarly, as the number of days, x, a book is overdue decreases, the fine, y cents, decreases *proportionally*, i.e. if x is halved, then y will be halved; if x is reduced to $\frac{1}{3}$ of its original value, then y will be reduced to $\frac{1}{3}$ of its original value.

This relationship is known as **direct proportion**. We say that the fine, y cents, is *directly proportional* to the number of days, x, a book is overdue.

Class Discussion

Real-Life Examples of Quantities in Direct Proportion

Work in pairs.

1. Give a few more real-life examples of quantities that are in direct proportion.
2. Explain why they are directly proportional to each other.

Complete Table 1.2.

Number of days (x)	1	2	3	4	5	6	7	8	9	10
Fine (y cents)	15	30	45	60	75	90	105	120	135	150
Rate $\left(\frac{y}{x}\right)$	$\frac{15}{1} = 15$	$\frac{30}{2} = 15$	$\frac{45}{3} = 15$							

Table 1.2

What can we observe about the rate $\frac{y}{x}$?

What does $\frac{y}{x}$ represent? What does the constant '15' mean in this context?

In direct proportion, the rate $\frac{y}{x}$ is a *constant*. In this case, $\frac{y}{x} = 15$.

Let the number of days a book is overdue be $x_1 = 3$. Then the corresponding fine is $y_1 = 45$.

Let the number of days a book is overdue be $x_2 = 6$. Then the corresponding fine is $y_2 = 90$.

From Table 1.2, $\frac{y_1}{x_1} = \frac{45}{3} = 15$ and $\frac{y_2}{x_2} = \frac{90}{6} = 15$.

$\therefore \frac{y_1}{x_1} = \frac{y_2}{x_2} = 15$ (constant), i.e. the two *rates* $\frac{y_1}{x_1}$ and $\frac{y_2}{x_2}$ are equal

Moreover, $\frac{x_2}{x_1} = \frac{6}{3} = 2$ and $\frac{y_2}{y_1} = \frac{90}{45} = 2$, i.e. y is doubled when x is doubled.

$\therefore \frac{x_2}{x_1} = \frac{y_2}{y_1}$, i.e. the two *ratios* $\frac{x_2}{x_1}$ and $\frac{y_2}{y_1}$ are equal (This can also be obtained by rearranging the equation $\frac{y_1}{x_1} = \frac{y_2}{x_2}$.)

To conclude, we have:

> If y is *directly proportional* to x, then $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ or $\frac{x_2}{x_1} = \frac{y_2}{y_1}$.

ATTENTION

In Book 1, we have learnt that a *rate* compares two or more quantities of *different kinds*. In this case, $\frac{y_1}{x_1}$ $\left(\text{or } \frac{y_2}{x_2}\right)$ compares the fine with the number of days a book is overdue. We have also learnt that a *ratio* compares two or more quantities of the *same kind*. In this case, $\frac{x_2}{x_1}$ (or $x_2 : x_1$) is a comparison of the number of days a book is overdue and $\frac{y_2}{y_1}$ (or $y_2 : y_1$) is a comparison of the fines.

Worked Example 1

(Simple Problem involving Direct Proportion)

If 6 kg of biscuits cost \$27, calculate the cost of 13 kg of biscuits.

Solution:

First, we note that the cost of the biscuits is directly proportional to the mass of the biscuits.

INFORMATION

Method 1 is called the *unitary* method because it involves finding the cost of 1 kg (or *1 unit*) of biscuits first.

Method 1: Unitary Method

6 kg of biscuits cost \$27.

1 kg of biscuits costs $\frac{\$27}{6}$.

13 kg of biscuits cost $\frac{\$27}{6} \times 13 = \$58.50$.

Method 2: Proportion Method

Let the cost of 13 kg of biscuits be \$$x$.

Then $\frac{x}{13} = \frac{27}{6}$. $\left(\frac{x_1}{y_1} = \frac{x_2}{y_2}\right)$

$$x = \frac{27}{6} \times 13$$

$$= \$58.50$$

Alternatively,

$\frac{x}{27} = \frac{13}{6}$ $\left(\frac{x_1}{x_2} = \frac{y_1}{y_2}\right)$

$$x = \frac{13}{6} \times 27$$

$$= \$58.50$$

$\therefore$ 13 kg of biscuits cost \$58.50.

PRACTISE NOW 1

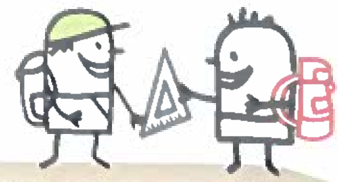
SIMILAR QUESTIONS

Exercise 1A Questions 1–2, 5(a)–(b), 6

(a) If 50 g of sweets cost \$2.10, find the cost of 380 g of sweets, giving your answer correct to the nearest 5 cents.

(b) $\frac{3}{4}$ of a piece of metal has a mass of 15 kg. What is the mass of $\frac{2}{5}$ of the piece of metal?

1.2 Algebraic and Graphical Representations of Direct Proportion



In the example on overdue books in Section 1.1, we have found that $\frac{y}{x} = 15$, which is a constant. If we represent this constant by k, then $\frac{y}{x} = k$ or $y = kx$, where $k \neq 0$. Hence, we have:

> If y is *directly proportional* to x, then $\frac{y}{x} = k$ or $y = kx$, where k is a constant and $k \neq 0$.

Thinking Time

If we substitute $k = 0$ into $y = kx$, what can we say about the relationship between x and y?

Investigation

Graphical Representation of Direct Proportion

Consider the example on overdue books in Section 1.1. Table 1.3 shows the fines, y cents, for various number of days, x, a book is overdue, where $\frac{y}{x} = 15$ or $y = 15x$. What does $y = 15x$ mean in this context?

Number of days (x)	0	1	2	3	4	5	6	7	8	9	10
Fine (y cents)	0	15	30	45	60	75	90	105	120	135	150

Table 1.3

1. Plot the graph of y against x in Fig. 1.1.

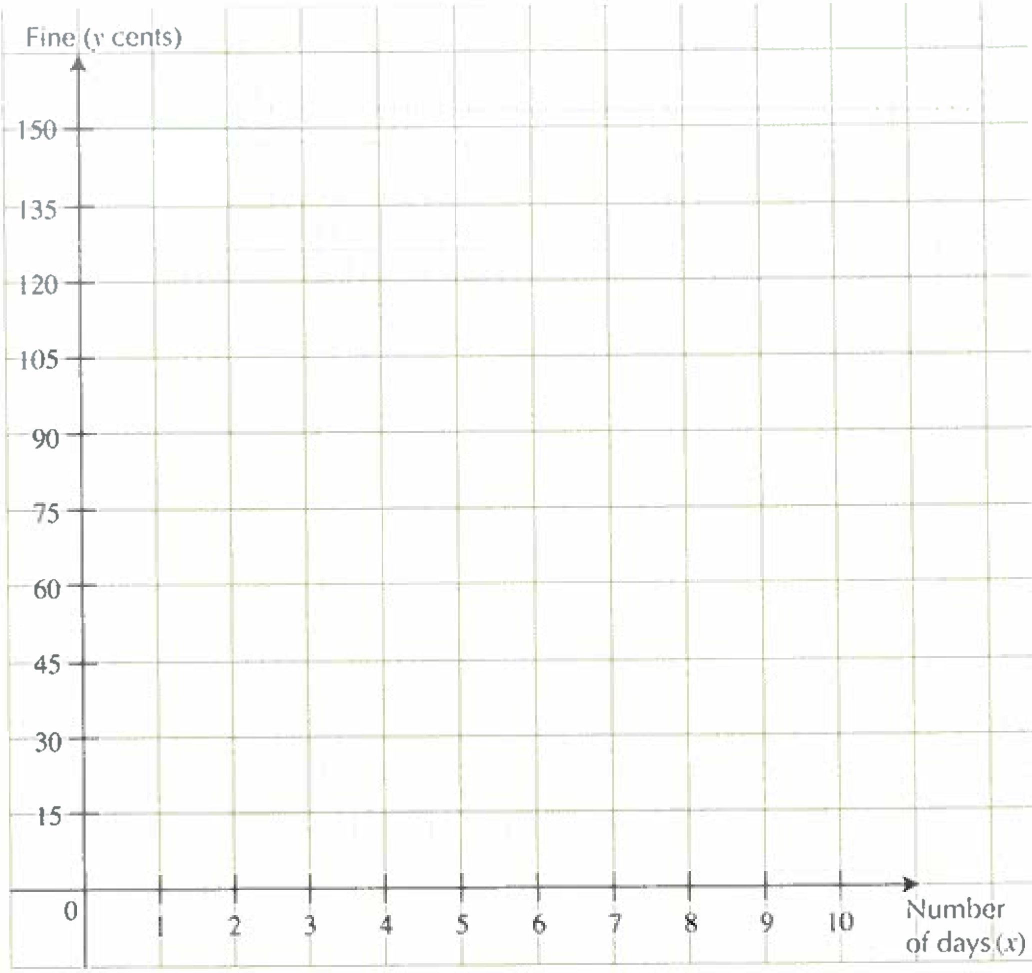
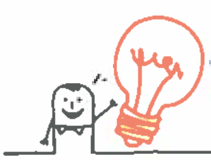


Fig. 1.1

2. What is the shape of the graph?
3. Does the graph pass through the origin?

To conclude, we have:

> If y is *directly proportional* to x, then the graph of y against x is a straight line that passes through the origin.



Thinking Time

1. If y is directly proportional to x, is x directly proportional to y? Explain your answer.
2. Suppose y is directly proportional to x. If we plot the graph of x against y, will we get a straight line that passes through the origin? Explain your answer.
3. If y is directly proportional to x, then the graph of y against x passes through the origin. If the graph of y against x does not pass through the origin, is y directly proportional to x? Explain your answer.
4. As x increases, y also increases. Can we conclude that y is directly proportional to x? Explain your answer.

Worked Example 2

Equation of Direct Proportion

If y is directly proportional to x and $y = 12$ when $x = 4$, find

(i) an equation connecting x and y,

(ii) the value of y when $x = 8$,

(iii) the value of x when $y = 21$.

Solution:

(i) Since y is directly proportional to x,
then $y = kx$, where k is a constant.
When $x = 4$, $y = 12$,

$12 = k \times 4$

$\therefore k = 3$

$\therefore y = 3x$

(ii) When $x = 8$,

$y = 3 \times 8$

$= 24$

Alternatively,
when $x = 8$, (x is doubled)

$y = 2 \times 12$ (y is doubled)

$= 24$

We can also use $\frac{y_2}{y_1} = \frac{x_2}{x_1}$,

i.e. $\frac{y}{12} = \frac{8}{4}$

$y = 2 \times 12$

$= 24$

(iii) Substitute $y = 21$ into $y = 3x$:

$21 = 3x$

$\therefore x = \frac{21}{3}$

$= 7$

ATTENTION

Since y is directly proportional to x, then $\frac{y}{x} = k$ or $y = kx$, where k is a constant and $k \neq 0$.

PRACTISE NOW 2

1. If y is directly proportional to x and $y = 10$ when $x = 2$, find
   (i) an equation connecting x and y,
   (ii) the value of y when $x = 10$,
   (iii) the value of x when $y = 60$.

2. If y is directly proportional to x and $y = 5$ when $x = 2$, find the value of y when $x = 7$.

3. Given that y is directly proportional to x, copy and complete the table.

x	4	5	7		
y		30		48	57

SIMILAR QUESTIONS

Exercise 1A Questions 3–4, 7–8, 9(a)–(b)

Worked Example 3

Problem involving Direct Proportion

The expenses, $\$E$, of a tea party are directly proportional to the number of guests, N, present. When there are 30 guests present at the tea party, the expenses incurred are \$210.

(i) Find an equation connecting E and N.

(ii) Calculate the expenses incurred when there are 80 guests present at the tea party.

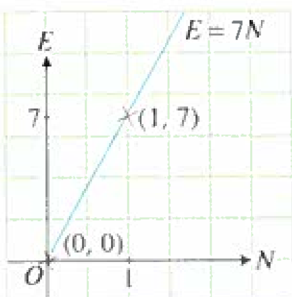
(iii) Draw the graph of E against N.

Solution:

(i) Since E is directly proportional to N,
then $E = kN$, where k is a constant.
When $N = 30$, $E = 210$,
$210 = k \times 30$
$\therefore k = 7$
$\therefore E = 7N$

(ii) When $N = 80$,
$E = 7 \times 80$
$= 560$
$\therefore$ The expenses incurred are \$560.

(iii)



$E = 7N$

When $N = 0$, $E = 0$.

When $N = 1$, $E = 7$.

ATTENTION

Worked Example 3 can also be solved using the Unitary Method or the Proportion Method (see Worked Example 1). However, as there are forms of direct proportion where these two methods will not work, we need to learn the algebraic method to solve them (see Worked Example 6).

PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 1A Questions 10–14

The cost, $\$C$, of transporting goods is directly proportional to the distance covered, d km. The cost of transporting goods over a distance of 60 km is \$100.

(i) Find an equation connecting C and d.

(ii) Find the cost of transporting goods over a distance of 45 km.

(iii) If the cost of transporting goods is \$120, calculate the distance covered.

(iv) Draw the graph of C against d.

Worked Example 4

(Non-Example of Direct Proportion)

The total monthly charges, $C, for a mobile plan consist of a fixed amount of $20 and a variable amount which depends on the usage. For every minute used, $0.20 is charged.

(i) If the duration of usage is 120 minutes, calculate the total monthly charges for the mobile plan.

(ii) If the total monthly charges for the mobile plan are $50, find the duration of usage.

(iii) Write down a formula connecting C and n, where n is the number of minutes of usage.

(iv) Draw the graph of C against n. Is C directly proportional to n? Use your graph to explain your answer.

Solution:

(i) Total monthly charges for the mobile plan $= \$20 + 120 \times \$0.20$

$= \$44$

(ii) Variable amount $= \$50 - \$20$

$= \$30$

Duration of usage $= \dfrac{30}{0.20}$

$= 150$ minutes

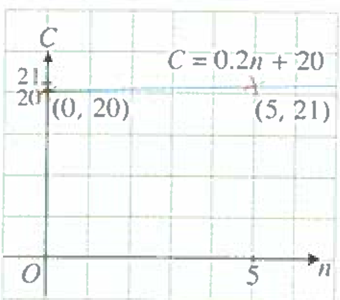
(iii) Variable amount $= n \times \$0.20$

$= \$0.2n$

Total monthly charges = variable amount + fixed amount

$\therefore C = 0.2n + 20$

(iv)



$C = 0.2n + 20$

When $n = 0$, $C = 20$.

When $n = 5$, $C = 21$.

C is *not* directly proportional to n because the line does not pass through the origin.

ATTENTION

- Since n cannot be negative, the line must start from $n = 0$.
- When $n = 120$, $C = 44$.
  When $n = 150$, $C = 50$.

  Since $\frac{44}{120} \neq \frac{50}{150}$, $\frac{C}{n}$ is not a constant.

  $\therefore C$ is not directly proportional to n.

PRACTISE NOW 4

The total monthly cost, $C, of running a kindergarten consists of a fixed amount of $5000 and a variable amount which depends on the enrolment. For every child enrolled, the monthly cost increases by $41.

(i) If the enrolment is 200, find the total monthly cost of running the kindergarten.

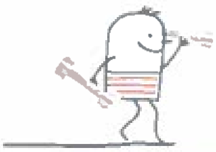
(ii) If the total monthly cost of running the kindergarten is $20 580, calculate the number of children enrolled in the kindergarten.

(iii) Write down a formula connecting C and n, where n is the number of children enrolled in the kindergarten.

(iv) Draw the graph of C against n. Is C directly proportional to n? Use your graph to explain your answer.

SIMILAR QUESTIONS

Exercise 1A Questions 15–16



Exercise 1A

BASIC LEVEL

1. 108 identical books have a mass of 30 kg. Find
- **(i)** the mass of 150 such books,
- **(ii)** the number of such books that have a mass of 20 kg.

2. In a bookstore, 60 identical books occupy a length of 1.5 m on a shelf. Find
- **(i)** the length occupied by 50 such books on a shelf,
- **(ii)** the number of such books needed to completely occupy a shelf that is 80 cm long.

3. If x is directly proportional to y and $x = 4.5$ when $y = 3$, find
- **(i)** an equation connecting x and y,
- **(ii)** the value of x when $y = 6$,
- **(iii)** the value of y when $x = 12$.

4. If Q is directly proportional to P and $Q = 28$ when $P = 4$,
- **(i)** express Q in terms of P,
- **(ii)** find the value of Q when $P = 5$,
- **(iii)** calculate the value of P when $Q = 42$.

INTERMEDIATE LEVEL

5. Find the cost of
- **(a)** 10 kg of tea leaves when 3 kg of tea leaves cost \$18,
- **(b)** a kg of sugar when b kg of sugar cost \$$c$.

6. $\frac{5}{9}$ of a piece of metal has a mass of 7 kg. What is the mass of $\frac{2}{7}$ of the piece of metal?

7. If z is directly proportional to x and $z = 12$ when $x = 3$, find the value of x when $z = 18$.

8. If B is directly proportional to A and $B = 3$ when $A = 18$, find the value of B when $A = 24$.

9. For each of the following, y is directly proportional to x. Copy and complete the tables.

(a)

x	4	20	24		
y			6	9	11

(b)

x	2	3	5.5		
y		3.6		9.6	11.4

10. If y is directly proportional to x and $y = 20$ when $x = 5$,
- **(i)** find an equation connecting x and y,
- **(ii)** draw the graph of y against x.

11. If z is directly proportional to y and $z = 48$ when $y = 6$,
- **(i)** find an equation connecting y and z,
- **(ii)** draw the graph of z against y.

12. The net force, F newtons, needed to push a block along a horizontal surface is directly proportional to the mass, m kg, of the block. When $m = 5$, $F = 49$.
- **(i)** Find an equation connecting F and m.
- **(ii)** Find the value of F when $m = 14$.
- **(iii)** Calculate the value of m when $F = 215.6$.
- **(iv)** Draw the graph of F against m.

13. The pressure, P pascals, of a gas in a container is directly proportional to its temperature, T kelvin. When $T = 10$, $P = 25$.
- **(i)** Find an equation connecting P and T.
- **(ii)** Find the value of P when $T = 24$.
- **(iii)** Calculate the value of T when $P = 12$.
- **(iv)** Draw the graph of P against T.

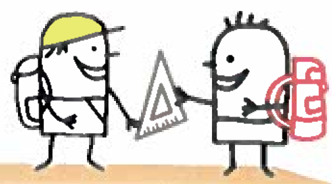
14. The amount of voltage, V volts, needed to send a fixed amount of current through a wire, is directly proportional to its resistance, R ohms. When $R = 6$, $V = 9$.
- **(i)** Find an equation connecting V and R.
- **(ii)** Find the value of V when $R = 15$.
- **(iii)** Calculate the value of R when $V = 15$.
- **(iv)** Draw the graph of V against R.

ADVANCED LEVEL

15. The total monthly income, $\$D$, of a salesman who sells tyres consists of a basic salary of \$600 and a variable amount which depends on the number of tyres he sells. For each tyre he sells, he receives \$8.
    (i) If the salesman sells 95 tyres in a particular month, find his total income for that month.
    (ii) If the salesman's monthly income for a particular month is \$1680, calculate the number of tyres he sells in that month.
    (iii) Write down a formula connecting D and n, where n is the number of tyres the salesman sells in a month.
    (iv) Draw the graph of D against n. Is D directly proportional to n? Use your graph to explain your answer.

16. A machine which manufactures ice needs to be run for 10 minutes to warm up *before* the production of ice begins. The mass, in tonnes, of ice produced is directly proportional to the number of hours of production. Given that 20 tonnes of ice are produced when the machine runs for half an hour, find the mass of ice manufactured when the machine runs for 1.75 hours.

1.3 Other Forms of Direct Proportion



Investigation

Other Forms of Direct Proportion

The variables x and y are connected by the equation $y = 3x^2$.

- Some values of x, and the corresponding values of y and $\frac{y}{x}$ are given in Table 1.4.

x	1	2	3	4
y	3	12	27	48
$\frac{y}{x}$	3	6	9	12

Table 1.4

Fig. 1.2 shows the graph of y against x.

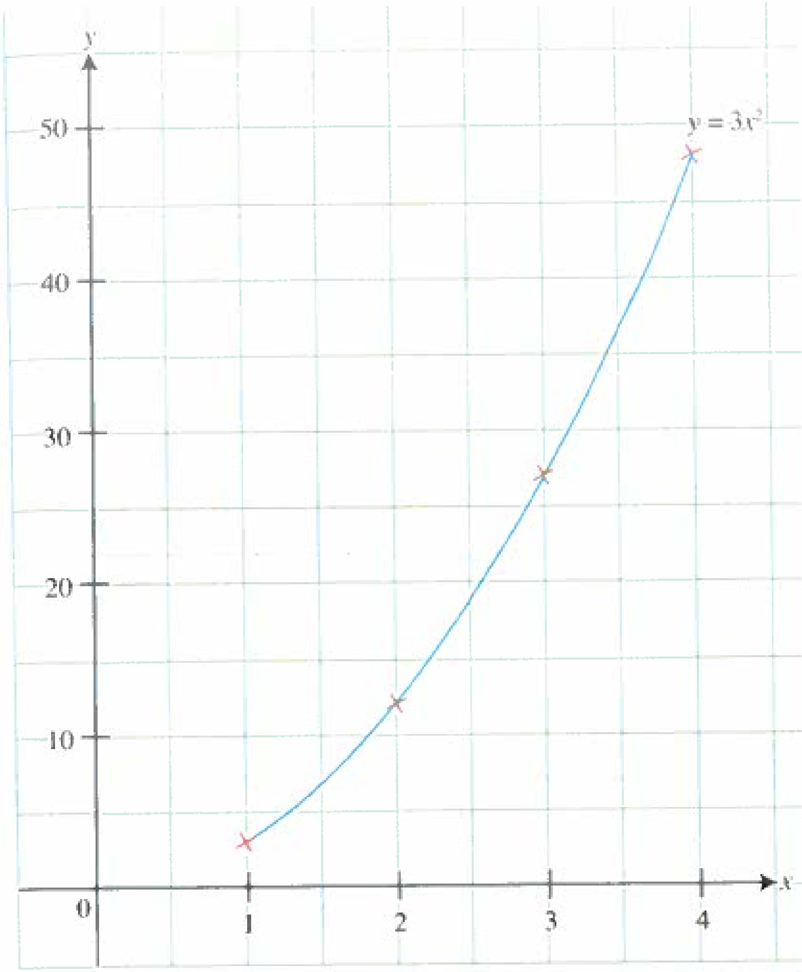


Fig. 1.2

Is y directly proportional to x? Explain your answer.

2. Now, let us plot the graph of y against x^2.

Some values of x, and the corresponding values of x^2, y and $\frac{y}{x^2}$ are given in Table 1.5.

x	1	2	3	4
x^2	1	4	9	16
y	3	12	27	48
$\frac{y}{x^2}$	3	3	3	3

Table 1.5

Plot the graph of y against x^2 in Fig. 1.3. The first three points have been plotted for you.

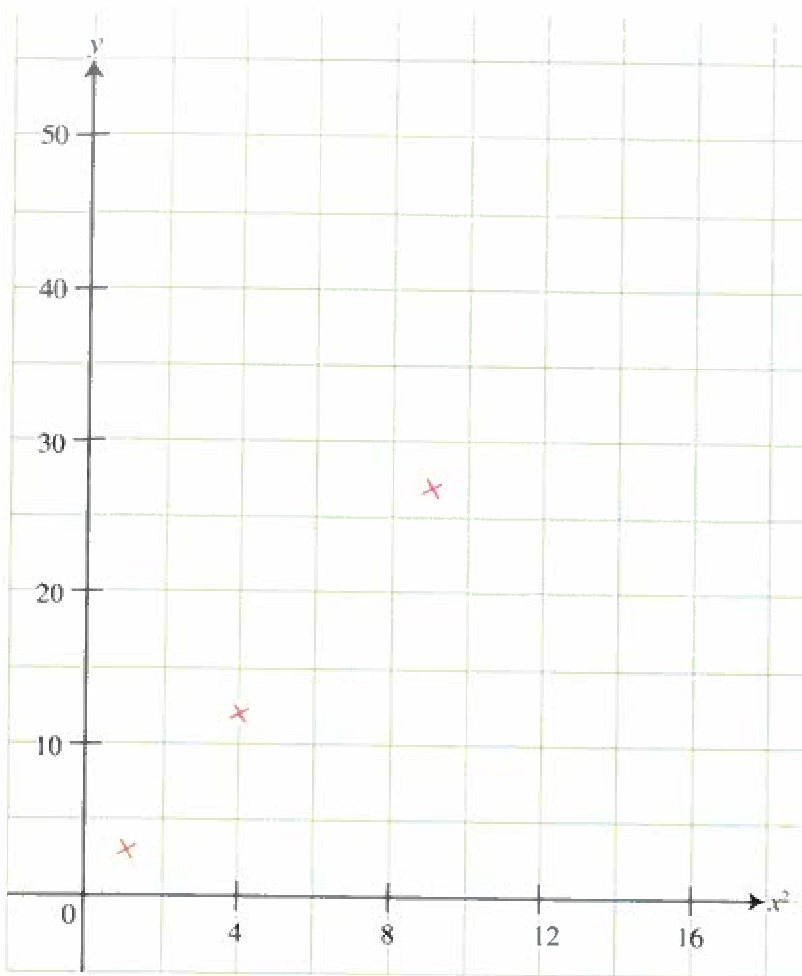


Fig. 1.3

Is y directly proportional to x^2? Explain your answer.

From the investigation, when $y = 3x^2$, y is not directly proportional to x because $\frac{y}{x} = 3x$ is not a constant. However, y is directly proportional to x^2 because $\frac{y}{x^2} = 3$ is a constant.

INFORMATION

Another way to look at this is to let $X = x^2$ such that $y = 3x^2$ becomes $y = 3X$, i.e. $\frac{y}{X} = 3$. Therefore, y is directly proportional to X $(= x^2)$.

Worked Example 5

Identifying Variables Which Are Directly Proportional to Each Other

For each of the following equations, state the two variables which are directly proportional to each other and explain your answer.

(a) $y = 5x^3$ **(b)** $y^2 = \sqrt{x}$

Solution:

(a) Since $y = 5x^3$, i.e. $\frac{y}{x^3} = 5$ is a constant, then y is directly proportional to x^3.

(b) Since $y^2 = \sqrt{x}$, i.e. $\frac{y^2}{\sqrt{x}} = 1$ is a constant, then y^2 is directly proportional to $\sqrt{x}$.

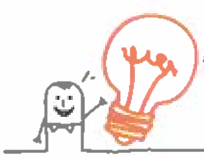
PRACTISE NOW 5

For each of the following equations, state the two variables which are directly proportional to each other and explain your answer.

(a) $y = 6x^2$ **(b)** $\sqrt{y} = x^3$

SIMILAR QUESTIONS

Exercise 1B Questions 4(a)–(d)



Thinking Time

1. In Worked Example 4, we have found that the formula connecting C and n is $C = 0.2n + 20$ and that C is not directly proportional to n. However, n is directly proportional to a variable. What is this variable?

2. In the equation $y - 1 = 4x$, state the two variables which are directly proportional to each other and explain your answer.

Worked Example 6

Equation of Another Form of Direct Proportion

If y is directly proportional to x^2 and $y = 20$ when $x = 2$,

(i) find an equation connecting x and y,

(ii) calculate the value of y when $x = 3$,

(iii) find the values of x when $y = 1.25$,

(iv) draw the graph of y against x^2.

Solution:

(i) Since y is directly proportional to x^2,
then $y = kx^2$, where k is a constant.

When $x = 2$, $y = 20$,

$20 = k \times 2^2$

$20 = 4k$

$\therefore k = 5$

$\therefore y = 5x^2$

(ii) When $x = 3$,

$y = 5 \times 3^2$

$= 45$

(iii) When $y = 1.25$,

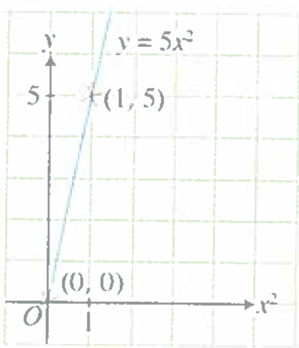
$1.25 = 5x^2$

$x^2 = 0.25$

$\therefore x = \pm\sqrt{0.25}$

$= \pm 0.5$

(iv) Since y is directly proportional to x^2, then the graph of y against x^2 is a straight line that passes through the origin.



$y = 5x^2$

When $x = 0$, $y = 0$.

When $x = 1$, $y = 5$.

ATTENTION

It is difficult to use the Unitary Method or the Proportion Method (see Worked Example 1) to solve direct proportion problems like in Worked Example 6. Thus we need to learn the algebraic method (see Worked Example 3).

ATTENTION

As x^2 cannot be negative, the line must start from the origin where $x^2 = 0$.

PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 1B Questions 1–3, 5–8, 11

1. If y is directly proportional to x^2 and $y = 18$ when $x = 3$,
   (i) find an equation connecting x and y,
   (ii) find the value of y when $x = 5$,
   (iii) calculate the values of x when $y = 32$,
   (iv) draw the graph of y against x^2.

2. If y is directly proportional to x^2 and $y = 21$ when $x = 2$, find the value of y when $x = 4$.

3. Given that y is directly proportional to x^2, where x is a positive real number, copy and complete the table.

x	2		3	5	
y		56.25	81		441

Worked Example 7

Problems involving Another Form of Direct Proportion

The volume, V cm^3, of a solid is directly proportional to the cube of its radius, r cm. When the radius of the solid is 6 cm, its volume is 905 cm^3.

(i) Find an equation connecting V and r.

(ii) Calculate the volume of the solid when its radius is 10 cm.

Solution:

(i) Since V is directly proportional to r^3,
then $V = kr^3$, where k is a constant.

When $r = 6$, $V = 905$,

$905 = k \times 6^3$

$\therefore\ k = \dfrac{905}{216}$

$\therefore\ V = \dfrac{905}{216} r^3$

(ii) When $r = 10$,

$V = \dfrac{905}{216} \times 10^3$

$= 4189\dfrac{22}{27}$

$\therefore$ The volume of the solid is $4189\dfrac{22}{27}$ cm^3.

PRACTISE NOW 7

SIMILAR QUESTIONS

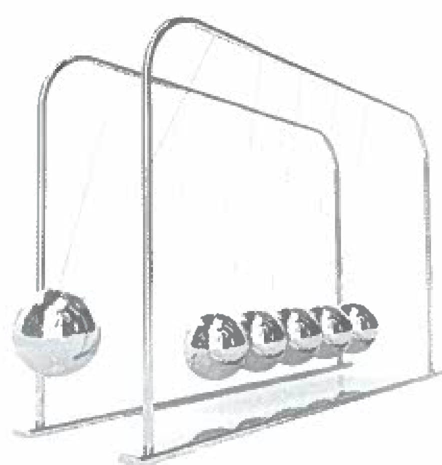
Exercise 1B Questions 9–10, 12

The length, l cm, of a simple pendulum is directly proportional to the square of its period (time taken to complete one oscillation), T seconds. A pendulum with a length of 220.5 cm has a period of 3 s.

(i) Find an equation connecting l and T.

(ii) Find the length of a pendulum which has a period of 5 s.

(iii) What is the period of a pendulum which has a length of 0.98 m?



Exercise 1B

BASIC LEVEL

1. If x is directly proportional to y^3 and $x = 32$ when $y = 2$,
   (i) find an equation connecting x and y,
   (ii) find the value of x when $y = 6$,
   (iii) calculate the value of y when $x = 108$,
   (iv) draw the graph of x against y^3.

2. If z^2 is directly proportional to w and $z = 4$ when $w = 8$,
   (i) find an equation connecting w and z,
   (ii) find the values of z when $w = 18$,
   (iii) calculate the value of w when $z = 5$,
   (iv) draw the graph of z^2 against w.

3. It is given that y is directly proportional to x^n. Write down the value of n when
   (i) y m² is the area of a square of length x m,
   (ii) y cm³ is the volume of a cube of length x cm.

INTERMEDIATE LEVEL

4. For each of the following equations, state the two variables which are directly proportional to each other and explain your answer.
   (a) $y = 4x^2$
   (b) $y = 3\sqrt{x}$
   (c) $y^2 = 5x^3$
   (d) $p^3 = q^2$

5. If z^2 is directly proportional to x^3 and $z = 8$ when $x = 4$, find the values of z when $x = 9$.

6. If q is directly proportional to $(p - 1)^2$ and $q = 20$ when $p = 3$, find the values of p when $q = 80$.

7. Given that y is directly proportional to x^3, copy and complete the table.

x	3	4		6	
y			375	648	1029

8. Given that the mass, m g, of a sphere is directly proportional to the cube of its radius, r cm, copy and complete the table.

r	0.2		0.7	1.5	
m		0.25		6.75	11.664

9. During a certain period in the life of an earthworm, its length, L cm, is directly proportional to the square root of N, where N is the number of hours after its birth. One hour after an earthworm is born, its length is 2.5 cm.
   (i) Find an equation connecting L and N.
   (ii) Find the length of an earthworm 4 hours after its birth.
   (iii) How long will it take for an earthworm to grow to a length of 15 cm?

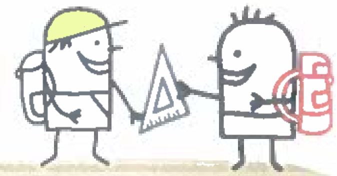
10. If y is directly proportional to x^2 and the difference in the values of y when $x = 1$ and $x = 3$ is 32, find the value of y when $x = -2$.

ADVANCED LEVEL

11. y is directly proportional to x^2 and $y = a$ for a particular value of x. Find an expression for y in terms of a, when this value of x is doubled.

12. The braking distance of a vehicle is directly proportional to the square of its speed. When the speed of the vehicle is b m/s, its braking distance is d m. If the speed of the vehicle is increased by 200%, find the percentage increase in its braking distance.

1.4 Inverse Proportion



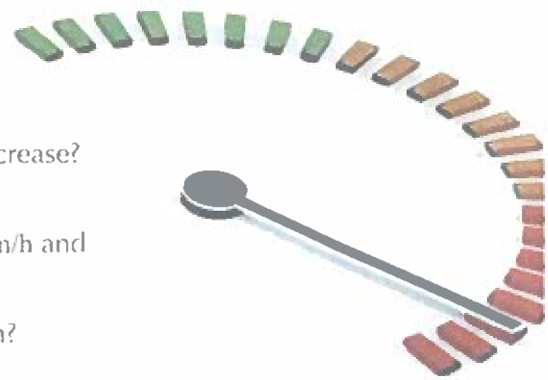
Investigation

Inverse Proportion

Table 1.6 shows the time taken for a car to travel a distance of 120 km at different speeds.

Speed (x km/h)	10	20	30	40	60	120
Time taken (y hours)	12	6	4	3	2	1

Table 1.6

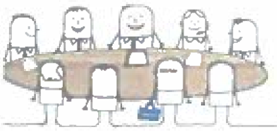


1. If the speed of the car increases, will the time taken increase or decrease?
2. If the speed of the car is doubled, how will the time taken change?
   Hint: Compare the time taken when the speeds of the car are 20 km/h and 40 km/h.
3. If the speed of the car is tripled, what will happen to the time taken?
4. If the speed of the car is halved, how will the time taken change?
   Hint: Compare the time taken when the speeds of the car are 60 km/h and 30 km/h.
5. If the speed of the car is reduced to $\frac{1}{3}$ of its original speed, what will happen to the time taken?

From the investigation, we notice that as the speed of the car, x km/h, increases, the time taken, y hours, decreases *proportionally*, i.e. if x is doubled, then y will be halved; if x is tripled, then y will be reduced to $\frac{1}{3}$ of its original value.

Similarly, as the speed of the car, x km/h, decreases, the time taken, y hours, increases *proportionally*, i.e. if x is halved, then y will be doubled; if x is reduced to $\frac{1}{3}$ of its original value, then y will be tripled.

This relationship is known as **inverse proportion**. We say that the speed of the car, x km/h, is *inversely proportional* to the time taken, y hours.



Class Discussion

Real-Life Examples of Quantities in Inverse Proportion

Work in pairs.

1. Give a few more real-life examples of quantities that are in inverse proportion.
2. Explain why they are inversely proportional to each other.

SIMILAR QUESTIONS

Exercise 1C Questions 1(a)–(e)

Complete Table 1.7.

Speed (x km/h)	10	20	30	40	60	120
Time taken (y hours)	12	6	4	3	2	1
Product xy km	$10 \times 12 = 120$	$20 \times 6 = 120$				

Table 1.7

Vhat can we observe about the product xy?

n inverse proportion, the product xy is a *constant*. In this case, xy = 120
= distance travelled.

et the speed of the car be $x_1 = 20$. Then the corresponding time taken is $y_1 = 6$.

et the speed of the car be $x_2 = 40$. Then the corresponding time taken is $y_2 = 3$.

rom Table 1.7, $x_1y_1 = 20 \times 6 = 120$ and $x_2y_2 = 40 \times 3 = 120$.

$\therefore x_1y_1 = x_2y_2 = 120$ (constant)

earranging,

$$\frac{y_2}{y_1} = \frac{x_1}{x_2}$$

o conclude, we have:

> If y is *inversely proportional* to x, then $\frac{y_2}{y_1} = \frac{x_1}{x_2}$ or $x_1y_1 = x_2y_2$.

ATTENTION

We have learnt that for direct proportion, $\frac{y_2}{y_1} = \frac{x_2}{x_1}$, but for inverse proportion, we have $\frac{y_2}{y_1} = \frac{x_1}{x_2}$ or $x_1y_1 = x_2y_2$. Note the order of x_1 and x_2.

Worked Example 8

(Problem involving Inverse Proportion)

10 identical taps can fill a tank in 4 hours. Calculate the time taken for 8 such taps to fill the same tank.

Solution:

First, we note that the time taken to fill the tank is inversely proportional to the number of taps used because as the number of taps increases, the time taken to fill the tank decreases.

Method 1: Unitary Method

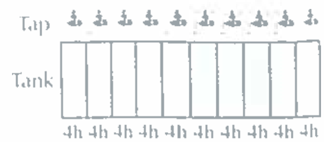
10 taps can fill the tank in 4 hours.

1 tap can fill the tank in (10×4) hours. (fewer taps require more time)

8 taps can fill the tank in $\frac{10 \times 4}{8}$ (more taps require less time)

$= 5$ hours.

Problem Solving Tip



Each tap can fill $\frac{1}{10}$ of the tank in 4 hours.

Method 2: Proportion Method

Let the time taken for 8 taps to fill the tank be y hours.

Then $8y = 10 \times 4$. $(x_1y_1 = x_2y_2)$

$$y = \frac{10 \times 4}{8}$$
$$= 5$$

$\therefore$ 8 taps can fill the tank in 5 hours.

Problem Solving Tip

It is more confusing to use $\frac{y_2}{y_1} = \frac{x_1}{x_2}$ to solve inverse proportion problems like in Worked Example 8 because we need to switch the order of x_1 and x_2.

PRACTISE NOW 8

Four identical taps can fill a tank in 70 minutes. Find the time taken for 7 such taps to fill the same tank.

SIMILAR QUESTIONS

Exercise 1C Questions 2, 5–7, 15

Worked Example 9

(Problem involving Direct and Inverse Proportions)

5 men can paint 2 identical houses in 3 days. Assuming that all the men work at the same rate, how long will it take 10 men to paint 8 such houses?

Solution:

The three variables are 'number of men', 'number of houses' and 'number of days'.

First, we keep the number of houses constant.

Number of men	Number of houses	Number of days	
5	2	3	
1	2	5×3	(fewer men require more days)
10	2	$\frac{5 \times 3}{10} = 1.5$	(more men require fewer days)

Next, we keep the number of men constant.

Number of men	Number of houses	Number of days	
10	2	1.5	
10	1	$\frac{1.5}{2}$	(fewer houses require fewer days)
10	8	$8 \times \frac{1.5}{2} = 6$	(more houses require more days)

∴ 10 men will take 6 days to paint 8 houses.

Problem Solving Tip

When three variables are involved, we keep one variable constant at a time.

ATTENTION

The number of days required to paint the houses is inversely proportional to the number of men.

ATTENTION

The number of days required to paint the houses is directly proportional to the number of houses that need to be painted.

PRACTISE NOW 9

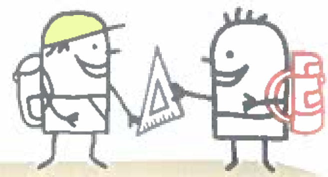
a) 3 men can dig 2 identical trenches in 5 hours. Assuming that all the men work at the same rate, how long will it take 5 men to dig 7 such trenches?

b) 7 identical taps can fill 3 identical tanks in 45 minutes. How long will it take 5 of the taps to fill one of the tanks?

SIMILAR QUESTIONS

Exercise 1C Questions 13–14, 16

1.5 Algebraic and Graphical Representations of Inverse Proportion



In the investigation on page 19 of Section 1.4, we have found that $xy = 120$, which is a constant. If we represent this constant by k, then $xy = k$ or $y = \frac{k}{x}$, where $k \neq 0$. Hence, we have:

> If y is *inversely proportional* to x, then $xy = k$ or $y = \frac{k}{x}$, where k is a constant and $k \neq 0$.

Thinking Time

If we substitute $k = 0$ into $y = \frac{k}{x}$, what can we say about the relationship between x and y?

Investigation

Graphical Representation of Inverse Proportion

Consider the example on the car in Section 1.4. Table 1.8 shows the time taken, y hours, for the car to travel a distance of 120 km at different speeds, x km/h, where $xy = 120$ or $y = \frac{120}{x}$.

Speed (x km/h)	10	20	30	40	50	60	70	80	90	100	110	120
Time taken (y hours)	12	6	4	3	2.4	2	1.7	1.5	1.3	1.2	1.1	1

Table 1.8

1. If we plot the graph of y against x, what type of graph would we obtain?
2. Plot the graph of y against x in Fig. 1.4.

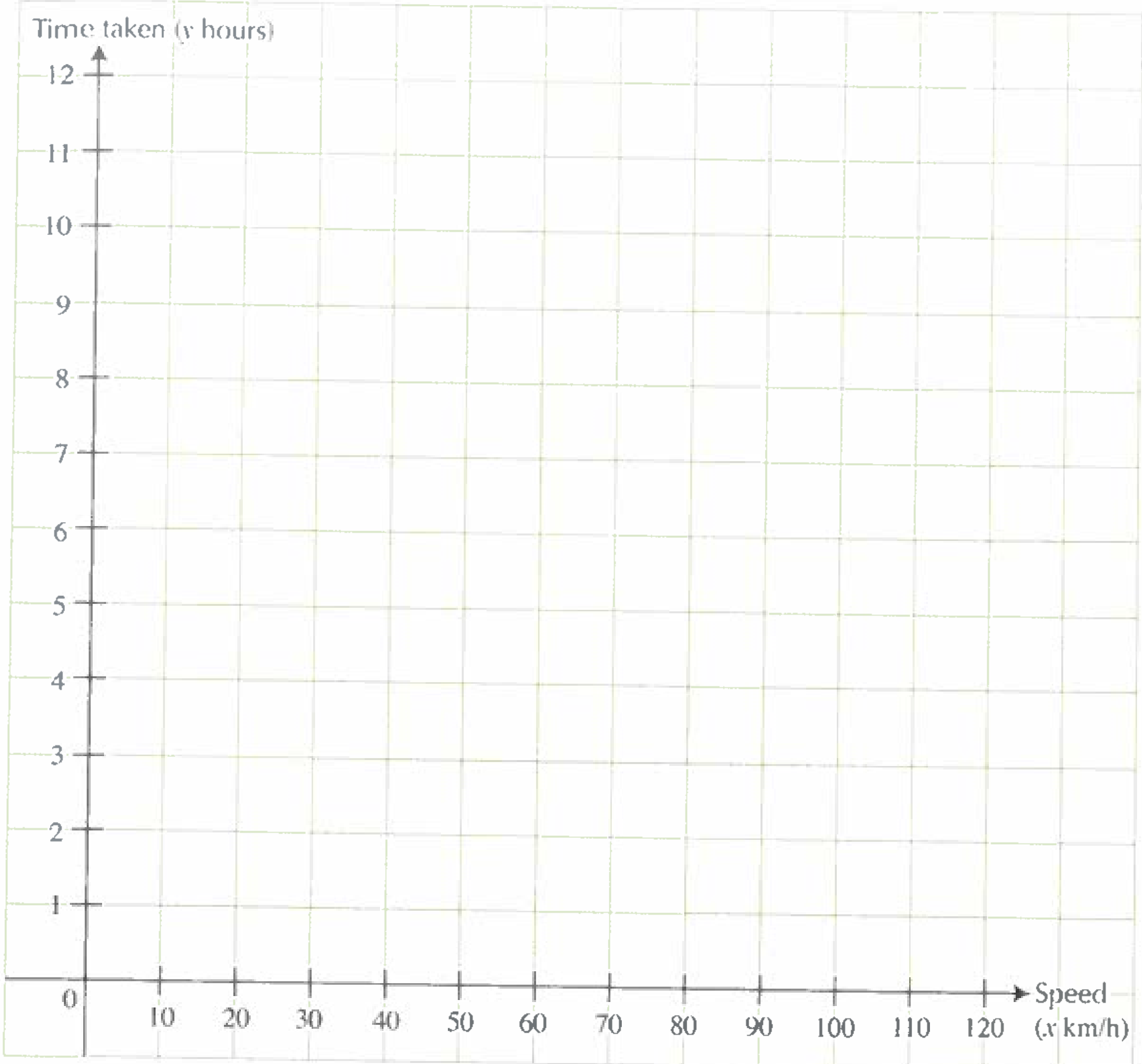


Fig. 1.4

3. If the value of x is doubled, how will the value of y change?
Illustrate this with an example by choosing two appropriate points on the graph you have drawn in Fig. 1.4.

What if we plot the graph of y against $\frac{1}{x}$? How would the graph look like?

Let $X = \frac{1}{x}$. Some values of x, and the corresponding values of X and y are given in Table 1.9. Complete the table.

Speed (x km/h)	10	20	30	40	50	60	70	80	90	100	110	120
$X = \frac{1}{x}$	0.1	0.05	0.033		0.02	0.017	0.014	0.013	0.011		0.009	0.008
Time taken (y hours)	12	6	4	3	2.4	2	1.7	1.5	1.3	1.2	1.1	1

Table 1.9

4. Plot the graph of y against X in Fig. 1.5.

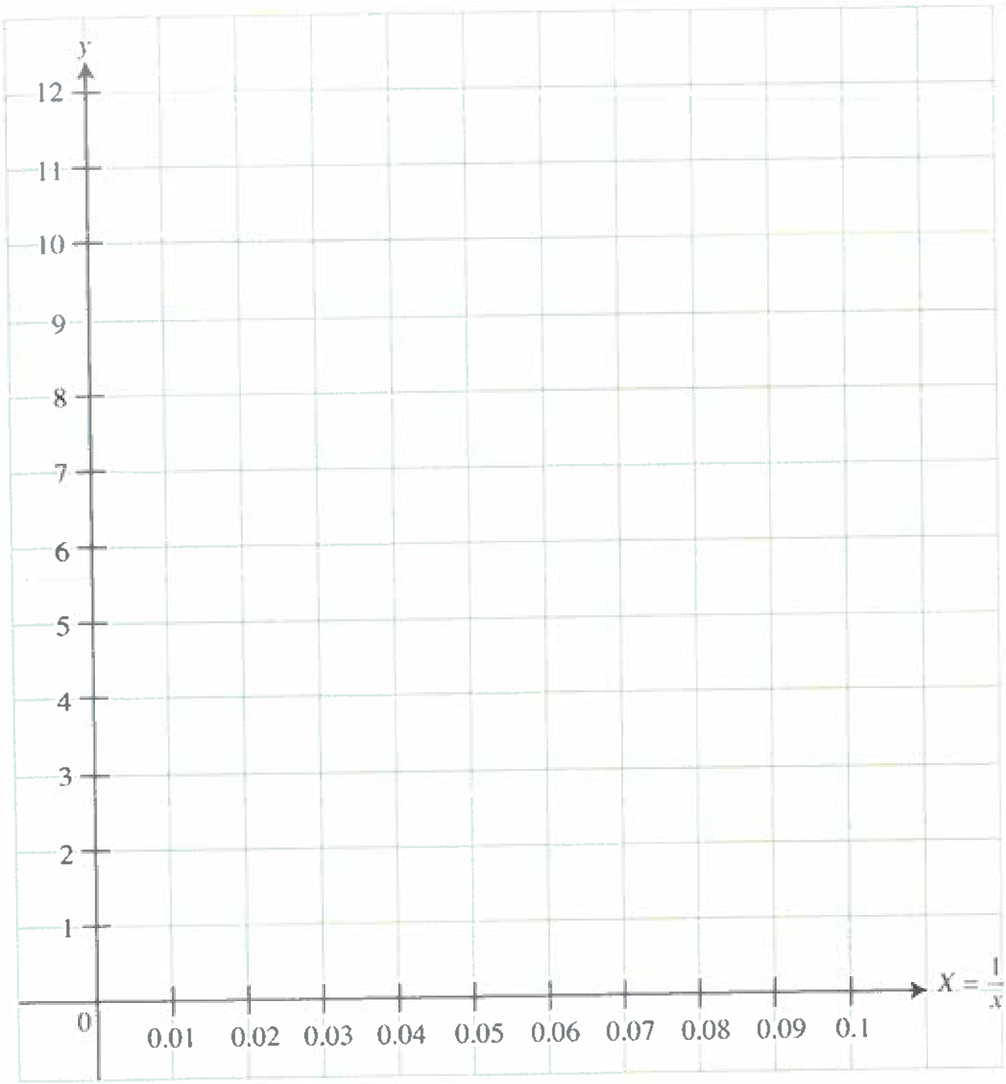


Fig. 1.5

5. Describe the graph obtained. What can we say about the relationship between y and X?
6. Although y is inversely proportional to x, what is the relationship between y and X?
7. Write down an equation connecting y and X. What does it tell you about the relationship between y and X?

In general,

if y is *inversely proportional* to x, then

- $xy = k$ or $y = \frac{k}{x}$, where k is a constant and $k \neq 0$,
- the graph of y against x (and of equation $y = \frac{k}{x}$) is a hyperbola (see Fig. 1.4),
- the graph of y against $\frac{1}{x}$ is a straight line (see Fig. 1.5).

Thinking Time

If y is inversely proportional to x, is x inversely proportional to y? Explain your answer.

Worked Example 10

(Equation of Inverse Proportion)

If y is inversely proportional to x and $y = 3$ when $x = 4$, find

(i) the value of y when $x = 8$,

(ii) an equation connecting x and y,

(iii) the value of x when $y = 48$.

Solution:

(i) When $x = 8$, (x is doubled)

$y = \frac{3}{2}$ (y is halved)

$= 1\frac{1}{2}$

Alternatively,

$x_2 y_2 = x_1 y_1$

$8 \times y = 4 \times 3$

$y = \frac{12}{8}$

$= 1\frac{1}{2}$

(ii) Since y is inversely proportional to x,

then $y = \frac{k}{x}$, where k is a constant.

When $x = 4$, $y = 3$,

$3 = \frac{k}{4}$

$\therefore k = 12$

$\therefore y = \frac{12}{x}$

(iii) When $y = 48$,

$48 = \frac{12}{x}$

$\therefore x = \frac{12}{48}$

$= \frac{1}{4}$

ATTENTION

Since y is inversely proportional to x, then $xy = k$ or $y = \frac{k}{x}$, where k is a constant and $k \neq 0$.

Just For Fun

Raj and Ethan are going to compete in a race. Raj's average speed is twice that of Ethan's.

Ethan wants to start 10 m in front of Raj. Ethan says, 'After Raj runs 10 m, I will be 5 m in front of him. After Raj runs another 5 m, I will be 2.5 m in front of him. Thus Raj will never catch up with me.' Is Ethan correct?

This is a variation of one of Zeno's paradoxes. Zeno (490 – 430 B.C.) was a Greek philosopher.

PRACTISE NOW 10

SIMILAR QUESTIONS

Exercise 1C Questions 3–4, 8–9, 10(a)–(b)

1. If y is inversely proportional to x and $y = 5$ when $x = 2$, find
   (i) the value of y when $x = 8$,
   (ii) an equation connecting x and y,
   (iii) the value of x when $y = 10$.

2. If y is inversely proportional to x and $y = 9$ when $x = 2$, find the value of y when $x = 3$.

3. Given that y is inversely proportional to x, copy and complete the table.

x	0.5		2	3	
y		4	2		0.8

Worked Example 11

Problem involving Inverse Proportion

Boyle's Law states that the volume, V dm^3, of a fixed mass of gas at constant temperature is inversely proportional to its pressure, P pascals (Pa). The pressure of 1 dm^3 of a gas in an airtight container is 50 Pa. Assuming that the temperature in the container is constant, calculate the volume of the gas when its pressure is 1250 Pa.

INFORMATION

1 dm^3 = 1000 cm^3
(dm^3 means 'cubic decimetre')

Solution:

Since V is inversely proportional to P,

then $V = \frac{k}{P}$, where k is a constant.

When $P = 50$, $V = 1$,

$$1 = \frac{k}{50}$$

$\therefore k = 50$

$\therefore \quad V = \frac{50}{P}$

When $P = 1250$,

$$V = \frac{50}{1250}$$
$$= 0.04$$

$\therefore$ The volume of the gas is 0.04 dm^3.

PRACTISE NOW 11

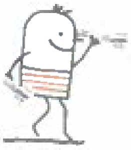
SIMILAR QUESTIONS

Exercise 1C Questions 11–12

The current, I amperes (A), flowing through a wire is inversely proportional to its resistance, R ohms (Ω). Given that the current flowing through a wire with a resistance of 0.5 Ω is 12 A, find

(i) the current flowing through the wire when its resistance is 3 Ω,

(ii) the resistance of the wire when the current flowing through it is 3 A.



Exercise 1C

BASIC LEVEL

Which of the following quantities are in inverse proportion? State the assumption made in each case.

(a) The number of pencils Farhan buys and the total cost of the pencils.

(b) The number of taps filling a tank and the time taken to fill the tank.

(c) The number of men laying a road and the time taken to finish laying the road.

(d) The number of cattle to be fed and the amount of fodder.

(e) The number of cattle to be fed and the time taken to finish a certain amount of the fodder.

Eight men can build a bridge in 12 days. Find the time taken for 6 men to build the same bridge. State the assumption made.

If x is inversely proportional to y and $x = 40$ when $y = 5$, find

(i) the value of x when $y = 25$,

(ii) an equation connecting x and y,

(iii) the value of y when $x = 400$.

If Q is inversely proportional to P and $Q = 0.25$ when $P = 2$,

(i) express Q in terms of P,

(ii) find the value of Q when $P = 5$,

(iii) calculate the value of P when $Q = 0.2$.

6. A consignment of fodder can feed 1260 cattle for 50 days. Given that all the cattle consume the fodder at the same rate, find

(i) the number of cattle an equal consignment of fodder can feed for 75 days,

(ii) the number of days an equal consignment of fodder can last if it is used to feed 1575 cattle.

7. At a sports camp, there is sufficient food for 72 athletes to last 6 days. If 18 athletes are absent from the camp, how many more days can the food last for the other athletes? State the assumption made.

8. If z is inversely proportional to x and $z = 5$ when $x = 7$, find the value of x when $z = 70$.

9. If B is inversely proportional to A and $B = 3.5$ when $A = 2$, find the value of B when $A = 1.4$.

10. For each of the following, y is inversely proportional to x. Copy and complete the tables.

(a)

x		2	2.5	3	
y	24			4	1.5

(b)

x	3	4			25
y		9	8	2.5	

11. The frequency, f kilohertz (kHz), of a radio wave is inversely proportional to its wavelength, λ m. The frequency of a radio wave that has a wavelength of 3000 m is 100 kHz. Find

(i) the frequency of a radio wave that has a wavelength of 500 m,

(ii) the wavelength of a radio wave that has a frequency of 800 kHz.

INTERMEDIATE LEVEL

35 workers are employed to complete a construction project in 16 days. Before the project starts, the boss of the company is told that he needs to complete the project in 14 days. Assuming that all the workers work at the same rate, how many more workers does he need to employ in order to complete the project on time?

12. The time, t hours, needed to complete a job is inversely proportional to N, where N is the number of men employed for the job. 3 men can complete the job in 8 hours.
 (i) Find an equation connecting t and N.
 (ii) Find the number of hours needed by 6 men to complete the job.
 (iii) If the job is to be completed in $\frac{3}{4}$ hour, how many men need to be employed?

ADVANCED LEVEL

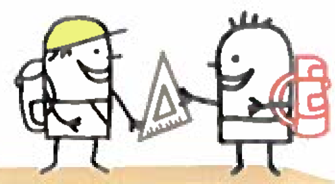
13. 12 glassblowers can make 12 identical vases in 9 minutes. Assuming that all the glassblowers work at the same rate, how long will it take 8 glassblowers to make 32 such vases?

14. A consignment of fodder can feed 1000 sheep for 20 days. Assuming that all the sheep consume the fodder at the same rate, how many consignments of fodder are needed to feed 550 sheep for 400 days?

15. Tap A takes 6 minutes to fill a tank and Tap B takes 9 minutes to fill the same tank. Pipe C can empty the tank in 15 minutes. How long will it take to fill up the tank if the pipe is in use when both taps are turned on?

16. A contractor agrees to lay a road 3000 m long in 30 days. 50 men are employed and they work for 8 hours per day. After 20 working days, he finds that only 1200 m of the road is completed. How many more men does he need to employ in order to finish the project on time if each man now works 10 hours a day?

1.6 Other Forms of Inverse Proportion



We have learnt that if y is inversely proportional to x, then

$$xy = k \text{ or } y = \frac{k}{x},$$

where k is a constant and $k \neq 0$.

Similarly, if y is inversely proportional to x^2, then

$$x^2y = k \text{ or } y = \frac{k}{x^2},$$

where k is a constant and $k \neq 0$.

INFORMATION

Another way to look at this is to let $X = x^2$ such that

$$x^2y = k \text{ becomes } Xy = k$$

or $$y = \frac{k}{x^2} \text{ becomes } y = \frac{k}{X},$$

where k is a constant and $k \neq 0$. Therefore, y is inversely proportional to X $(= x^2)$.

Vorked Example 12

Identifying Variables Which are Inversely Proportional to Each Other

For each of the following equations, state the two variables which are inversely proportional to each other and explain your answer.

(a) $y = \dfrac{2}{x^3}$ (b) $y = \dfrac{3}{\sqrt{x}}$

olution:

) Since $y = \dfrac{2}{x^3}$, i.e. $x^3y = 2$ is a constant, then y is inversely proportional to x^3.

) Since $y = \dfrac{3}{\sqrt{x}}$, i.e. $y\sqrt{x} = 3$ is a constant, then y is inversely proportional to $\sqrt{x}$.

PRACTISE NOW 12

SIMILAR QUESTIONS

Exercise 1D Questions 3(a)–(e)

r each of the following equations, state the two variables which are inversely oportional to each other and explain your answer.

) $y = \dfrac{4}{x^2}$ (b) $y^2 = \dfrac{1}{\sqrt[3]{x}}$ (c) $y = \dfrac{5}{x+2}$

RECALL

The square root sign $\sqrt{}$ is used to denote the *positive square root* only. Thus $\sqrt{4} = 2$. However, if $x^2 = 4$, then $x = \pm\sqrt{4} = \pm 2$. If we want both the positive and the negative square roots, we need to write $\pm\sqrt{}$.

Vorked Example 13

Equation of Another Form of Inverse Proportion

If y is inversely proportional to $\sqrt{x}$ and $y = 6$ when $x = 4$,

(i) calculate the value of y when $x = 16$,

(ii) find an equation connecting x and y,

(iii) find the value of x when $y = 4$.

olution:

When $x = 16 = 4 \times 4$ (16 is 4 times of 4),

$y = \dfrac{1}{\sqrt{4}} \times 6$ (y is $\dfrac{1}{\sqrt{4}}$ times of 6 since y is inversely proportional to $\sqrt{x}$)

$= \dfrac{1}{2} \times 6$

$= 3$

(ii) Since y is inversely proportional to $\sqrt{x}$, then $y = \frac{k}{\sqrt{x}}$, where k is a constant.

When $x = 4$, $y = 6$,

$$6 = \frac{k}{\sqrt{4}}$$

$$6 = \frac{k}{2}$$

$$\therefore k = 12$$

$$\therefore y = \frac{12}{\sqrt{x}}$$

(iii) When $y = 4$,

$$4 = \frac{12}{\sqrt{x}}$$

$$\sqrt{x} = \frac{12}{4}$$

$$= 3$$

$$\therefore x = 9$$

PRACTISE NOW 13

SIMILAR QUESTIONS

Exercise 1D Questions 1–2, 4–6, 9–10

1. If y is inversely proportional to x^2 and $y = 2$ when $x = 4$,
   (i) find the value of y when $x = 8$,
   (ii) find an equation connecting x and y,
   (iii) calculate the values of x when $y = 8$.

2. If y is inversely proportional to $\sqrt{x}$ and $y = 6$ when $x = 9$, find the value of y when $x = 25$.

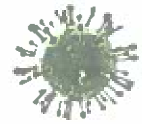
3. Given that y is inversely proportional to $\sqrt{x}$, copy and complete the table.

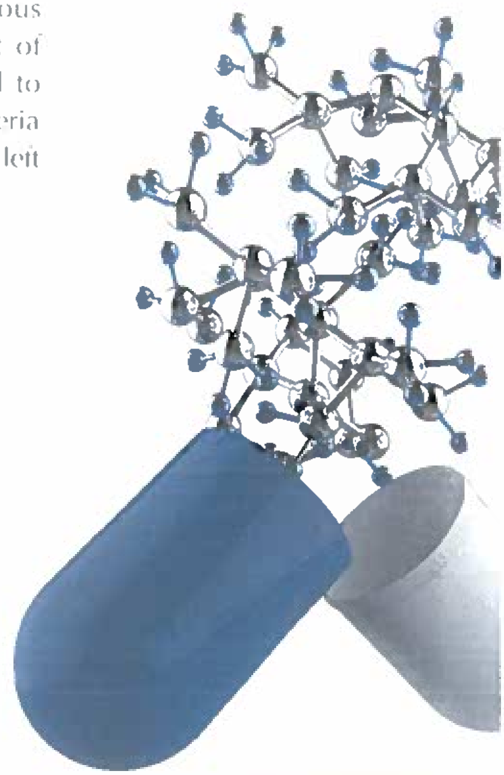
x		1	4	16	
y	16	8			$1\frac{1}{3}$

Worked Example 14

(Problem involving Another Form of Inverse Proportion)

In a computer simulation of an experiment, a drug is added to two identical flasks, each containing the same amount of a certain bacteria. The drug is allowed to react with the bacteria for various times in t hours. It is found that the amount of bacteria left, s units, is inversely proportional to $(t - 2)$ hours. In one flask, there are 6 units of bacteria left after 5 hours. Calculate the amount of bacteria left in the other flask after 7 hours.





Solution:

Since s is inversely proportional to $t - 2$,

then $s = \frac{k}{t-2}$, where k is a constant.

When $t = 5$, $s = 6$,

$$6 = \frac{k}{5-2}$$

$$6 = \frac{k}{3}$$

$\therefore k = 18$

$\therefore s = \frac{18}{t-2}$

When $t = 7$,

$$s = \frac{18}{7-2}$$

$$= 3.6$$

$\therefore$ The amount of bacteria left in the other flask after 7 hours is 3.6 units.

PRACTISE NOW 14

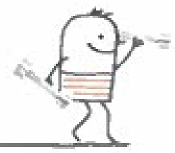
SIMILAR QUESTIONS

Exercise 1D Questions 7–8, 11

The force, F newtons (N), between two particles is inversely proportional to the square of the distance, d m, between them. When the particles are 2 m apart, the force between them is 10 N. Find

(i) the force between the particles when they are 5 m apart,

(ii) the distance between the particles when the force between them is 25 N.



Exercise 1D

BASIC LEVEL

1. If x is inversely proportional to y^3 and $x = 50$ when $y = 2$,
   (i) find the value of x when $y = 4$,
   (ii) find an equation connecting x and y,
   (iii) calculate the value of y when $x = 3.2$.

2. If z is inversely proportional to $\sqrt{w}$ and $z = 9$ when $w = 9$,
   (i) find an equation connecting w and z,
   (ii) find the value of z when $w = 16$,
   (iii) calculate the value of w when $z = 3$.

INTERMEDIATE LEVEL

3. For each of the following equations, state the two variables which are inversely proportional to each other and explain your answer.
   (a) $y = \frac{3}{x^2}$
   (b) $y = \frac{1}{\sqrt{x}}$
   (c) $y^2 = \frac{5}{x^3}$
   (d) $n = \frac{7}{m-1}$
   (e) $q = \frac{4}{(p+1)^2}$

4. If z is inversely proportional to $\sqrt{x}$ and $z = 5$ when $x = 64$, find the value of z when $x = 216$.

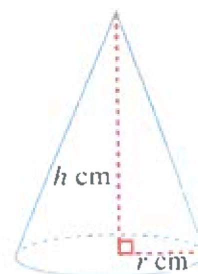
5. If q^2 is inversely proportional to $p + 3$ and $q = 5$ when $p = 2$, find the values of q when $p = 17$.

6. Given that t is inversely proportional to s^3, copy and complete the table.

s	1	2	4		
t	80			0.08	0.01

7. The force of repulsion, F newtons (N), between two particles is inversely proportional to the square of the distance, d m, between the particles.
   (i) Write down a formula connecting F and d.
   (ii) When the particles are a certain distance apart, the force of repulsion is 20 N. Find the force when the distance is halved.

8. For a fixed volume, the height, h cm, of a cone is inversely proportional to the square of the base radius, r cm. Cone A has a base radius of 6 cm and a height of 5 cm. The base radius of Cone B is 3 cm and the height of Cone C is 1.25 cm. If all the cones have the same volume, find
   (i) the height of Cone B,
   (ii) the base radius of Cone C.



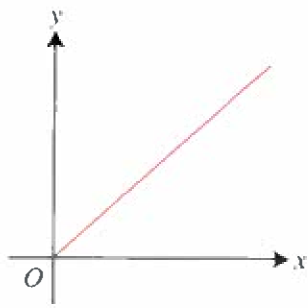
ADVANCED LEVEL

9. If y is inversely proportional to $2x + 1$ and the difference in the values of y when $x = 0.5$ and $x = 2$ is 0.9, find the value of y when $x = -0.25$.

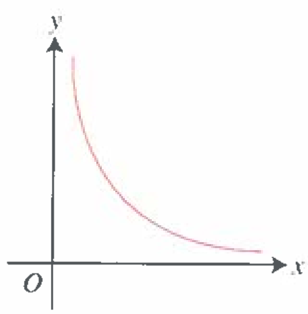
10. y is inversely proportional to x^2 and $y = b$ for a particular value of x. Find an expression in terms of b for y when this value of x is tripled.

11. The force of attraction between two magnets is inversely proportional to the square of the distance between them. When the magnets are r cm apart, the force of attraction between them is F newtons (N). If the distance between the magnets is increased by 400%, the force of attraction between them becomes cF N. Find the value of c.

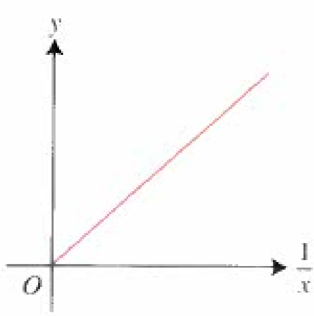
1. If y is *directly proportional* to x, then
   - $\frac{y}{x} = k$ or $y = kx$, where k is a constant and $k \neq 0$,
   - $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ or $\frac{x_2}{x_1} = \frac{y_2}{y_1}$,
   - the graph of y against x is a straight line that passes through the origin.



2. If y is *inversely proportional* to x, then
   - $xy = k$ or $y = \frac{k}{x}$, where k is a constant and $k \neq 0$,
   - $\frac{y_2}{y_1} = \frac{x_1}{x_2}$ or $x_1 y_1 = x_2 y_2$,
   - the graph of y against x is a hyperbola,



   - the graph of y against $\frac{1}{x}$ is a straight line that starts from the origin ($x \neq 0$).



Review Exercise 1

1. If y is directly proportional to x and $y = 6$ when $x = 2$,
   (i) express y in terms of x,
   (ii) find the value of y when $x = 11$,
   (iii) calculate the value of x when $y = 12$,
   (iv) draw the graph of y against x.

2. If A is directly proportional to B and $A = 1\frac{2}{3}$ when $B = \frac{5}{6}$, find
   (i) the value of A when $B = \frac{1}{3}$,
   (ii) the value of B when $A = 7\frac{1}{2}$.

3. If y is directly proportional to x^3 and $y = 108$ when $x = 3$,
   (i) find an equation connecting x and y,
   (ii) find the value of y when $x = 7$,
   (iii) calculate the value of x when $y = 4000$,
   (iv) draw the graph of y against x^3.

4. If n is directly proportional to m^2 and $n = 9.375$ when $m = 2.5$, find
   (i) the value of n when $m = 3$,
   (ii) the values of m when $n = 181.5$.

5. If t is directly proportional to $\sqrt{s}$ and $t = 4$ when $s = 64$, find
   (i) the value of t when $s = 125$,
   (ii) the value of s when $t = 2$.

6. If y is inversely proportional to x and $y = 4$ when $x = 3$,
   (i) express y in terms of x,
   (ii) find the value of y when $x = 6$,
   (iii) calculate the value of x when $y = 24$.

7. If q is inversely proportional to p^2 and $q = 3$ when $p = 5$, find
   (i) an equation connecting p and q,
   (ii) the value of q when $p = 10$,
   (iii) the negative value of p when $q = \frac{1}{3}$.

8. If z is inversely proportional to $w + 3$ and $z = 4$ when $w = 3$, find
   (i) the value of z when $w = 9$,
   (ii) the value of w when $z = 2.4$.

9. Given that y is inversely proportional to $2x^2$, copy and complete the table.

x	0.2	0.5			2
y			1.5	0.96	0.375

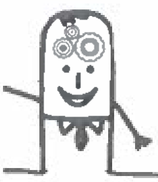
10. The total monthly charges, \$$C$, for a fixed phone line consists of a fixed amount of \$9.81 and a variable amount which depends on the usage. For every minute used, \$0.086 is charged.
    (i) If the duration of usage is 300 minutes, find the total monthly charges for the fixed phone line.
    (ii) If the total monthly charges for the fixed phone line are \$20.56, calculate the duration of usage.
    (iii) Write down a formula connecting C and n, where n is the number of minutes of usage. Hence, state the two variables which are directly proportional to each other.

11. The gravitational potential energy, G joules (J), of an object is directly proportional to its height, h m, above the surface of the Earth. When the object is at a height of 40 m above the surface of the Earth, its gravitational potential energy is 2200 J. Find
    (i) an equation connecting G and h,
    (ii) the gravitational potential energy of the object when it is at a height of 22 m above the surface of the Earth,
    (iii) the height of the object above the surface of the Earth when it has a gravitational potential energy of 3025 J.

12. Kate makes a donation to a charitable organisation on a monthly basis. Her monthly donation is directly proportional to the square of her monthly savings. If she saves \$900 and \$1200 in January and February respectively, her donation increases by \$35 from January to February. Find the amount of money she donates to the charitable organisation in each of the two months.

13. Boyle's Law states that the pressure, P pascals (Pa), of a fixed mass of gas at constant temperature is inversely proportional to its volume, V dm^3. The pressure of 4000 dm^3 of a gas in an airtight container is 250 Pa. Assuming that the temperature in the container is constant, find
   (i) the pressure of the gas when its volume is 5000 dm^3,
   (ii) the volume of the gas when its pressure is 125 Pa.

14. 5 men are hired to complete a job. If one more man is hired, the job can be completed 8 days earlier. Assuming that all the men work at the same rate, how many more men should be hired so that the job can be completed 28 days earlier?



Challenge Yourself

1. If A is directly proportional to C and B is directly proportional to C, prove that each of the following is directly proportional to C.
   (a) $A + B$
   (b) $A - B$
   (c) $\sqrt{AB}$

2. It is given that z is directly proportional to x^2 and inversely proportional to $\sqrt{y}$.
   (i) Write down an equation connecting x, y and z.
   (ii) If $z = 16$ when $x = 2$ and $y = 9$, find the value of z when $x = 5$ and $y = 4$.

3. The time, T days, needed to paint some buildings is directly proportional to the number of buildings, B, that need to be painted and inversely proportional to the number of painters employed, P. 18 painters can paint 3 buildings in 20 days.
   (i) Find an equation connecting T, B and P.
   (ii) Find the number of days needed by 16 painters to paint 4 buildings.
   (iii) If 10 buildings are to be painted in 24 days, how many painters need to be employed?

Linear Graphs and Simultaneous Linear Equations

During festive seasons, people usually buy gifts for their friends and relatives. When shopping for gifts, if we narrow down our gift choices to two items and we have a fixed budget, how do we determine the number of each item to buy?

Chapter Two

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- find the gradient of a straight line,
- state the y-intercept of a straight line,
- state the equation of a horizontal line and of a vertical line,
- draw graphs of linear equations in the form $ax + by = k$,
- solve simultaneous linear equations in two variables using
  (a) the graphical method,
  (b) the elimination method,
  (c) the substitution method.
- formulate a pair of linear equations in two variables to solve mathematical and real-life problems.

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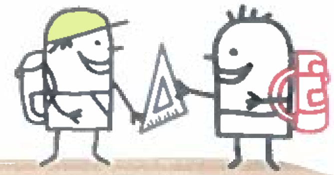
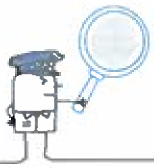
2.1 Gradient of a Straight Line



The equation of a straight line is in the form $y = mx + c$, where m and c are constants. Now, we shall take a look at how changing the value of m and of c affects the line.



Investigation

Equation of a Straight Line

In this investigation, we shall explore how the graph of a straight line in the form $y = mx + c$ changes when either m or c varies.

Go to http://www.shinglee.com.sg/StudentResources/ and open the spreadsheet 'Equation of a Straight Line'.

1. Change the value of c from –3 to 3 in steps of 1 by clicking on the scroll bar. What happens to the line? State the coordinates of the point where the line cuts the y-axis.
2. Change the value of m from 0 to 5 in steps of 1. What happens to the line?
3. Change the value of m from 0 to –5 in steps of –1. What happens to the line?
4. What is the difference between a line with a positive value for m and a line with a negative value for m?

Graph of y = mx + c

x	-3	-2	-1	0	1	2	3
y	-11	-9	-7	-5	-3	-1	1

m = 2 c = -5

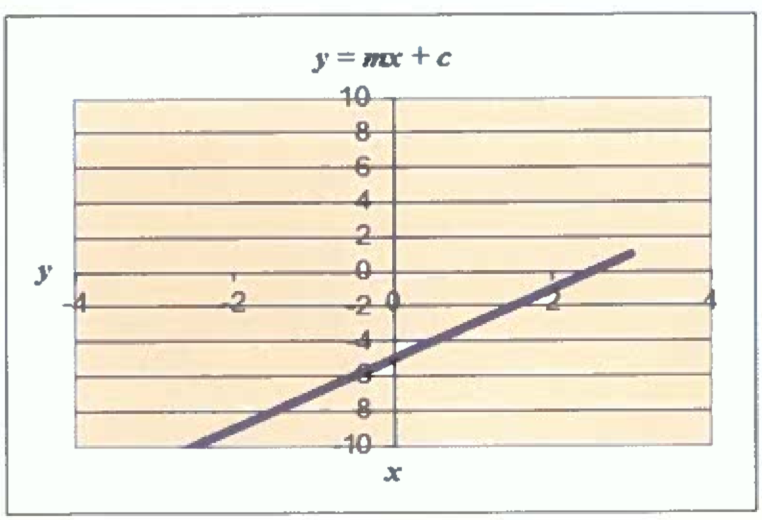


Fig. 2.1

Gradient of a Straight Line

Look at the roads around you. Some roads are steeper than others. The steeper a road is, the harder it is to walk up the road. **Gradient** is a measure of how steep a slope is.

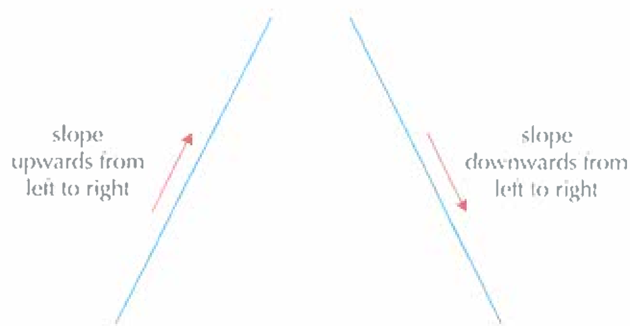
From the investigation on the previous page,

> The equation of a straight line is in the form of $y = mx + c$, where the constant m is the **gradient** of the line and the constant c is the ***y*-intercept**.

Note: The y-intercept refers to the y-coordinate of the point of intersection of the line with the y-axis.

From the investigation, we have discovered that the gradient of a line can be either *positive* or *negative*.

- Fig. 2.2(a) shows a line that slopes __________ from *the left to the right* and its gradient is *positive*.
- Fig. 2.2(b) shows a line that slopes __________ from *the left to the right* and its gradient is *negative*.



(a) Positive gradient

(b) Negative gradient

Fig. 2.2

ATTENTION

In Fig. 2.2(a), the line slopes downwards from the right to the left but its gradient is positive. Thus it is important to specify whether the line is sloping upwards or downwards from the left to the right.

In real life, can the gradient of a slope be negative?

From the investigation, we can also see that as the *absolute value* of the gradient m *increases*, the steepness of the line *increases*. For example, a line with a gradient of 3 is steeper than a line with a gradient of 2 since $3 > 2$; but a line with a gradient of -3 is also __________ than a line with a gradient of -2 although $-3 < -2$.

How do we find the gradient of a line?

The gradient of a straight line is the measure of the ratio of the vertical change (or rise) to the horizontal change (or run), i.e.

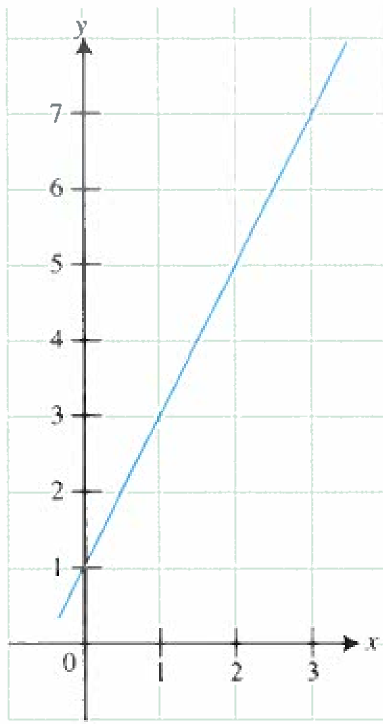
$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}} \text{ or } \frac{\text{rise}}{\text{run}}.$$

Worked Example 1

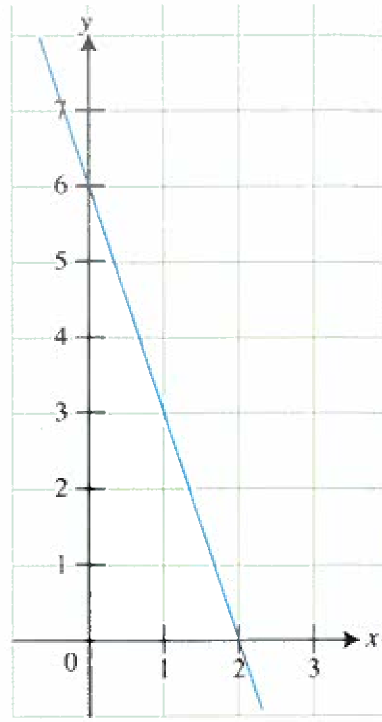
(Finding Gradients of Straight Lines)

Find the gradient of each of the following lines.

(a)

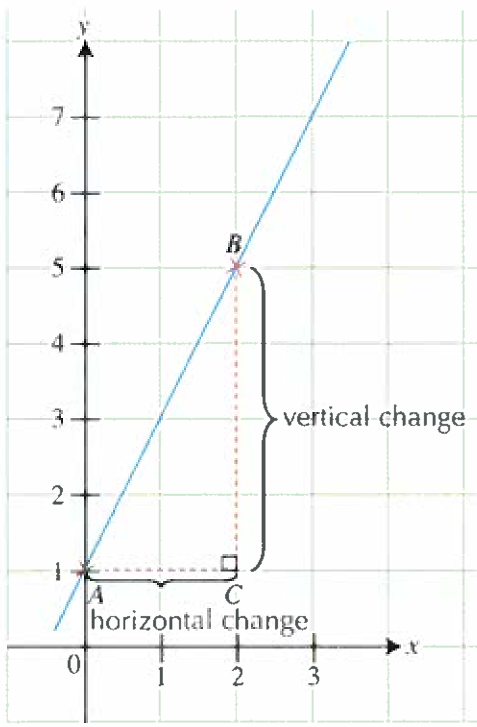


(b)



Solution:

(a)



Take two points A and B on the line and draw dotted lines to form the right-angled triangle ABC.

Vertical change (or rise) $BC = 5 - 1$

$= 4$

Horizontal change (or run) $AC = 2 - 0$

$= 2$

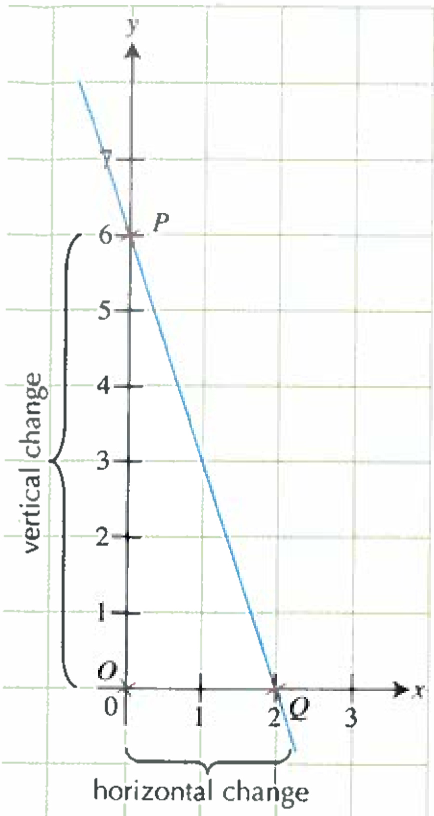
Since the line slopes upwards from the left to the right, its gradient is positive.

$\therefore$ Gradient $= \dfrac{\text{rise}}{\text{run}}$

$= \dfrac{4}{2}$

$= 2$

(b)



Take two points P and Q where the line cuts the y-axis and x-axis respectively.
Let O be the origin (0, 0).
Vertical change (or rise) $OP = 6$
Horizontal change (or run) $OQ = 2$
Since the line slopes downwards from the left to the right, its gradient is negative.

$$\therefore \text{ Gradient} = \frac{\text{rise}}{\text{run}} = -\frac{6}{2} = -3$$

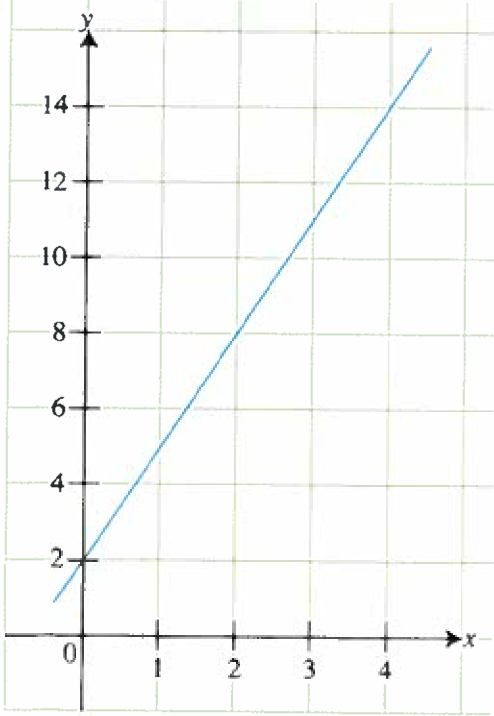
PRACTISE NOW 1

SIMILAR QUESTIONS

Exercise 2A Questions 2(a)–(h)

Find the gradient of each of the following lines.

(a)



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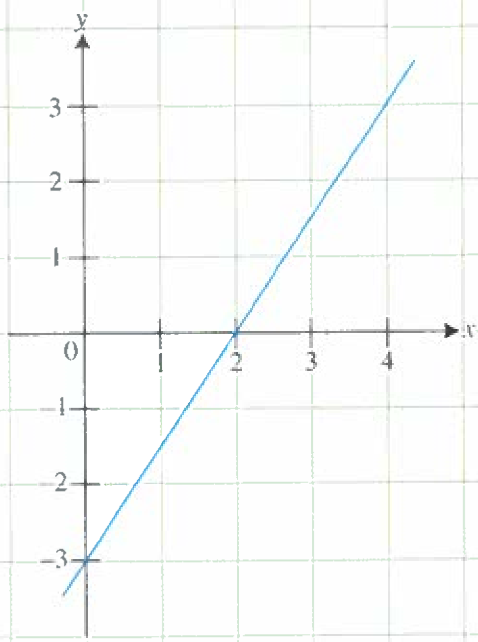
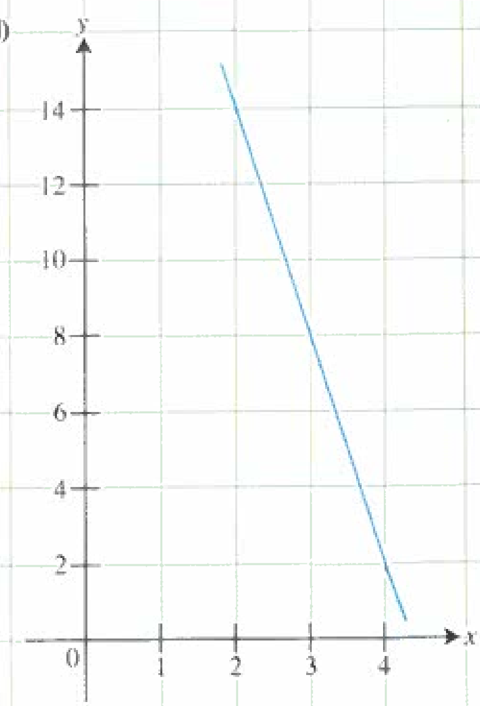
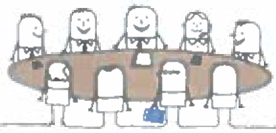
(b)

(c)



(d)





Class Discussion

Gradients of Straight Lines

Work in pairs.

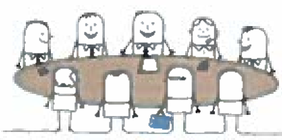
Consider the points $D(1, 3)$ and $E(2.5, 6)$ on the line in Worked Example 1(a).

(i) Find the gradient of DE.

(ii) Is gradient of DE = gradient of AB?

(iii) Hence, we can choose *any* two points on a line to find its gradient because the gradient of a straight line is __________.

After learning how to find the gradient of a straight line, we need to have a sense of the magnitude of the steepness of a line.



Class Discussion

Gradients in the Real World

1. How steep is a road with a gradient of 1? Fig. 2.3 shows a line with a gradient of 1. Measure the angle of inclination between the line and the horizontal dotted line.

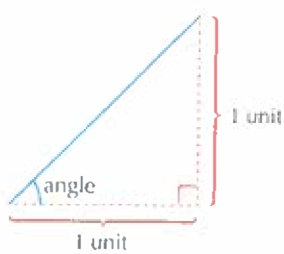


Fig. 2.3

2. How steep is a road with a gradient of 2? Make an accurate drawing of a line with a gradient of 2 and, indicate the vertical change and horizontal change clearly. Measure the angle of inclination.

3. Repeat Step 2 for a road with a gradient of $\frac{1}{2}$.

4. Do you consider a road with a gradient of 1 steep or gentle? Discuss with your classmates if there are many roads in Pakistan that have a gradient of 1.

5. Do you consider a road with a gradient of $\frac{1}{2}$ steep or gentle? Discuss with your classmates whether the gradients of most roads in Pakistan are greater than or less than $\frac{1}{2}$.

INFORMATION

The steepest street in the world is Baldwin Street in Dunedin, New Zealand, with a gradient of about 0.38.

Horizontal Line

Investigation

Gradient of a Horizontal Line

In this investigation, we shall find out what the gradient of a horizontal line is.

Fig. 2.4 shows a horizontal line.

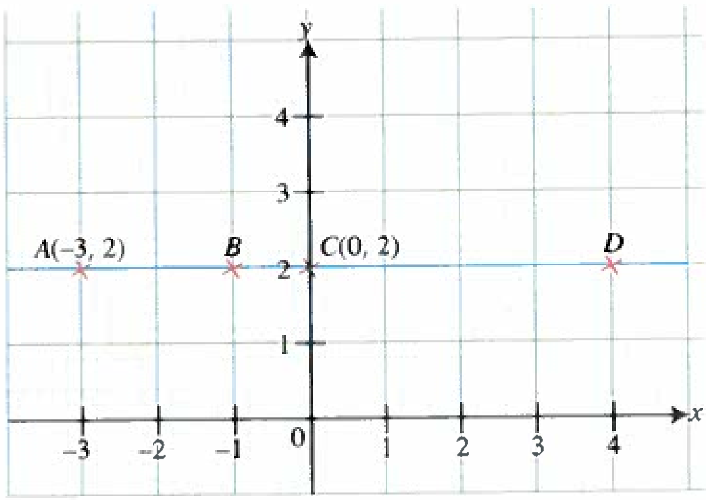


Fig. 2.4

1. There are 4 points on the line. The coordinates of *A* and *C* are given. Write down the coordinates of *B* and of *D*.
2. In the line segment *AC*,

   rise = ________ and run = ________.
3. In the line segment *BD*,

   rise = ________ and run = ________.
4. What can you conclude about the gradient of a horizontal line?

INFORMATION

In this case, the equation of the horizontal line is $y = 2$, which is a linear function. Notice that the y-coordinates of all the points on the horizontal line are equal to 2.

Vertical Line

Investigation

Gradient of a Vertical Line

In this investigation, we shall find out what the gradient of a vertical line is. Fig. 2.5 shows a vertical line.

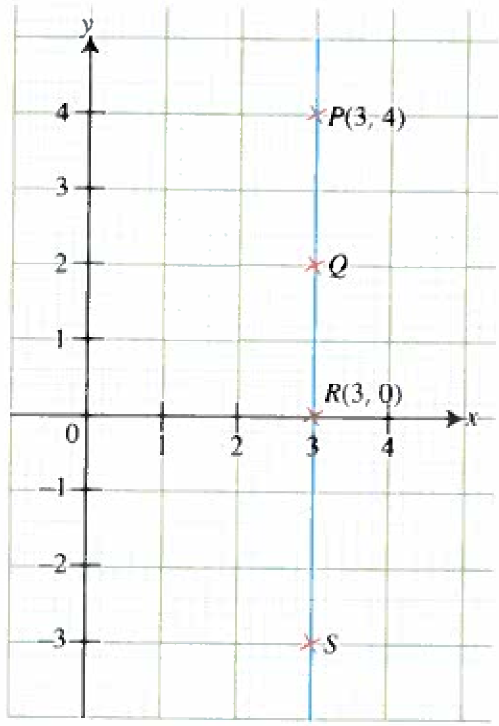


Fig. 2.5

1. There are 4 points on the line. The coordinates of P and of R are given. Write down the coordinates of Q and of S.
2. In the line segment PR,
   rise = ________ and run = ________.
3. In the line segment QS,
   rise = ________ and run = ________.
4. What can you conclude about the gradient of a vertical line?

INFORMATION

In this case, the equation of the vertical line is $x = 3$, which is not a linear function. Notice that the x-coordinates of all the points on the vertical line are equal to 3.

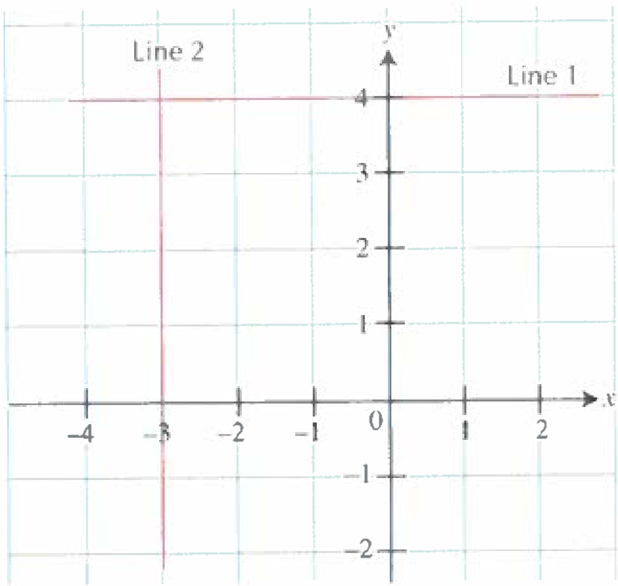
SIMILAR QUESTIONS

Exercise 2A Questions 1, 3–4

Exercise 2A

BASIC LEVEL

1.

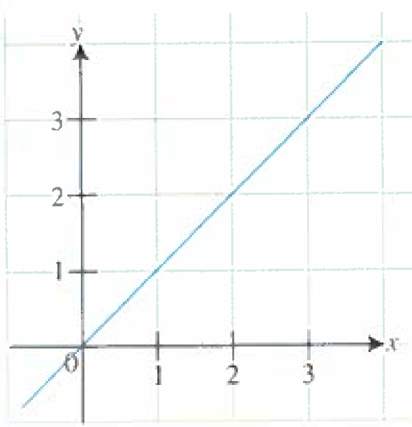


Write down the gradient of each of the given lines.

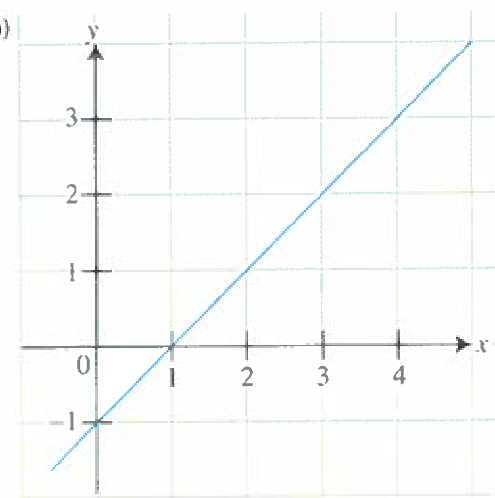
INTERMEDIATE LEVEL

2. Given that the equation of the line representing each of the following linear graphs is in the form $y = mx + c$, find the gradient m and state the y-intercept c.

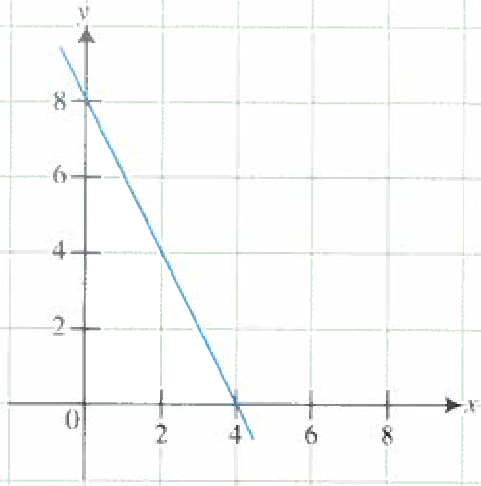
(a)



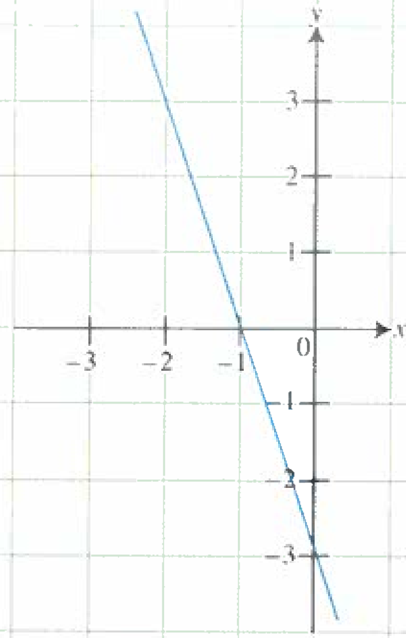
(b)



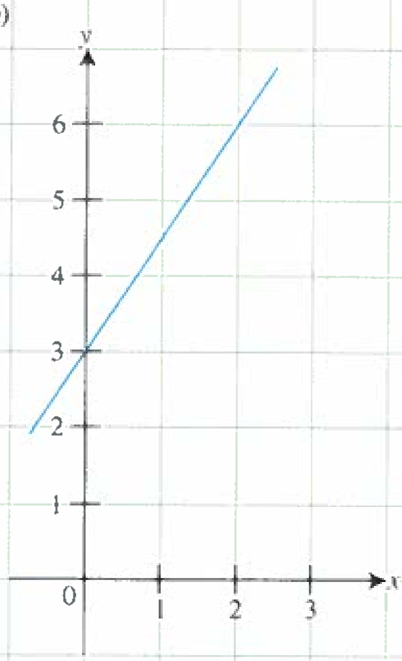
(c)



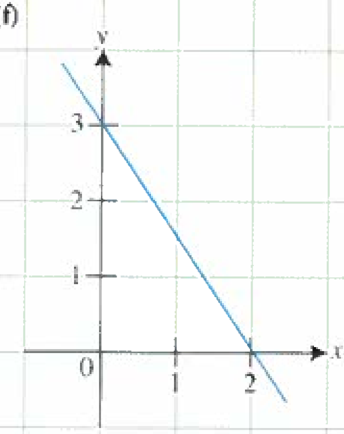
(d)



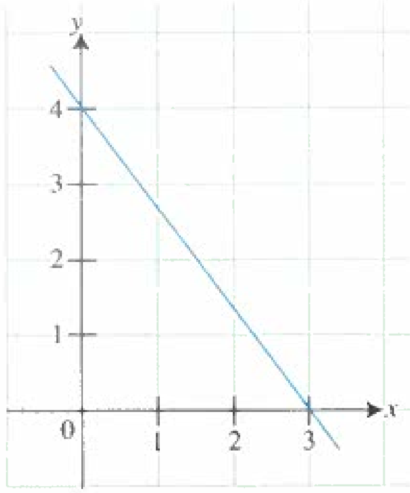
(e)



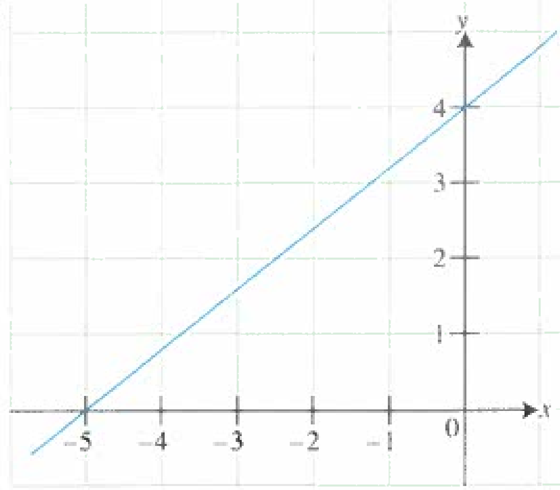
(f)



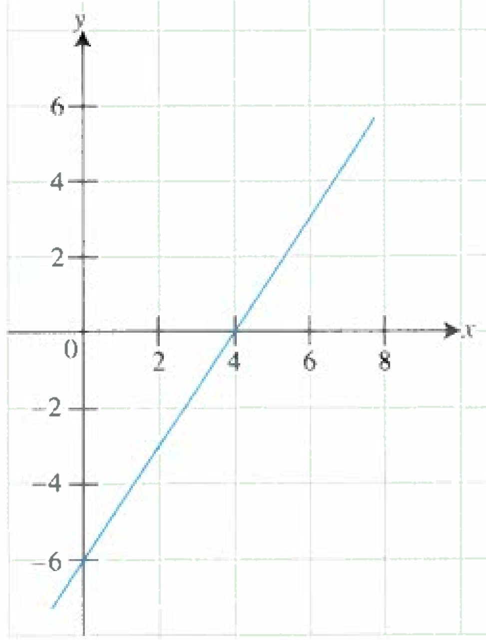
(g)



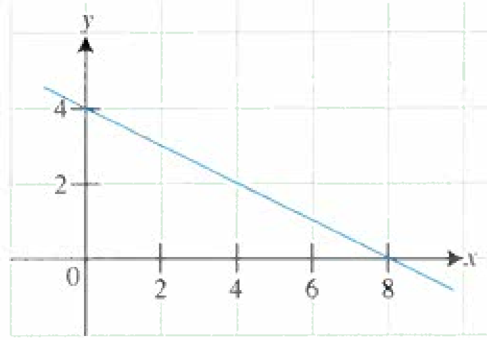
(h)



(i)



(j)



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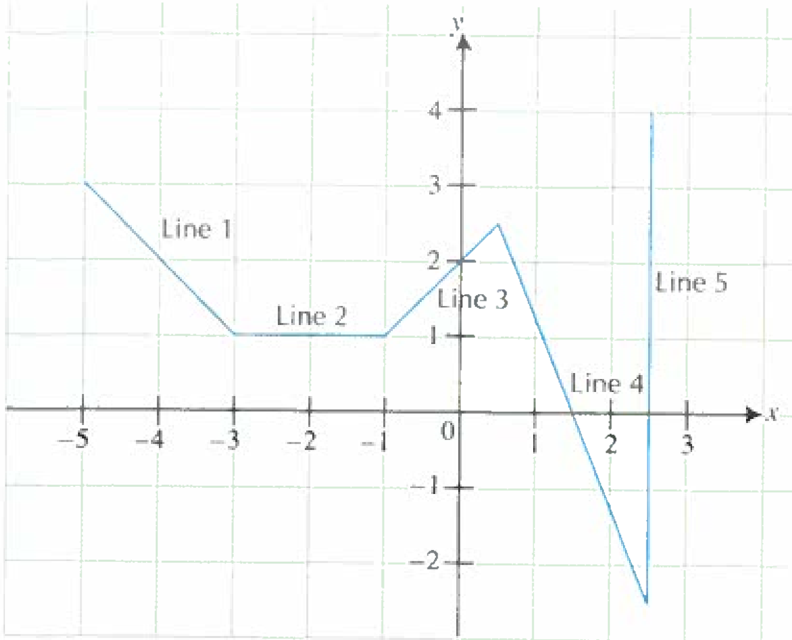
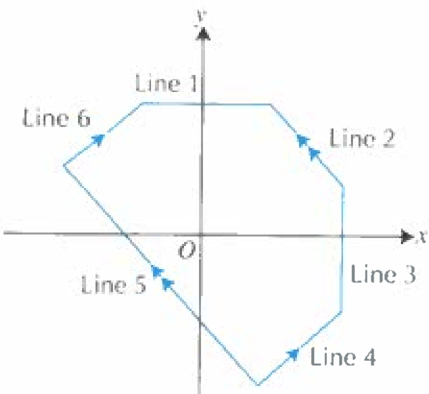
3. The figure shows five line segments.



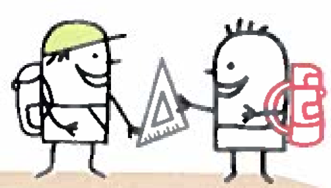
Find the gradient of each of the line segments.

ADVANCED LEVEL

4. 

In the figure, Line 1 is parallel to the x-axis and Line 3 is parallel to the y-axis. Line 2 is parallel to Line 5 and Line 4 is parallel to Line 6. If the gradients of Line 5 and Line 6 are -3 and $\frac{1}{2}$ respectively, write down the gradients of Line 1, Line 2, Line 3 and Line 4.

2.2 Further Applications of Linear Graphs in Real-World Contexts



In this section, we will learn how to apply the concepts of gradient and y-intercept to solve linear graphs used in daily situations.

Worked Example 2

(Distance-Time Graph)

The travel graph shows a journey taken by a cyclist. He started his 50-km journey at 0800 hours. At 0900 hours, his bicycle tyre suffered a puncture and he spent half an hour repairing it. He then continued his journey and reached his destination at 1130 hours.

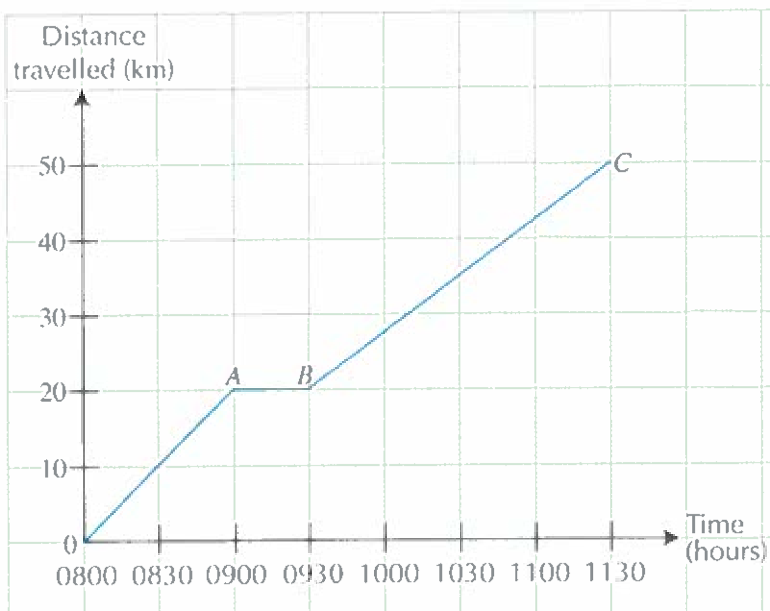
(a) How far did the cyclist travel before his bicycle tyre suffered a puncture?

(b) Find the gradient of each of the following line segments, stating clearly what each gradient represents.

(i) OA

(ii) AB

(iii) BC



Solution:

(a) 20 km

(b) (i) Gradient of $OA = \frac{20}{1}$

$= 20$

The cyclist travelled 20 km in 1 hour, i.e. his average speed was 20 km/h before his bicycle tyre suffered a puncture.

(ii) Gradient of $AB = 0$

He stopped cycling, i.e. his average speed was zero.

(iii) Gradient of $BC = \frac{30}{2}$

$= 15$

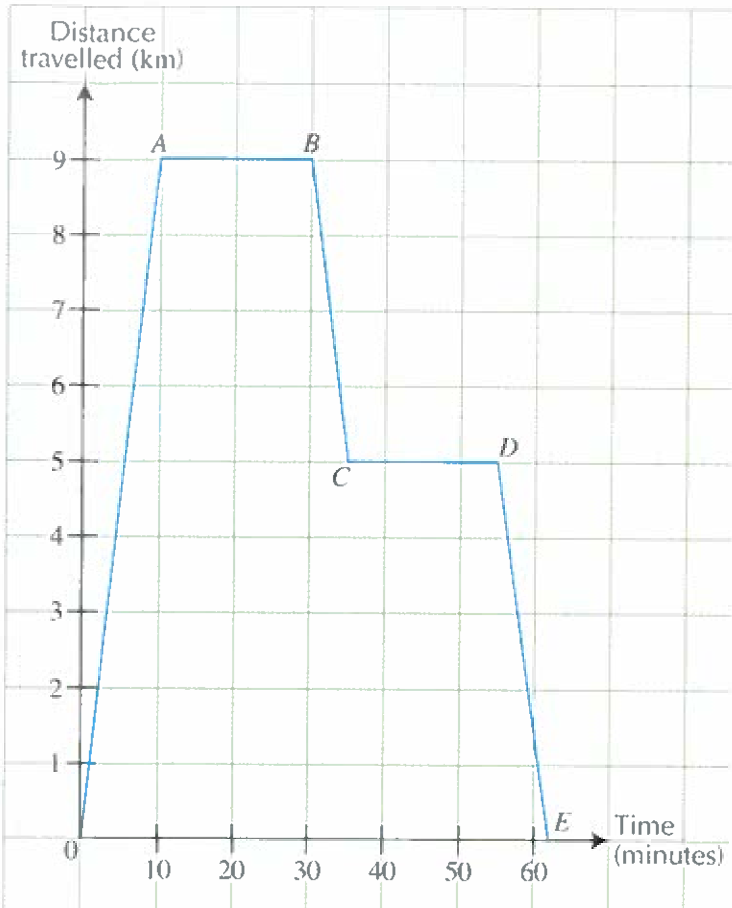
The average speed during the last part of his journey was 15 km/h.

PRACTISE NOW 2

SIMILAR QUESTIONS

Exercise 2B Questions 1-2

A technician in a computer firm drove from his workshop to repair a customer's computer. On his way back, he stopped to repair another customer's computer. The distance-time graph shows his entire journey.



(a) How long did he take to repair each computer?

(b) How far from his workshop was his first customer?

(c) Find the gradient of each of the following line segments, stating clearly what each gradient represents.

(i) *OA*

(ii) *AB*

(iii) *BC*

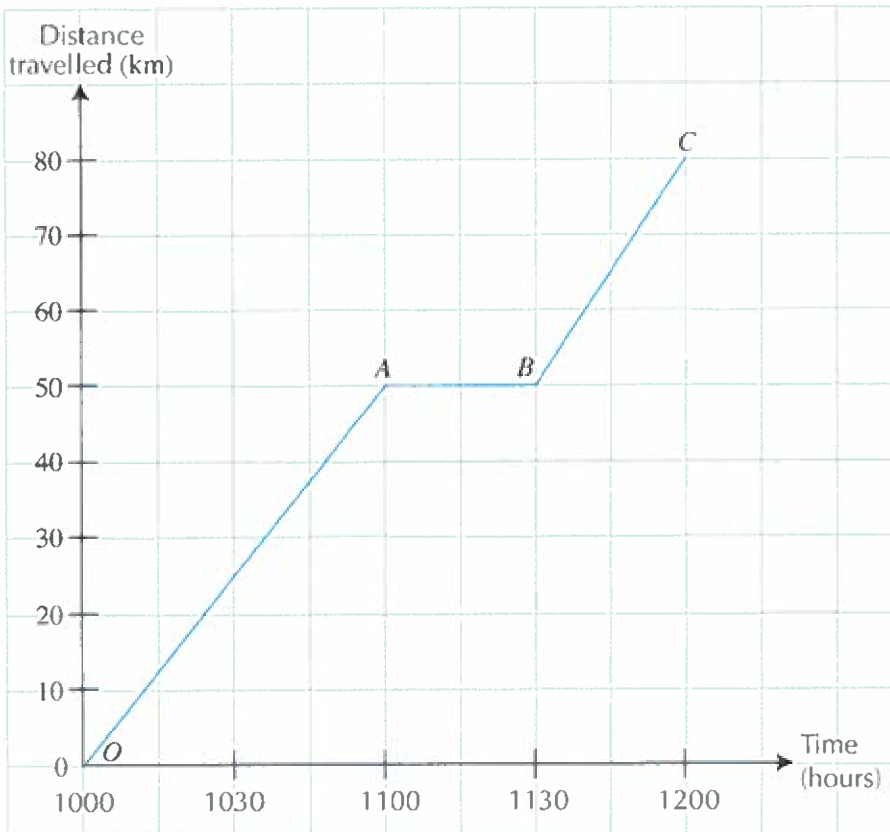
(iv) *CD*

(v) *DE*

Exercise 2B

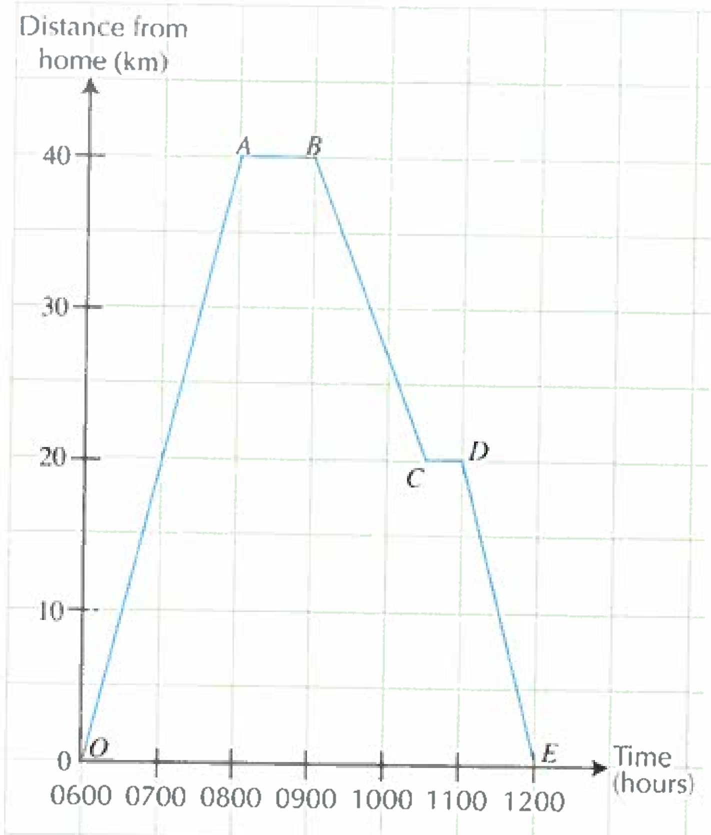
BASIC LEVEL

1. The graph shows Khairul's journey when he visited a friend in Town C. During the journey, he stopped for breakfast at a cafeteria, after which he continued to drive to Town C.

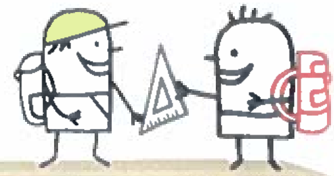


(a) At what time did he leave home?
(b) How far did he travel before he reached the cafeteria?
(c) Find the gradient of each of the following line segments, stating clearly what each gradient represents.
(i) OA (ii) AB (iii) BC

2. Ethan cycled from home to a post office. On his way back, he stopped at a hawker centre to have his breakfast. The distance-time graph shows his entire journey.



(a) How far from his home was the post office?
(b) Find the total time he stayed at the post office and at the hawker centre.
(c) Find the gradient of each of the following line segments, stating clearly what each gradient represents.
(i) OA (ii) BC (iii) DE



2.3 Horizontal and Vertical Lines

In section 2.1, we have learnt that the equation of a straight line is in the form $y = mx + c$, where the constant m is the gradient of the line and the constant c is the y-intercept.

Consider the equation of the straight line $y = 2x + 1$. For each value of x, there is a corresponding value of y.

When $x = 0$, $y = 2(0) + 1 = 1$.
When $x = 2$, $y = 2(2) + 1 = 5$.

If we plot these points on a sheet of graph paper, we obtain a straight line. Since the points (0, 1) and (2, 5) lie on the line, the coordinates of these points satisfy the equation $y = 2x + 1$.

Does the point (4, 10) lie on the line with equation $y = 2x + 1$?

We replace x by 4 and y by 10 in the equation $y = 2x + 1$.

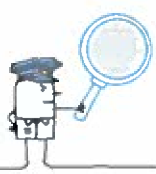
LHS = 10

RHS = 2(4) + 1

= 9 ≠ LHS

Since the coordinates of (4, 10) do not satisfy the equation $y = 2x + 1$, the point does not lie on the line.

What are the equations of a horizontal line and a vertical line?



Investigation

Equation of a Horizontal Line

Fig. 2.6 shows a horizontal line.

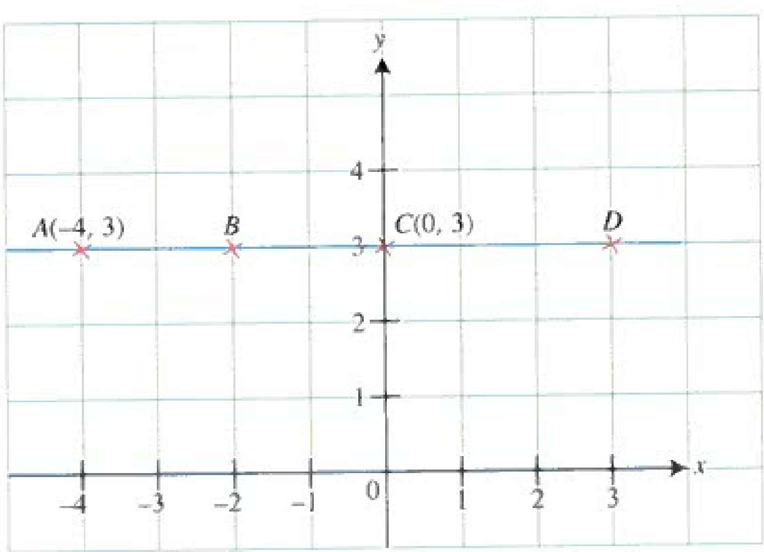


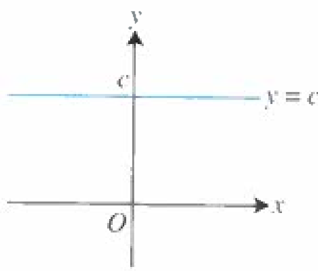
Fig. 2.6

1. What is the gradient of the horizontal line?
2. There are 4 points on the line. The coordinates of A and of C are given. Write down the coordinates of B and of D.
3. What do you notice about the y-coordinates of all the four points on the horizontal line?
4. What do you think the equation of the horizontal line is?

From the investigation, since the gradient m of a horizontal line is 0, the equation of a horizontal line is

$$\boxed{y = c.}$$

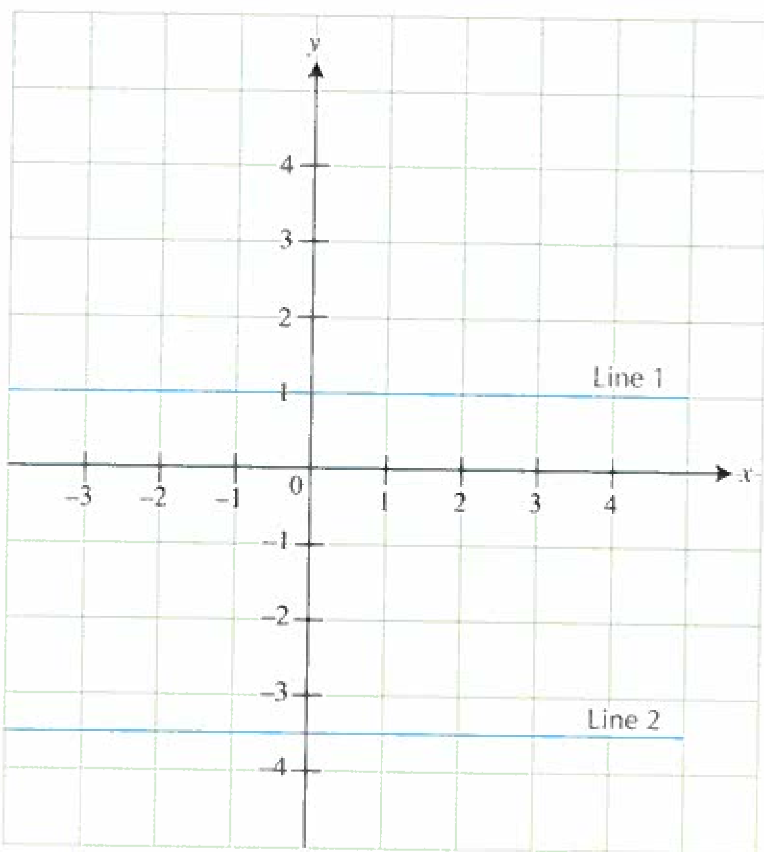
The y-coordinates of all the points on a horizontal line are equal to c (a constant).



PRACTISE NOW

SIMILAR QUESTIONS

Exercise 2C Question 1



(a) Write down the equation of each of the given horizontal lines.

(b) In the graph, draw each of the lines with the following equations.

(i) $y = 2$

(ii) $y = 0$

Describe the lines.

Investigation

Equation of a Vertical Line

Fig. 2.7 shows a vertical line.

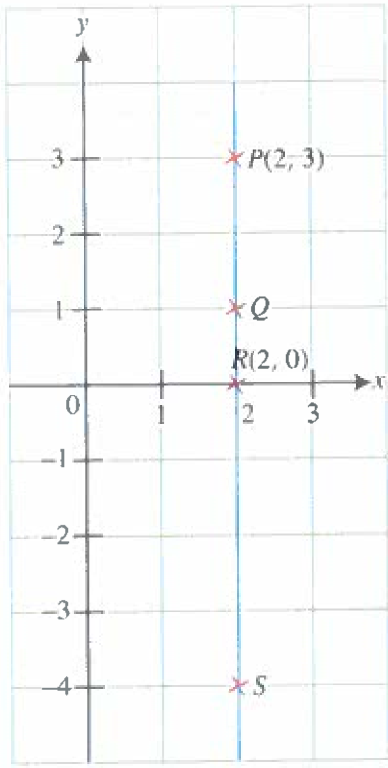
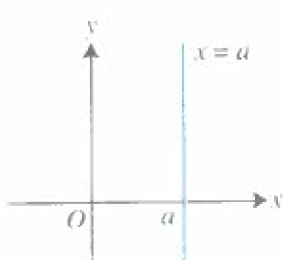


Fig. 2.7

1. What is the gradient of the vertical line?
2. There are 4 points on the line. The coordinates of P and of R are given. Write down the coordinates of Q and of S.
3. What do you notice about the x-coordinates of all the four points on the vertical line?
4. What do you think the equation of the vertical line is?

From the investigation, since the gradient m of a vertical line is undefined, we cannot write the equation of a vertical line in the form $y = mx + c$. As the x-coordinates of all the points on a vertical line are equal to the same constant value a, the equation of a vertical line is

$$\boxed{x = a.}$$



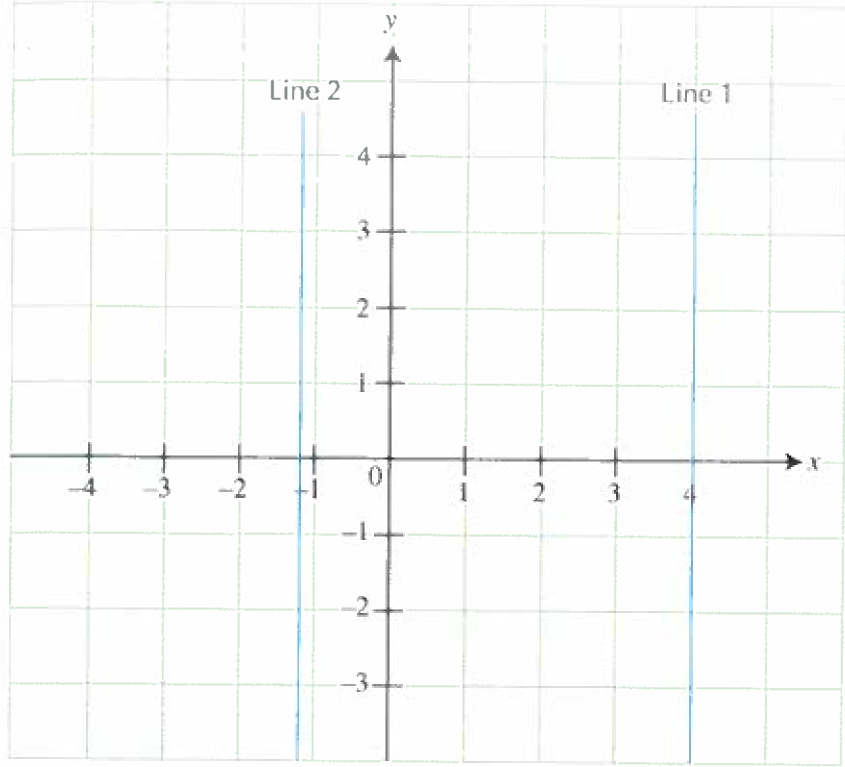
INFORMATION

$x = a$ is not a linear function because there are many values of y for one value of x.

PRACTISE NOW

SIMILAR QUESTIONS

Exercise 2C Question 2



(a) Write down the equation of each of the given vertical lines.

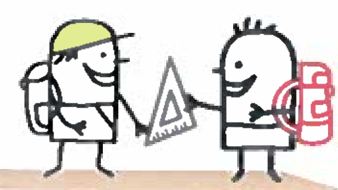
(b) In the graph, draw each of the lines with the following equations.

(i) $x = -3.5$

(ii) $x = 0$

Describe the lines.

2.4 Graphs of Linear Equations in the form $ax + by = k$



We have learnt how to draw graphs of linear equations in the form $y = mx + c$, where m and c are constants. In this section, we shall take a look at the graphs of linear equations in the form $ax + by = k$, where a, b and k are constants.

Investigation

Graphs of $ax + by = k$

1. Consider the equation $2x + y = 3$.
   - **(i)** Using a graphing software, draw the graph of $2x + y = 3$.
   - **(ii)** Do the points $A(2, -1)$ and $B(-2, 5)$ lie on the graph in **(i)**?
     Do the coordinates of each of the points satisfy the equation $2x + y = 3$?
     Explain your answers.
   - **(iii)** The point $(1, p)$ lies on the graph in **(i)**. Determine the value of p.
   - **(iv)** The point $(q, -7)$ lies on the graph in **(i)**. Determine the value of q.
   - **(v)** On the same axes in **(i)**, draw the graph of $y = -2x + 3$. What do you notice?
     Hence, show algebraically that $y = -2x + 3$ can be obtained from $2x + y = 3$.
2. Consider the equation $3x - 4y = 6$.
   - **(i)** Using a graphing software, draw the graph of $3x - 4y = 6$.
   - **(ii)** The point $(2, r)$ lies on the graph in **(i)**. Determine the value of r.
   - **(iii)** The point $(s, -1.5)$ lies on the graph in **(i)**. Determine the value of s.
   - **(iv)** State the coordinates of two other points that satisfy the equation $3x - 4y = 6$.
   - **(v)** On the same axes in **(i)**, draw the graph of $y = \frac{3}{4}x - \frac{3}{2}$. What do you notice?
     Hence, show algebraically that $y = \frac{3}{4}x - \frac{3}{2}$ can be obtained from $3x - 4y = 6$.

Worked Example 3

(Drawing the Graph of $ax + by = k$)

The variables x and y are connected by the equation $2x - 3y = 2$. Some values of x and the corresponding values of y are given in the table.

x	-2	-0.5	4
y	-2	p	2

(a) Calculate the value of p.

(b) On a sheet of graph paper, using a scale of 1 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $2x - 3y = 2$ for $-2 \leqslant x \leqslant 4$.

(c) The point $(1, q)$ lies on the graph in **(b)**. Find the value of q.

(d) **(i)** On the same axes in **(b)**, draw the graph of $y = 1$.

(ii) State the x-coordinate of the point on the graph of $2x - 3y = 2$ that has a y-coordinate of 1.

ATTENTION

$-2 \leqslant x \leqslant 4$ represents values of x that are more than or equal to -2 but less than or equal to 4.

Solution:

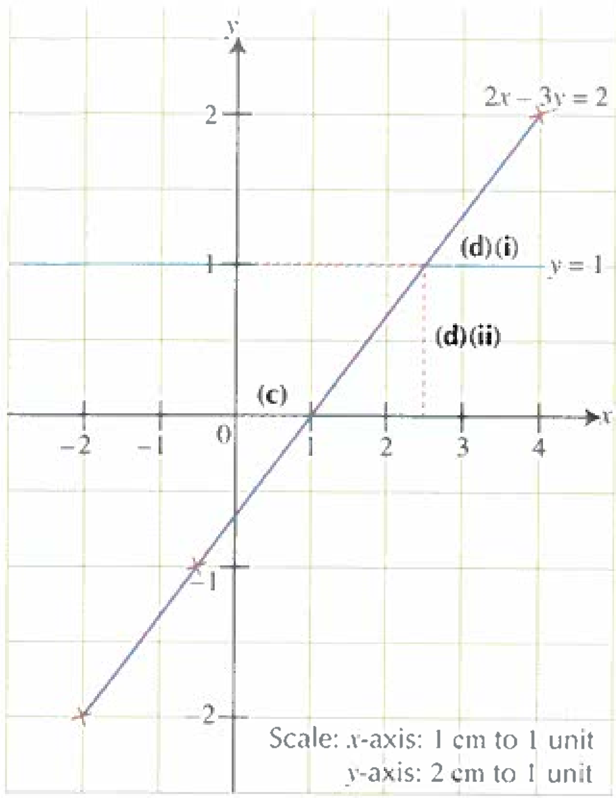
(a) When $x = -0.5$, $y = p$,

$2(-0.5) - 3p = 2$

$3p = -3$

$\therefore p = -1$

(b)



(c) From the graph in **(b)**,

When $x = 1$,

$q = y = 0$

(d) **(ii)** x-coordinate of point = 2.5

PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 2C Questions 3–4

The variables x and y are connected by the equation $3x + y = 1$. Some values of x and the corresponding values of y are given in the table.

x	-2	0	2
y	p	1	-5

(a) Find the value of p.

(b) On a sheet of graph paper, using a scale of 4 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $3x + y = 1$ for $-2 \leqslant x \leqslant 2$.

(c) The point $(-1, q)$ lies on the graph in **(b)**. Find the value of q.

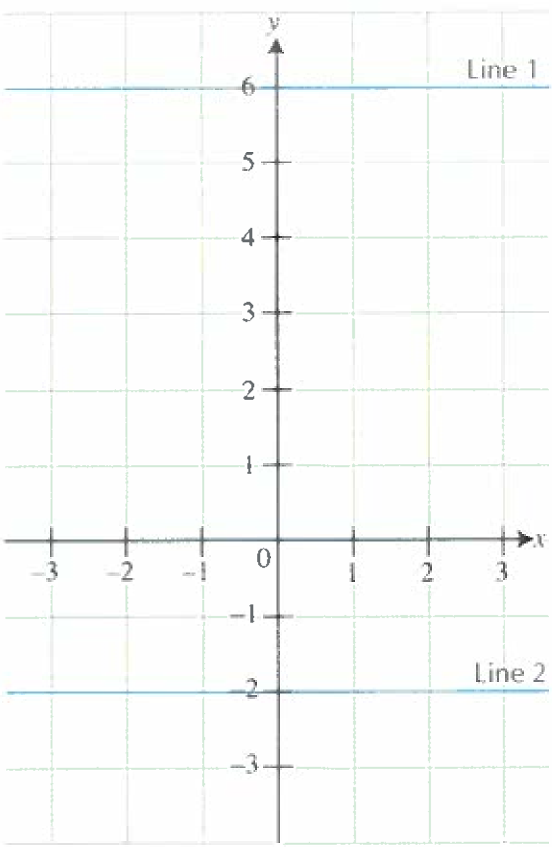
(d) **(i)** On the same axes in **(b)**, draw the graph of $y = -0.5$.

(ii) State the x-coordinate of the point on the graph of $3x + y = 1$ that has a y-coordinate of -0.5.

Exercise 2C

BASIC LEVEL

1.



(a) Write down the equation of each of the given horizontal lines.

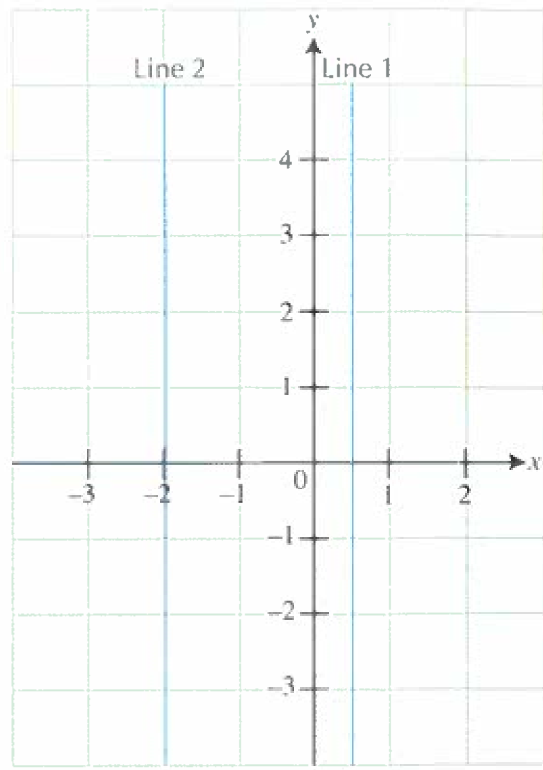
(b) In the graph, draw each of the lines with the following equations.

(i) $y = -3$

(ii) $y = 3\frac{1}{2}$

Describe the lines.

2.



(a) Write down the equation of each of the given vertical lines.

(b) In the graph, draw each of the lines with the following equations.

(i) $x = 1$

(ii) $x = -2\frac{1}{2}$

Describe the lines.

INTERMEDIATE LEVEL

3. The variables x and y are connected by the equation $-x + 2y = 4$. Some values of x and the corresponding values of y are given in the table.

x	-5	0	5
y	p	2	q

(a) Find the value of p and of q.

(b) On a sheet of graph paper, using a scale of 1 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $-x + 2y = 4$ for $-5 \leqslant x \leqslant 5$.

(c) The point $(r, 0.5)$ lies on the graph in **(b)**. Find the value of r.

(d) **(i)** On the same axes in **(b)**, draw the graph of $x = 3$.

(ii) State the y-coordinate of the point on the graph of $-x + 2y = 4$ that has an x-coordinate of 3.

ADVANCED LEVEL

4. Consider the equation $-2x + y = -3$.

(a) Copy and complete the table.

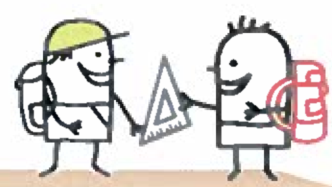
x	-1	0	2
y			

(b) On a sheet of graph paper, using a scale of 4 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $-2x + y = -3$ for $-1 \leqslant x \leqslant 2$.

(c) **(i)** On the same axes in **(b)**, draw the graph of $y = -1$.

(ii) Find the area of the trapezium bounded by the lines $-2x + y = -3$, $y = -1$, and the x- and y-axes.

2.5 Solving Simultaneous Linear Equations Using Graphical Method



We have learnt how to draw graphs of $2x - 3y = 2$, $y = -3$, $x = -2\frac{1}{2}$, $-x + 2y = 4$, etc.

What is the relationship between the graphs of two linear equations when we draw them on the same axes? What is the connection between the coordinates of the point of intersection of the two graphs and the pair of values of x and y that satisfies both the equations?

Investigation

Solving Simultaneous Linear Equations Graphically

1. Consider the linear equations $2x + 3y = 5$ and $3x - y = 2$.

    (i) Using a graphing software, draw the graphs of $2x + 3y = 5$ and $3x - y = 2$ on the same axes.

    (ii) What are the coordinates of the point of intersection of the two graphs?

    (iii) Five pairs of values of x and y are given in Table 2.1.

x	-2	0	1	2	4
y	3	-2	1	4	-1

Table 2.1

Determine the pair of values of x and y that satisfies both the equations $2x + 3y = 5$ and $3x - y = 2$. What do you notice?

2. Consider the linear equations $3x - 4y = 10$ and $5x + 7y = 3$.

    (i) Using a graphing software, draw the graphs of $3x - 4y = 10$ and $5x + 7y = 3$ on the same axes.

    (ii) What are the coordinates of the point of intersection of the two graphs?

    (iii) Hence, state the pair of values of x and y that satisfies both the equations $3x - 4y = 10$ and $5x + 7y = 3$.

What can we conclude about the coordinates of the point of intersection of the two graphs and the pair of values of x and y that satisfies both the equations? Explain your answer.

In the investigation, the graphs of $2x + 3y = 5$ and $3x - y = 2$ intersect at the point (1, 1). $x = 1$ and $y = 1$ satisfies the two linear equations simultaneously. We say that $x = 1$ and $y = 1$ is the **solution** of the **simultaneous linear equations** $2x + 3y = 5$ and $3x - y = 2$.

Can we say the same for the linear equations $3x - 4y = 10$ and $5x + 7y = 3$?

Choice of Appropriate Scales for Graphs

Before we proceed to draw a graph, we have to choose a suitable scale. The following guidelines may be useful:

- Use a convenient scale for both the x-axis and the y-axis. For example, we may use 1 cm to represent 1 unit, 2 units, 4 units, 5 units or 10 units. Avoid using awkward scales such as 1 cm to represent 3 units or 1 cm to represent 4.3 units.
- The scale used for the x-axis need not be the same as the scale used in the y-axis.
- Choose a suitable scale so that the graph will occupy more than half the size of the graph paper.
- Look at the largest and the smallest value of x and estimate the scale to be used. Repeat the process for the values of y.

Class Discussion

Choice of Appropriate Scales for Graphs and Accuracy of Graphs

Work in pairs.

1. Using a suitable scale, draw the graph of $y = 3x - 1$. Compare your graph with that of your classmate. Do the graphs look different?
2. Use your graph in Question 1 to find
   (i) the value of y when $x = 1.3$,
   (ii) the value of x when $y = -2.8$.
3. How do you check for the accuracy of your answers in Question 2?
4. If your answers in Question 2 are inaccurate, how can you improve your graph?

Worked Example 4

(Solving Simultaneous Linear Equations Using Graphical Method)

Using the graphical method, solve the simultaneous equations

$$2x - 5y = 32,$$
$$2x + 3y = 0.$$

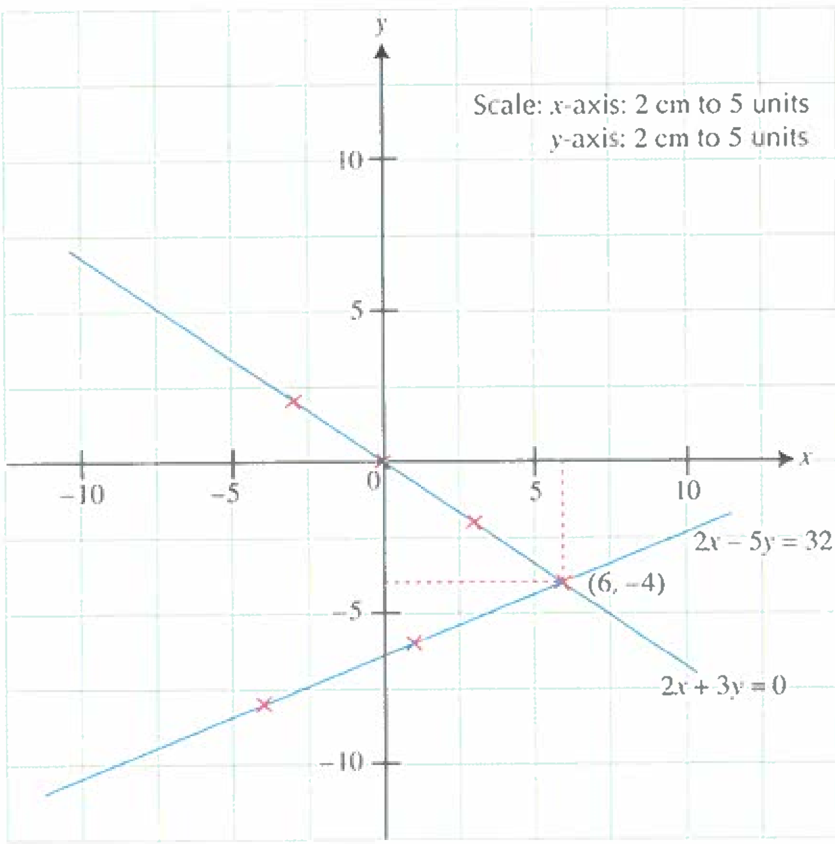
Solution:

$2x - 5y = 32$

x	–4	1	6
y	–8	–6	–4

$2x + 3y = 0$

x	–3	0	3
y	2	0	–2



The graphs intersect at the point (6, –4).

∴ The solution is $x = 6$ and $y = -4$.

RECALL

We only need to plot 3 points to obtain the graph of a linear equation. In fact, a straight line can be determined by plotting 2 points. We use the 3rd point to check for mistakes in the graph.

PRACTISE NOW 4

1. Using the graphical method, solve the simultaneous equations

$$x + y = 3,$$
$$3x + y = 5.$$

2. Using the graphical method, solve the simultaneous equations

$$7x - 2y + 11 = 0,$$
$$6x + y + 4 = 0.$$

SIMILAR QUESTIONS

Exercise 2D Questions 1(a)–(f), 2(a)–(d), 3

Class Discussion

Coincident Lines and Parallel Lines

Work in pairs.

1. (a) Using a graphing software, on separate axes, draw the graphs of each of the following pairs of simultaneous equations.
   (i) $x + y = 1$
   $3x + 3y = 3$
   (ii) $2x + 3y = -1$
   $20x + 30y = -10$
   (iii) $x - 2y = 5$
   $5x - 10y = 25$
   (b) What do you notice about the graphs of each pair of simultaneous equations?
   (c) Does each pair of simultaneous equations have any solutions? If yes, what are the solutions?
2. (a) Using a graphing software, on separate axes, draw the graphs of each of the following pairs of simultaneous equations.
   (i) $x + y = 1$
   $3x + 3y = 15$
   (ii) $2x + 3y = -1$
   $20x + 30y = -40$
   (iii) $x - 2y = 5$
   $5x - 10y = 30$
   (b) What do you notice about the graphs of each pair of simultaneous equations?
   (c) Does each pair of simultaneous equations have any solutions? If yes, what are the solutions?

From the class discussion, we notice that the graphs of each pair of simultaneous equations in Question 1 are *identical*, i.e. the two lines coincide. Since every point on each line is a point of intersection of the graphs, the graphs have an infinite number of points of intersection. Hence, the simultaneous equations have an *infinite number of solutions*.

SIMILAR QUESTIONS

Exercise 2D Questions 4(a)–(d), 5(a)–(b)

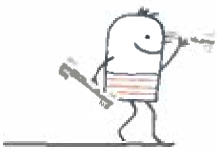
The graphs of each pair of simultaneous equations in Question 2 are *parallel lines*. Since the graphs do not intersect, they have no point of intersection. Hence, the simultaneous equations have *no solution*.

Thinking Time

What type of simultaneous equations have

(a) infinitely many solutions?

(b) no solution?



Exercise 2D

BASIC LEVEL

1. Using the graphical method, solve each of the following pairs of simultaneous equations.
   (a) $3x - y = 0$
   $2x - y = 1$
   (b) $x - y = -3$
   $x - 2y = -1$
   (c) $3x - 2y = 7$
   $2x + 3y = 9$
   (d) $3x + 2y = 4$
   $5x + y = 2$
   (e) $2x + 5y = 25$
   $3x - 2y = 9$
   (f) $3x - 4y = 25$
   $4x - y = 16$

INTERMEDIATE LEVEL

2. Using the graphical method, solve each of the following pairs of simultaneous equations.
   (a) $x + 4y - 12 = 0$
   $4x + y - 18 = 0$
   (b) $3x + y - 2 = 0$
   $2x - y - 3 = 0$
   (c) $3x - 2y - 13 = 0$
   $2x + 2y = 0$
   (d) $2x + 4y + 5 = 0$
   $-x + 5y + 1 = 0$

3. (a) Consider the equation $y = 2x + 9$.
   (i) Copy and complete the table.

x	-8	0	4
y			

   (ii) On a sheet of graph paper, using a scale of 1 cm to represent 1 unit on the x-axis and 1 cm to represent 4 units on the y-axis, draw the graph of $y = 2x + 9$ for $-8 \leqslant x \leqslant 4$.

   (b) Consider the equation $y = \frac{1}{4}x + 2$.
   (i) Copy and complete the table.

x	-8	0	4
y			

   (ii) On the same axes in (a)(ii), draw the graph of $y = \frac{1}{4}x + 2$ for $-8 \leqslant x \leqslant 4$.

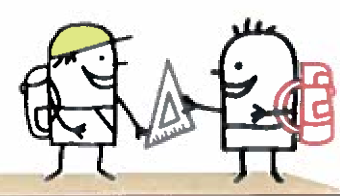
   (c) Hence, solve the simultaneous equations $2x - y = -9$ and $x - 4y = -8$.

4. Using the graphical method, solve each of the following pairs of simultaneous equations.
   (a) $x + 2y = 3$
   $2x + 4y = 6$
   (b) $4x + y = 2$
   $4x + y = -3$
   (c) $2y - x = 2$
   $4y - 2x = 4$
   (d) $2y + x = 4$
   $2y + x = 6$

ADVANCED LEVEL

5. Using the graphical method, solve each of the following pairs of simultaneous equations.
   (a) $y = 3 - 5x$
   $5x + y - 1 = 0$
   (b) $3y + x = 7$
   $15y = 35 - 5x$

2.6 Solving Simultaneous Linear Equations Using Algebraic Methods



In Book 1, we have learnt how to solve linear equations in one variable such as $3x - 4 = 11$ and $4x - 10 = 5x + 7$.

To solve a linear equation in one variable x means to find the value of x so that the values on both sides of the equation are equal, i.e. x satisfies the equation.

Thinking Time

What are the solutions to a linear equation in two variables, e.g. $2x + y = 13$? Do you obtain the same solutions as your classmates?

In Section 2.5, we have learnt that from the graphs of two linear equations, the coordinates of the point(s) of intersection give the solution(s) to the pair of simultaneous linear equations. In this section, we shall take a look at two algebraic methods that can be used to solve a pair of simultaneous equations: the **elimination method** and the **substitution method**.

Solving Simultaneous Linear Equations Using Elimination Method

Let us use the **elimination method** to solve the following pair of equations. We shall label the equations as equation (1) and equation (2).

$$3x - y = 12 \text{ ---------- (1)}$$
$$2x + y = 13 \text{ ---------- (2)}$$

The elimination method is usually used when the absolute values of the coefficients of one variable in both the equations are the same. For example, the absolute values of the coefficients of y in equations (1) and (2) are the same. What happens when we add equation (2) to equation (1)?

(1) + (2): $(3x - y) + (2x + y) = 12 + 13$

$$3x + 2x - y + y = 25$$

Notice that the terms in y are eliminated. We are left with a linear equation in one variable x.

$$5x = 25$$
$$x = 5$$

Substitute $x = 5$ into (1): $3(5) - y = 12$

$$y = 3$$

$\therefore$ The solution of the simultaneous equations is $x = 5$ and $y = 3$.

Check: Substitute $x = 5$ and $y = 3$ into (1) and (2):

In (1), LHS $= 3(5) - 3$
$= 12$
$=$ RHS

In (2), LHS $= 2(5) + 3$
$= 13$
$=$ RHS

Since $x = 5$ and $y = 3$ satisfies both the equations, it is the solution of the simultaneous equations.

Problem Solving Tip

It is a good practice to check your solution by substituting the values of the unknowns which you have found into the original equations.

Worked Example 5

(Solving Simultaneous Linear Equations Using Elimination Method)

Using the elimination method, solve the simultaneous equations

$$3x + 7y = 17,$$
$$3x - 6y = 4.$$

Solution:

$3x + 7y = 17$ ---------- (1)

$3x - 6y = 4$ ---------- (2)

(1) – (2): $(3x + 7y) - (3x - 6y) = 17 - 4$

$3x + 7y - 3x + 6y = 13$

$13y = 13$

$y = 1$

Substitute $y = 1$ into (1): $3x + 7(1) = 17$

$3x = 10$

$x = 3\frac{1}{3}$

$\therefore$ The solution is $x = 3\frac{1}{3}$ and $y = 1$.

Check: Substitute $x = 3\frac{1}{3}$ and $y = 1$ into (1) and (2):

In (1), LHS $= 3\left(3\frac{1}{3}\right) + 7(1)$

$= 17$

$=$ RHS

In (2), LHS $= 3\left(3\frac{1}{3}\right) - 6(1)$

$= 4$

$=$ RHS

ATTENTION

The coefficient of x in both the equations is 3. Hence, when we subtract equation (2) from equation (1), the terms in x are eliminated.

Problem Solving Tip

To eliminate a variable, the absolute values of the coefficients of the variable in both the equations must be the same.

PRACTISE NOW 5

SIMILAR QUESTIONS

Exercise 2E Questions 1(a)–(f), 5(a), 6(a)–(b)

1. Using the elimination method, solve each of the following pairs of simultaneous equations.

   (a) $x - y = 3$
   $4x + y = 17$

   (b) $7x + 2y = 19$
   $7x + 8y = 13$

   (c) $13x + 9y = 4$
   $17x - 9y = 26$

   (d) $4x - 5y = 17$
   $x - 5y = 8$

2. Using the elimination method, solve the simultaneous equations

$$3x - y + 14 = 0,$$
$$2x + y + 1 = 0.$$

It is sometimes necessary to manipulate one of the equations before we can eliminate a variable by addition or subtraction.

Worked Example 6

(Solving Simultaneous Linear Equations Using Elimination Method)

Using the elimination method, solve the simultaneous equations

$$3x + 2y = 8,$$
$$4x - y = 7.$$

Solution:

$3x + 2y = 8$ ---------- (1)

$4x - y = 7$ ---------- (2)

$2 \times$ (2): $8x - 2y = 14$ ---------- (3)

(1) + (3): $(3x + 2y) + (8x - 2y) = 8 + 14$

$11x = 22$

$x = 2$

Substitute $x = 2$ into (2): $4(2) - y = 7$

$y = 1$

$\therefore$ The solution is $x = 2$ and $y = 1$.

ATTENTION

- In this case, it is easier to eliminate y first.
- We multiply equation (2) by 2 so that the absolute values of the coefficients of y in both the equations are the same.

PRACTISE NOW 6

Using the elimination method, solve each of the following pairs of simultaneous equations.

(a) $2x + 3y = 18$
$3x - y = 5$

(b) $4x + y = 11$
$3x + 2y = 7$

SIMILAR QUESTIONS

Exercise 2E Questions 2(a)-(f), 5(b), 6(c)-(d)

It is sometimes necessary to manipulate both the equations before we can eliminate a variable.

Worked Example 7

(Solving Simultaneous Linear Equations Using Elimination Method)

Using the elimination method, solve the simultaneous equations

$$13x - 6y = 20,$$
$$7x + 4y = 18.$$

Solution:

$13x - 6y = 20$ ---------- (1)

$7x + 4y = 18$ ---------- (2)

$2 \times$ (1): $26x - 12y = 40$ ---------- (3)

$3 \times$ (2): $21x + 12y = 54$ ---------- (4)

(3) + (4): $(26x - 12y) + (21x + 12y) = 40 + 54$

$47x = 94$

$x = 2$

Substitute $x = 2$ into (1): $13(2) - 6y = 20$

$6y = 6$

$y = 1$

$\therefore$ The solution is $x = 2$ and $y = 1$.

ATTENTION

- The LCM of 6 and 4 is 12.
- We multiply equation (1) by 2 and equation (2) by 3 so that the absolute values of the coefficients of y in both the equations are the same.

Thinking Time

In Worked Example 7, is it easier to eliminate x first? Explain your answer by showing how x can be eliminated.

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 2E Questions 3(a)–(f), 5(c)–(d), 6(e)–(f)

Using the elimination method, solve each of the following pairs of simultaneous equations.

(a) $9x + 2y = 5$
$7x - 3y = 13$

(b) $5x - 4y = 17$
$2x - 3y = 11$

Worked Example 8

(Solving Simultaneous Fractional Equations Using Elimination Method)

Using the elimination method, solve the simultaneous equations

$$\frac{2}{3}x - \frac{y}{9} = 6,$$
$$x - \frac{y}{3} = 6.$$

Solution:

Method 1:

$\frac{2}{3}x - \frac{y}{9} = 6$ ---------- (1)

$x - \frac{y}{3} = 6$ ---------- (2)

$\frac{3}{2} \times$ (1): $x - \frac{y}{6} = 9$ ---------- (3)

(3) – (2): $\left(x - \frac{y}{6}\right) - \left(x - \frac{y}{3}\right) = 9 - 6$

$-\frac{y}{6} + \frac{y}{3} = 3$

$\frac{y}{6} = 3$

$y = 18$

Substitute $y = 18$ into (2): $x - \frac{18}{3} = 6$

$x = 12$

$\therefore$ The solution is $x = 12$ and $y = 18$.

Method 2:

$\frac{2}{3}x - \frac{y}{9} = 6$ ---------- (1)

$x - \frac{y}{3} = 6$ ---------- (2)

$9 \times$ (1): $6x - y = 54$ ---------- (3)

$3 \times$ (2): $3x - y = 18$ ---------- (4)

(3) – (4): $(6x - y) - (3x - y) = 54 - 18$

$3x = 36$

$x = 12$

Substitute $x = 12$ into (4): $3(12) - y = 18$

$y = 18$

$\therefore$ The solution is $x = 12$ and $y = 18$.

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 2E Questions 7(a)–(d)

Using the elimination method, solve the simultaneous equations

$$\frac{x}{2} - \frac{y}{3} = 4,$$
$$\frac{2}{5}x - \frac{y}{6} = 3\frac{1}{2}.$$

Solving Simultaneous Linear Equations Using Substitution Method

Now, we shall take a look at how we can solve a pair of simultaneous equations using the **substitution method**.

In this method, we first rearrange one equation to express one variable in terms of the other variable. Next, we substitute this expression into the other equation to obtain an equation in only one variable.

Worked Example 9

(Solving Simultaneous Linear Equations Using Substitution Method)

Using the substitution method, solve the simultaneous equations

$$7x - 2y = 21,$$
$$4x + y = 57.$$

Solution:

$7x - 2y = 21$ ---------- (1)

$4x + y = 57$ ---------- (2)

From (2), $y = 57 - 4x$ ---------- (3) (rearrange (2) to express y in terms of x)

Substitute (3) into (1): $7x - 2(57 - 4x) = 21$ (replace y in (1) with expression in x to obtain an equation in x)

$7x - 114 + 8x = 21$

$15x = 135$ (solve the equation in x)

$x = 9$

Substitute $x = 9$ into (3): $y = 57 - 4(9)$ (substitute the solution of x into the expression for y)

$= 21$

$\therefore$ The solution is $x = 9$ and $y = 21$.

Problem Solving Tip

It is easier to obtain the value of y by substituting the value of x into equation (3) instead of equation (1) or (2).

Thinking Time

If we make x the subject of equation (1) or (2) in Worked Example 9, will we get the same solution? Which way is easier?

PRACTISE NOW 9

Using the substitution method, solve the simultaneous equations

$$3y - x = 7,$$
$$2x + 3y = 4.$$

SIMILAR QUESTIONS

Exercise 2F Questions 4(a)–(h)

Worked Example 10

(Solving Simultaneous Linear Equations Using Substitution Method)

Using the substitution method, solve the simultaneous equations

$$3x + 2y = 7,$$
$$9x + 8y = 22.$$

Solution:

$3x + 2y = 7$ ---------- (1)

$9x + 8y = 22$ ---------- (2)

From (1), $2y = 7 - 3x$

$$y = \frac{7 - 3x}{2} \text{ ---------- (3)}$$

Substitute (3) into (2): $9x + 8\left(\frac{7 - 3x}{2}\right) = 22$

$$9x + 4(7 - 3x) = 22$$
$$9x + 28 - 12x = 22$$
$$-3x = -6$$
$$3x = 6$$
$$x = 2$$

Substitute $x = 2$ into (3): $y = \frac{7 - 3(2)}{2}$

$$= \frac{1}{2}$$

$\therefore$ The solution is $x = 2$ and $y = \frac{1}{2}$.

PRACTISE NOW 10

Using the substitution method, solve the simultaneous equations

$$3x - 2y = 8,$$
$$4x + 3y = 5.$$

SIMILAR QUESTIONS

Exercise 2F Questions 8(a)–(f), 11–13

Thinking Time

Devi was asked to solve the simultaneous equations

$$2x + y = 6,$$
$$x = 1 - \frac{1}{2}y,$$

using the substitution method.

She did it this way:

$2x + y = 6$ ---------- (1)

$x = 1 - \frac{1}{2}y$ ---------- (2)

Substitute (2) into (1): $2\left(1 - \frac{1}{2}y\right) + y = 6$

$$2 - y + y = 6$$
$$2 = 6$$

What was wrong? Explain your answer.

Worked Example 11

(Solving Simultaneous Fractional Equations Using Substitution Method)

Using the substitution method, solve the simultaneous equations

$$\frac{x+1}{y+2} = \frac{1}{2},$$
$$\frac{x-2}{y-1} = \frac{1}{3}.$$

Solution:

$\frac{x+1}{y+2} = \frac{1}{2}$ ---------- (1)

$\frac{x-2}{y-1} = \frac{1}{3}$ ---------- (2)

From (1), $2(x + 1) = y + 2$

$2x + 2 = y + 2$

$y = 2x$ ---------- (3)

From (2), $3(x - 2) = y - 1$

$3x - 6 = y - 1$

$3x - y = 5$ ---------- (4)

Substitute (3) into (4): $3x - 2x = 5$

$x = 5$

Substitute $x = 5$ into (3): $y = 2(5)$

$= 10$

$\therefore$ The solution is $x = 5$ and $y = 10$.

RECALL

Consider $\frac{a}{b} = \frac{c}{d}$, where $b, d \neq 0$.

Multiply by bd on both sides,

$bd \times \frac{a}{b} = bd \times \frac{c}{d}$

$\therefore ad = bc$

PRACTISE NOW 11

SIMILAR QUESTIONS

Exercise 2E Questions 9(a)–(d), 10(a)–10(d)

Using the substitution method, solve each of the following pairs of simultaneous equations.

(a) $\frac{x-1}{y-3} = \frac{2}{3}$

$\frac{x-2}{y-1} = \frac{1}{2}$

(b) $3x + 2y = 3$

$\frac{1}{x+y} = \frac{3}{x+2y}$

Exercise 2E

BASIC LEVEL

1. Using the elimination method, solve each of the following pairs of simultaneous equations.

(a) $x + y = 16$, $x - y = 0$ **(b)** $x - y = 5$, $x + y = 19$

(c) $11x + 4y = 12$, $9x - 4y = 8$ **(d)** $4y + x = 11$, $3y - x = 3$

(e) $3x + y = 5$, $x + y = 3$ **(f)** $2x + 3y = 5$, $2x + 7y = 9$

(g) $7x - 3y = 15$, $11x - 3y = 21$ **(h)** $3y - 2x = 9$, $2y - 2x = 7$

(i) $3a - 2b = 5$, $2b - 5a = 9$ **(j)** $5c - 2d = 9$, $3c + 2d = 7$

(k) $3f + 4h = 1$, $5f - 4h = 7$ **(l)** $6j - k = 23$, $3k + 6j = 11$

2. Using the elimination method, solve each of the following pairs of simultaneous equations.

(a) $7x - 2y = 17$, $3x + 4y = 17$ **(b)** $16x + 5y = 39$, $4x - 3y = 31$

(c) $x + 2y = 3$, $3x + 5y = 7$ **(d)** $3x + y = -5$, $7x + 3y = 1$

(e) $7x - 3y = 13$, $2x - y = 3$ **(f)** $9x - 5y = 2$, $3x - 4y = 10$

3. Using the elimination method, solve each of the following pairs of simultaneous equations.

(a) $7x - 3y = 18$, $6x + 7y = 25$ **(b)** $4x + 3y = -5$, $3x - 2y = 43$

(c) $2x + 3y = 8$, $5x + 2y = 9$ **(d)** $5x + 4y = 11$, $3x + 5y = 4$

(e) $4x - 3y = -1$, $5x - 2y = 4$ **(f)** $5x - 4y = 23$, $2x - 7y = 11$

4. Using the substitution method, solve each of the following pairs of simultaneous equations.

(a) $x + y = 7$, $x - y = 5$ **(b)** $3x - y = 0$, $2x + y = 5$

(c) $2x - 7y = 5$, $3x + y = -4$ **(d)** $5x - y = 5$, $3x + 2y = 29$

(e) $5x + 3y = 11$, $4x - y = 2$ **(f)** $3x + 5y = 10$, $x - 2y = 7$

(g) $x + y = 9$, $5x - 2y = 4$ **(h)** $5x + 2y = 3$, $x - 4y = -6$

INTERMEDIATE LEVEL

5. Using the elimination method, solve each of the following pairs of simultaneous equations.

(a) $x + y = 0.5$, $x - y = 1$ **(b)** $2x + 0.4y = 8$, $5x - 1.2y = 9$

(c) $10x - 3y = 24.5$, $3x - 5y = 13.5$ **(d)** $6x + 5y = 10.5$, $5x - 3y = -2$

6. Using the elimination method, solve each of the following pairs of simultaneous equations.

(a) $4x - y - 7 = 0$
$4x + 3y - 11 = 0$

(b) $7x + 2y - 33 = 0$
$3y - 7x - 17 = 0$

(c) $5x - 3y - 2 = 0$
$x + 5y - 6 = 0$

(d) $5x - 3y - 13 = 0$
$7x - 6y - 20 = 0$

(e) $7x + 3y - 8 = 0$
$3x - 4y - 14 = 0$

(f) $3x + 5y + 8 = 0$
$4x + 13y - 2 = 0$

7. Using the elimination method, solve each of the following pairs of simultaneous equations.

(a) $\frac{x+1}{y+2} = \frac{3}{4}$
$\frac{x-2}{y-1} = \frac{3}{5}$

(b) $\frac{x}{3} - \frac{y}{2} = \frac{5}{6}$
$3x - \frac{2}{5}y = 3\frac{2}{5}$

(c) $\frac{x}{4} - \frac{3}{8}y = 3$
$\frac{5}{3}x - \frac{y}{2} = 12$

(d) $\frac{x-3}{5} = \frac{y-7}{2}$
$11x = 13y$

8. Using the substitution method, solve each of the following pairs of simultaneous equations.

(a) $2x + 5y = 12$
$4x + 3y = -4$

(b) $4x - 3y = 25$
$6x + 5y = 9$

(c) $3x + 7y = 2$
$6x - 5y = 4$

(d) $9x + 2y = 5$
$7x - 3y = 13$

(e) $2y - 5x = 25$
$4x + 3y = 3$

(f) $3x - 5y = 7$
$4x - 3y = 3$

9. Using the substitution method, solve each of the following pairs of simultaneous equations.

(a) $\frac{x}{5} + y + 2 = 0$
$\frac{x}{3} - y - 10 = 0$

(b) $\frac{x+y}{3} = 3$
$\frac{3x+y}{5} = 1$

(c) $3x - y = 23$
$\frac{x}{3} + \frac{y}{4} = 4$

(d) $\frac{x}{3} + \frac{y}{2} = 4$
$\frac{2}{3}x - \frac{y}{6} = 1$

ADVANCED LEVEL

10. Using either the elimination or the substitution method, solve each of the following pairs of simultaneous equations.

(a) $\frac{2}{x+y} = \frac{1}{2x+y}$
$3x + 4y = 9$

(b) $\frac{1}{5}(x-2) = \frac{1}{4}(1-y)$
$\frac{1}{7}\left(x + 2\frac{2}{3}\right) = \frac{1}{3}(3-y)$

(c) $\frac{5x+y}{9} = 2 - \frac{x+y}{5}$
$\frac{7x-3}{2} = 1 + \frac{y-x}{3}$

(d) $\frac{x+y}{3} = \frac{x-y}{5} = 2x - 3y + 5$

11. If $x = 3$ and $y = -1$ is the solution of the simultaneous equations

$$3px + qy = 11,$$
$$-qx + 5y = p,$$

find the value of p and of q.

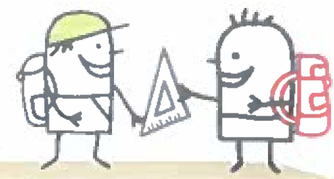
12. If $x = -11$ and $y = 5$ is the solution of the simultaneous equations

$$px + 5y = q,$$
$$qx + 7y = p,$$

find the value of p and of q.

13. A computer animation shows a cat moving in a straight line. Its height, h metres, above the ground, is given by $8s - 3h = -9$, where s is the time in seconds after it starts moving. In the same animation, a mouse starts to move at the same time as the cat and its movement is given by $-29s + 10h = 16$. Find the height above the ground and the time when the cat meets the mouse.

2.7 Applications of Simultaneous Equations in Real-World Contexts



In this section, we will learn how to apply the concept of simultaneous equations to solve mathematical and real-life problems.

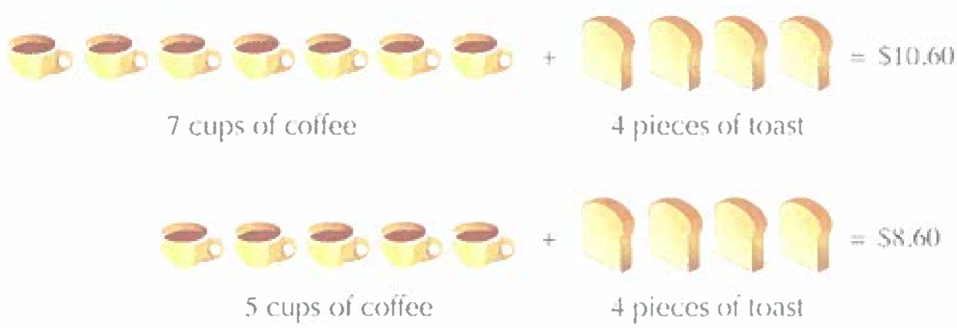
A familiar problem which we have learnt how to solve in primary school is used to illustrate how to formulate a pair of simultaneous equations to solve the problem.

Consider the following problem:

> 7 cups of coffee and 4 pieces of toast cost \$10.60. 5 cups of coffee and 4 pieces of toast cost \$8.60. Find the cost of each item.

In primary school, we have learnt how to solve the problem by using the following representation (or drawing a diagram).

Method 1:



$\therefore$ Cost of 2 cups of coffee = \$10.60 – \$8.60
= \$2

$\therefore$ Cost of 1 cup of coffee = \$1

5 × \$1 + cost of 4 pieces of toast = \$8.60

Cost of 4 pieces of toast = \$8.60 – \$5
= \$3.60

$\therefore$ Cost of 1 piece of toast = \$0.90

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We shall now take a look at how we can solve the problem by using the algebraic method.

Method 2:

Let the cost of 1 cup of coffee be $\$x$ and the cost of 1 piece of toast be $\$y$.

$$7x + 4y = 10.6 \text{ ---------- (1)}$$
$$5x + 4y = 8.6 \text{ ---------- (2)}$$

(1) – (2): $2x = 10.6 - 8.6$

$2x = 2$

$x = 1$

Substitute $x = 1$ into (2): $5(1) + 4y = 8.6$

$4y = 8.6 - 5$

$4y = 3.6$

$y = 0.9$

∴ Cost of 1 cup of coffee = \$1

Cost of 1 piece of toast = \$0.90

Worked Example 12

(Finding Two Numbers Given Sum and Difference)

The sum of two numbers is 67 and their difference is 3. Find the two numbers.

Solution:

Let the smaller number be x and the greater number be y.

$$x + y = 67 \text{ ---------- (1)}$$
$$y - x = 3 \text{ ---------- (2)}$$

(1) + (2): $2y = 70$

$y = 35$

Substitute $y = 35$ into (1): $x + 35 = 67$

$x = 32$

The two numbers are 32 and 35.

Thinking Time

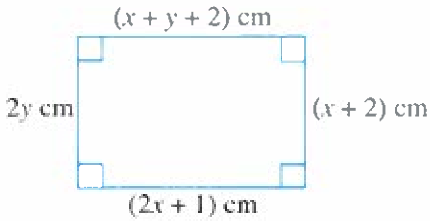
Can you solve Worked Example 12 by using only one variable x?

PRACTISE NOW 12

SIMILAR QUESTIONS

Exercise 2F Questions 1–2, 5–10, 18

1. The sum of two numbers is 36 and their difference is 9. Find the two numbers.
2. One third of the sum of two angles is 60° and one quarter of their difference is 28°. Find the two angles.
3. The figure shows a rectangle with its length and breadth as indicated. Find the perimeter of the rectangle.



Worked Example 13

(Finding a Fraction)

If 1 is added to the numerator and 2 to the denominator of a fraction, the value obtained is $\frac{2}{3}$. If 2 is subtracted from its numerator and 1 from its denominator, the resulting value is $\frac{1}{3}$. Find the fraction.

Solution:

Let the numerator of the fraction be x and its denominator be y, i.e. let the fraction be $\frac{x}{y}$.

$$\frac{x+1}{y+2} = \frac{2}{3} \quad \text{----------} (1)$$

$$\frac{x-2}{y-1} = \frac{1}{3} \quad \text{----------} (2)$$

From (1), $3(x + 1) = 2(y + 2)$

$3x + 3 = 2y + 4$

$3x - 2y = 1$ ---------- (3)

From (2), $3(x - 2) = y - 1$

$3x - 6 = y - 1$

$3x - y = 5$ ---------- (4)

(4) – (3): $y = 4$

Substitute $y = 4$ into (4): $3x - 4 = 5$

$3x = 9$

$x = 3$

$\therefore$ The fraction is $\frac{3}{4}$.

PRACTISE NOW 13

SIMILAR QUESTIONS

Exercise 2F Question 11

If 1 is added to the numerator and to the denominator of a fraction, the value obtained is $\frac{4}{5}$. If 5 is subtracted from its numerator and from its denominator, the resulting value is $\frac{1}{2}$. Find the fraction.

Worked Example 14

(Finding ages)

The sum of the ages of Jun Wei and his mother is 60. Two years ago, Jun Wei's mother was three times as old as Jun Wei. Calculate

(i) Jun Wei's present age,

(ii) the age of Jun Wei's mother when he was born.

Solution:

(i) Let the present age of Jun Wei's mother be x years and that of Jun Wei be y years. Then two years ago, Jun Wei's mother was $(x - 2)$ years old and Jun Wei was $(y - 2)$ years old.

$$x + y = 60 \text{ ---------- (1)}$$

$$x - 2 = 3(y - 2) \text{ ---------- (2)}$$

From (2), $x - 2 = 3y - 6$

$x = 3y - 4$ ---------- (3)

Substitute (3) into (1): $(3y - 4) + y = 60$

$4y = 64$

$y = 16$

Jun Wei's present age = 16 years

(ii) Substitute $y = 16$ into (3): $x = 3(16) - 4$

$= 44$

$\therefore$ Age of Jun Wei's mother when he was born $= 44 - 16$

$= 28$ years

PRACTISE NOW 14

SIMILAR QUESTIONS

Exercise 2F Questions 3–4, 12–17, 20–21

1. In five years' time, Kate's father will be three times as old as Kate. Four years ago, her father was six times as old as her. Find their present ages.
2. To visit the two conservatories at Gardens by the Bay, 11 adults and 5 children have to pay \$280 whereas 14 adults and 9 children have to pay \$388. Find the total amount a family of 2 adults and 3 children has to pay to visit the two conservatories.

Worked Example 15

(Finding a Two-digit Number)

The sum of the digits of a two-digit number is 8. When the digits of the number are reversed and the number is subtracted from the original number, the result obtained is 18. Find the original number.

Solution:

Let the tens digit of the original number be x and its ones digit be y.
Then the original number is $10x + y$,
the number obtained when the digits of the original number are reversed is $10y + x$.

$$x + y = 8 \text{ ---------- (1)}$$

$$10x + y - (10y + x) = 18 \text{ ---------- (2)}$$

From (2), $10x + y - 10y - x = 18$

$$9x - 9y = 18$$

$$x - y = 2 \text{ (divide by 9 throughout) ---------- (3)}$$

(1) + (3): $2x = 10$

$$x = 5$$

Substitute $x = 5$ into (1): $5 + y = 8$

$$y = 3$$

$\therefore$ The original number is 53.

Search on the Internet for 'Psychic Mind Reader'. This is an interactive applet that claims to read your mind. Follow the instructions on the applet. Do you believe that the applet can read your mind? If not, try to show, by using algebra, how this fascinating trick works! Explain why the symbols for the numbers 90 and 99 are different from the symbol that corresponds to your final number.

PRACTISE NOW 15

A two-digit number is such that the sum of its digits is 11. When the digits of the number are reversed and the number is subtracted from the original number, the result obtained is 9. Find the original number.

SIMILAR QUESTIONS

Exercise 2F Question 19

Exercise 2F

In each of the following questions, formulate a pair of linear equations in two variables to solve the problem.

BASIC LEVEL

1. The sum of two numbers is 138 and their difference is 88. Find the two numbers.

2. The difference between two numbers is 10 and their sum is four times the smaller number. Find the two numbers.

3. A belt and a wallet cost \$42. 7 belts and 4 wallets cost \$213. Find the cost of each item.

4. 8 kg of potatoes and 5 kg of carrots cost \$28 whereas 2 kg of potatoes and 3 kg of carrots cost \$11.20. Find the cost of 1 kg of each item.

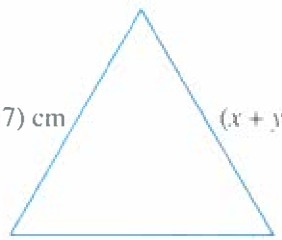
INTERMEDIATE LEVEL

5. Two numbers are such that if 7 is added to the first number, a number twice the second number is obtained. If 20 is added to the second number, the number obtained is four times the first number. Find the two numbers.

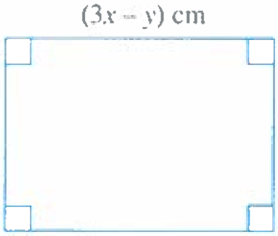
6. The sum of two numbers is 48. If the smaller number is one fifth of the larger number, find the two numbers.

7. One fifth of the sum of two angles is 24° and half their difference is 14°. Find the two angles.

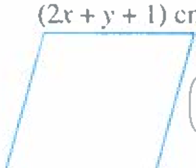
8. The figure shows an equilateral triangle with its sides as indicated. Find the length of each side of the triangle.



9. The figure shows a rectangle with its length and breadth as indicated. Given that the perimeter of the rectangle is 120 cm, find the area of the rectangle.



10. The figure shows a rhombus with its sides as indicated. Find the perimeter of the figure.



11. If 1 is subtracted from the numerator and from the denominator of a fraction, the value obtained is $\frac{1}{2}$. If 1 is added to its numerator and to its denominator, the resulting value is $\frac{2}{3}$. Find the fraction.

12. The giant pandas Kai Kai and Jia Jia reside at the River Safari. The sum of their ages when they first arrived in Singapore in 2013 was 11 years. In 2022, Kai Kai will be three times as old as Jia Jia was in 2013. Find their ages in 2014.

13. 6 adults and 4 senior citizens have to pay $228 while 13 adults and 7 senior citizens have to pay $459 to visit an exhibition at the ArtScience Museum. Find the total amount 2 adults and a senior citizen have to pay to visit the exhibition.

14. Lixin intends to buy either Gift *A*, which costs $10, or Gift *B*, which costs $8, as Christmas gifts for each of her parents, 2 siblings, 13 relatives and 10 friends. Given that she intends to spend $230, find the number of each gift she should buy.

15. There are some chickens and goats on a farm. Given that the animals have a total of 50 heads and 140 legs, how many more chickens than goats are there?

16. $80 is divided between Ethan and Michael such that one quarter of Ethan's share is equal to one sixth of Michael's share. How much does each of them receive?

17. Rui Feng deposited a total of $25 000 in Bank *A* and Bank *B* at the beginning of 2013. Bank *A* and Bank *B* pay simple interest at rates of 0.6% and 0.65% per annum respectively. He withdrew all his money from the two banks at the end of 2013. If the amount of interest he earned from each bank is the same, find the amount of money he deposited in each bank.

ADVANCED LEVEL

18. Two numbers are such that when the larger number is divided by the smaller number, both the quotient and the remainder are equal to 2. If five times the smaller number is divided by the larger number, both the quotient and the remainder are also equal to 2. Find the two numbers.

19. A two-digit number is such that the sum of its digits is $\frac{1}{8}$ of the number. When the digits of the number are reversed and the number is subtracted from the original number, the result obtained is 45. Find the original number.

20. Raj has \$10. If he buys 8 pears and 5 mangoes, he will be short of \$1.10. If he buys 5 pears and 4 mangoes, he will receive \$1.75 in change. Find the price of 1 pear and 1 mango.

21. Huixian's mother buys some shares of Company *A* on Day 0. On Day 7, the share price of Company *A* is \$4.60. If she sells all her shares of Company *A* and buys 2000 shares of Company *B* on Day 7, she would receive \$7400. On Day 12, the share price of Company *A* is \$4.80 and the share price of Company *B* is \$0.50 less than that on Day 7. If she sells all her shares of Company *A* and buys 5000 shares of Company *B* on Day 12, she would have to pay \$5800. Find

(i) the number of shares of Company *A* Huixian's mother has,

(ii) the share price of Company *B* on Day 12.

1. The equation of a straight line is in the form $y = mx + c$, where the constant m is the **gradient** of the line and the constant c is the **y-intercept**.

2. The gradient of a straight line is the measure of the ratio of the vertical change (or rise) to the horizontal change (or run), i.e.

$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}} \text{ or } \frac{\text{rise}}{\text{run}}.$$

3. If $x = a$ and $y = b$ satisfy each of the two **simultaneous equations** in two variables, then $x = a$ and $y = b$ is known as a **solution** of the two equations.

4. A pair of simultaneous linear equations in two variables can be solved by

- the **graphical method**,
- the **elimination method**,
- the **substitution method**.

5. The solution of a pair of simultaneous linear equations is given by the coordinates of the *point of intersection* of the graphs of the two equations.

6. A pair of simultaneous linear equations has an *infinite number of solutions* if the graphs of the two equations are *identical*.

7. A pair of simultaneous linear equations has *no solution* if the graphs of the two equations are *parallel*.

8. For mathematical and real-life problems that involve simultaneous equations, we formulate a pair of linear equations in two variables before solving for the unknowns in the problems.

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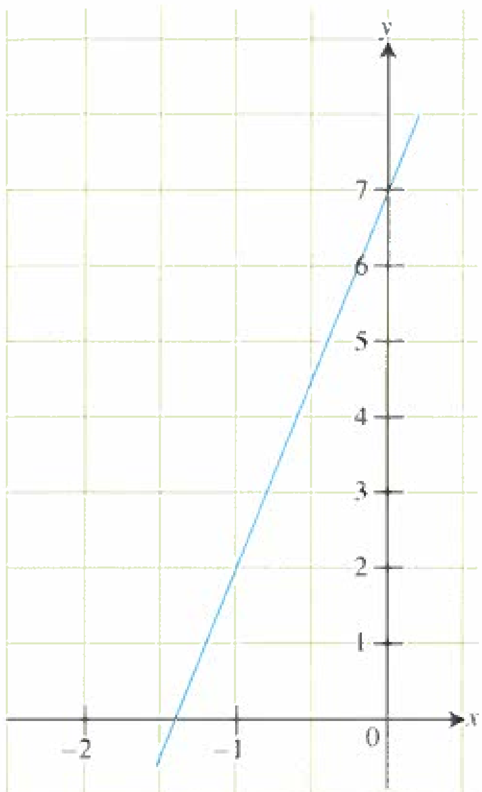
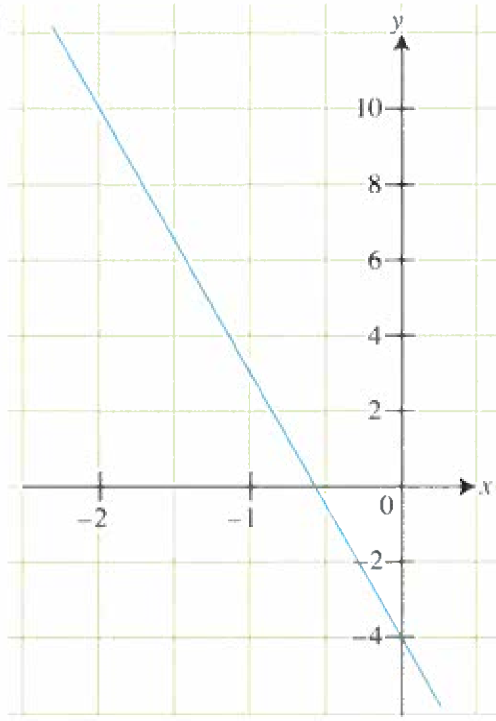
Review Exercise 2

1. Given that the equation of the line representing each of the following linear graphs is in the form $y = mx + c$, find the gradient m and state the y-intercept c.

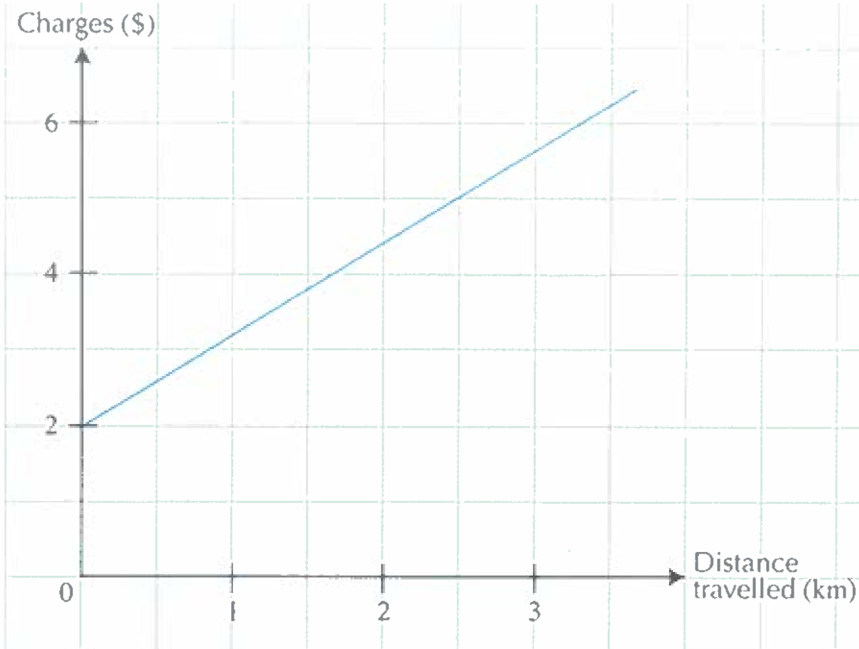
(a)



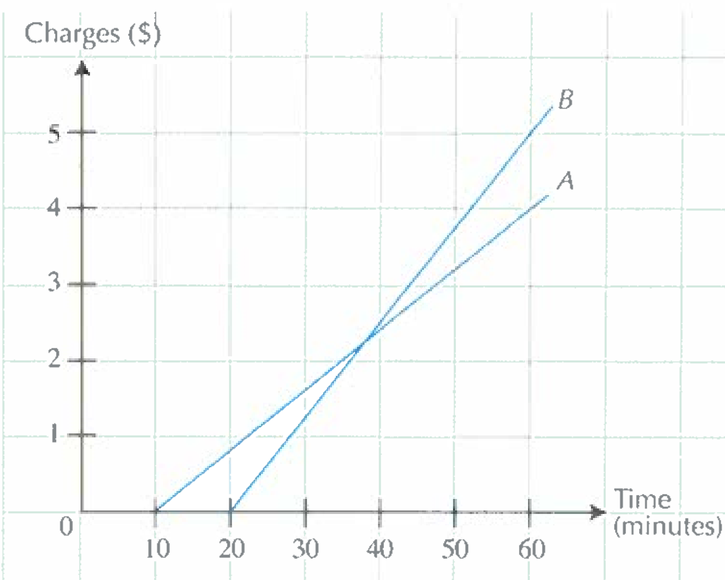
(b)



2. The flag-down fare of a taxi is $$m$. The taxi charges $$n$ for each kilometre it travels. Use the graph to find the value of m and of n.



3. Two mobile phone companies, A and B, offer plans with a talk time rate as shown in the graph.



(i) How much does Company A charge for 20 minutes of talk time?

(ii) How much does Company B charge for 50 minutes of talk time?

(iii) If Jun Wei uses less than 30 minutes of talk time per month, which company would be able to offer him a better price? Explain your answer.

(iv) Which company has a greater rate of increase in charges? Explain your answer.

(v) If Michael wants to pay only $4 for a talk time plan per month, which company should he choose? Explain your answer.

4. Consider the equation $2x + y = 2$.

(a) Copy and complete the table.

x	−4	0	4
y			

(b) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $2x + y = 2$ for $-4 \leqslant x \leqslant 4$.

(c) The point $(p, -2)$ lies on the graph in **(b)**. Find the value of p.

(d) (i) On the same axes in **(b)**, draw the graph of $x = -0.5$.

(ii) State the coordinates of the point of intersection of the graphs of $2x + y = 2$ and $x = -0.5$.

5. (a) The variables x and y are connected by the equation $5x - 3y = 2$. Some values of x and the corresponding values of y are given in the table.

x	−5	−2	7
y	p	−4	q

(i) Find the value of p and of q.

(ii) On a sheet of graph paper, using a scale of 1 cm to represent 1 unit on both axes, draw the graph of $5x - 3y = 2$ for $-5 \leqslant x \leqslant 7$.

(b) Consider the equation $3x + 4y = 7$.

(i) Copy and complete the table.

x	−5	3	7
y			

(ii) On the same axes in **(a)(ii)**, draw the graph of $3x + 4y = 7$ for $-5 \leqslant x \leqslant 7$.

(c) Hence, solve the simultaneous equations $5x - 3y = 2$ and $3x + 4y = 7$.

6. Solve each of the following pairs of simultaneous equations.

(a) $7x + 2y = 10$
$5x + 2y = 6$

(b) $9x + 4y = 28$
$4y - 11x = -12$

(c) $2x - 5y = 22$
$2x - 3y = 14$

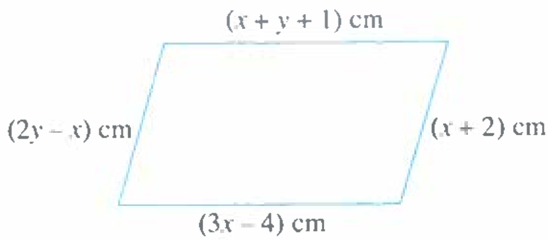
(d) $6x - y = 16$
$3x + 2y = -12$

(e) $4x + 3y = 0$
$5y + 53 = 11x$

(f) $5x - 4y = 4$
$2x - y = 2.5$

7. Two numbers are such that if 11 is added to the first number, a number twice the second number is obtained. If 20 is added to the second number, the number obtained is twice the first number. Find the two numbers.

8. The figure shows a parallelogram with its sides as indicated. Find the perimeter of the parallelogram.



9. If 1 is subtracted from the numerator and 2 is added to the denominator of a fraction, the value obtained is $\frac{1}{2}$. If 3 is added to its numerator and 2 is subtracted from its denominator, the resulting value is $1\frac{1}{4}$. Find the fraction.

10. A two-digit number is such that the sum of its digits is 12 and the ones digit is twice its tens digit. Find the number.

11. In four years' time, Khairul's mother will be three times as old as Khairul. Six years ago, his mother was seven times as old as him. Find

(i) Khairul's present age,

(ii) the age of Khairul's mother when he was born.

12. If Shirley gives $3 to Priya, Priya will have twice as much as Shirley. If Priya gives $5 to Shirley, Shirley will have twice as much as Priya. How much does each of them have?

13. A vendor buys 36 smartphones and tablet computers for $28 065. Given that a smartphone costs $895 and a tablet computer costs $618, find the number of each item the vendor buys.

14. 5 cups of ice-cream milk tea and 4 cups of citron tea cost $26.80 whereas 7 cups of ice-cream milk tea and 6 cups of citron tea cost $38.60. Find the difference between the cost of 1 cup of ice-cream milk tea and 1 cup of citron tea.

15. Vishal mixes coffee powder that costs $2.50 per kg with coffee powder that costs $3.50 per kg. Given that he sold 20 kg of the mixture at $2.80 per kg such that he does not make any profit or incur any loss, find the mass of each type of coffee powder that he uses for the mixture.

16. A mobile company charges a fixed rate of x cents per minute for the first 120 minutes of talk time and another rate of y cents per minute for each additional minute of talk time. Ethan paid $26.80 and $32.40 for 175 minutes and 210 minutes of talk time on two different occasions respectively. Find the amount he has to pay if he uses 140 minutes of talk time.

17. In a Mathematics test, the average score obtained by Class 2A is 72 and the average score obtained by Class 2B is 75. The average score obtained by the two classes is 73.48. Given that there is a total of 75 students in the two classes, find the number of students in each class.

Challenge Yourself

1. Consider the simultaneous equations

$$px - y = 6,$$
$$8x - 2y = q.$$

Determine the conditions that p and q must satisfy if the simultaneous equations have

(i) an infinite number of solutions,
(ii) no solution,
(iii) a unique solution, i.e. only one solution.

2. Solve the simultaneous equations

$$\frac{4}{x} + \frac{15}{y} = 15,$$
$$\frac{7}{5x} - \frac{6}{y} = 3.$$

3. Two positive numbers are such that the sum of 11 times the square of the first number and 13 times the cube of the second number is 395. If 218 is subtracted from 26 times the cube of the second number, the number obtained is 121 times the square of the first number. Find the two numbers.

4. Some spiders, dragonflies and houseflies are kept in three separate enclosures. They have a total of 20 heads, 136 legs and 19 pairs of wings. Given that a spider has 8 legs and 0 pairs of wings, a dragonfly has 6 legs and 2 pairs of wings, and a housefly has 6 legs and 1 pair of wings, find

(i) the number of spiders,
(ii) the number of dragonflies,
(iii) the number of houseflies.

5. A rooster costs \$5 and a hen costs \$3. Chicks are sold at 3 for \$1. A farmer bought 100 birds of these three types for \$100. How many of each type of bird did he buy?
Hint: There are three possible sets of answers.

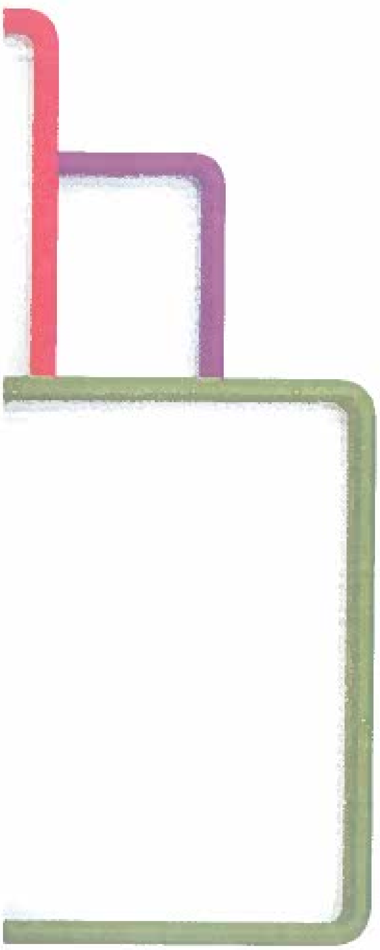
Expansion and Factorisation of Quadratic Expressions

A teacher asks his students, 'If the area of a rectangle is $(x^2 + 8x + 12)$ cm^2 and its lengt $(x + 6)$ cm, what is the breadth of the rectangle?'

A student replies, 'The breadth of the rectangle is $(x + 2)$ cm.'

After you have learnt how to factorise quadratic expressions, you will be able to determi how the student arrives at his answer.

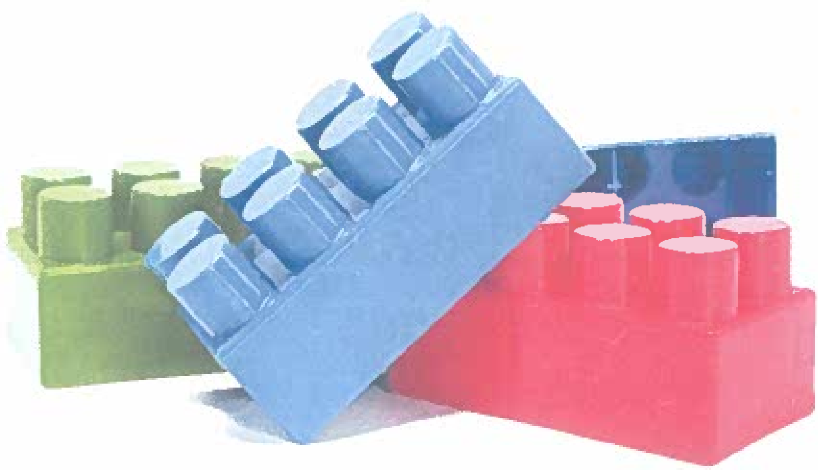
Chapter Three



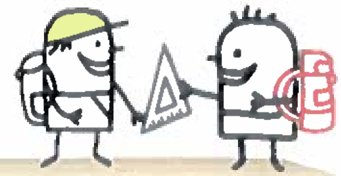
LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- recognise quadratic expressions,
- expand and simplify quadratic expressions,
- use a multiplication frame to factorise quadratic expressions.



3.1 Quadratic Expressions



Recap (Algebraic Expressions)

In Book 1, we have learnt that an **algebraic expression** is an expression that consists of algebraic terms, operation symbols (+, –, ×, ÷) or brackets. An algebraic expression has no equal sign.

We have also learnt that in the algebraic expression $3x - 5$, there are:

- 2 terms: $3x$, -5
- 1 variable: x
- 1 constant term: -5

The coefficient of x is 3.

The algebraic expression $3x - 5$ is an example of a **linear expression** of the form $ax + b$, where x is the only variable, and a and b are constants.

Now, we shall learn how to manipulate **quadratic expressions** in one variable. The general form of a quadratic expression in one variable is $ax^2 + bx + c$, where x is the variable, a, b and c are constants and $a \neq 0$.

Representation of Quadratic Expressions

In Book 1, we have learnt that an algebra disc has two sides:

1	-1
front	back
x	$-x$
front	back

Similarly, for an algebra disc where one side shows the variable x^2, the other side will show $-x^2$.

x^2	$-x^2$
front	back

To obtain the negative of x^2, we flip the disc with x^2 as shown:

x^2 —flip (change sign)→ $-x^2$ $\qquad -(x^2) = -x^2$

To obtain the negative of $-x^2$, we flip the disc with $-x^2$ as shown:

$-x^2$ —flip (change sign)→ x^2 $\qquad -(-x^2) = x^2$

If we put two discs x^2 and $-x^2$ together, we will get a zero pair:

x^2 $-x^2$ $\qquad x^2 + (-x^2) = 0$

We use three x^2 discs to represent $3x^2$.

x^2 x^2 x^2 $\quad 3x^2 = x^2 + x^2 + x^2$

We use three $-x^2$ discs to represent $-3x^2$.

$-x^2$ $-x^2$ $-x^2$ $\quad -3x^2 = (-x^2) + (-x^2) + (-x^2)$

To obtain the negative of $3x^2$, i.e. $-(3x^2)$, we flip the three x^2 discs as shown:

x^2 x^2 x^2 —flip (change sign)→ $-x^2$ $-x^2$ $-x^2$ $\quad -(3x^2) = -3x^2$

To obtain the negative of $-3x^2$, i.e. $-(-3x^2)$, we flip the three $-x^2$ discs as shown:

$-x^2$ $-x^2$ $-x^2$ —flip (change sign)→ x^2 x^2 x^2 $\quad -(-3x^2) = 3x^2$

What happens if we put three x^2 discs and three $-x^2$ discs together?

x^2 $-x^2$

x^2 $-x^2$ $\qquad 3x^2 + (-3x^2) = 0$

x^2 $-x^2$

We will get zero pairs.

We can also use algebra discs to represent quadratic expressions.

Example: $2x^2 + x - 3$

x^2 x^2 x -1 -1 -1 $\quad 2x^2 + x - 3 = x^2 + x^2 + x + (-1) + (-1) + (-1)$

Example: $-2x^2 - x + 1$

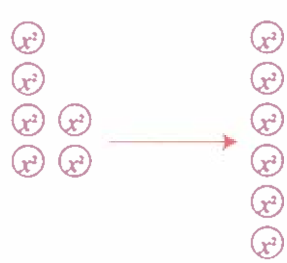
$-x^2$ $-x^2$ $-x$ 1 $\quad -2x^2 - x + 1 = (-x^2) + (-x^2) + (-x) + 1$

What is the quadratic expression represented by x^2 x^2 x^2 $-x$ -1 -1?

Addition and Subtraction of Quadratic Expressions

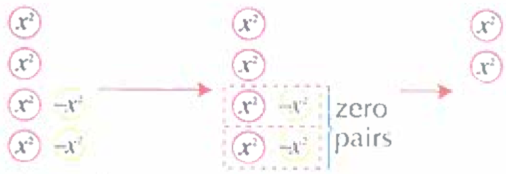
We will show how to carry out addition and subtraction of quadratic expressions using algebra discs.

Example: $4x^2 + 2x^2$



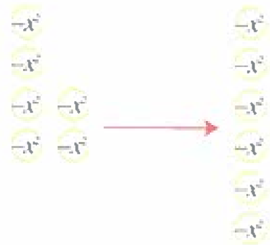
Therefore, $4x^2 + 2x^2 = 6x^2$.

Example: $4x^2 + (-2x^2)$



Therefore, $4x^2 + (-2x^2) = 2x^2$.

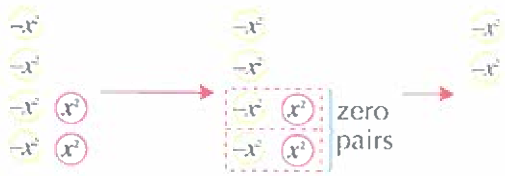
Example: $-4x^2 - 2x^2 = -4x^2 + (-2x^2)$



Therefore, $-4x^2 - 2x^2 = -6x^2$.

Example: $-4x^2 - (-2x^2)$

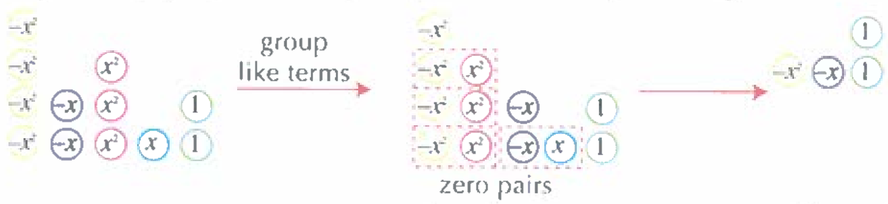
As $-(-2x^2) = 2x^2$, $-4x^2 - (-2x^2)$ can be represented by:



Therefore, $-4x^2 - (-2x^2) = -4x^2 + 2x^2$
$= -2x^2$.

Example: $-4x^2 - 2x - (-3x^2) + x + 2$

As $-(-3x^2) = 3x^2$, $-4x^2 - 2x - (-3x^2) + x + 2$ can be represented by:



$-4x^2 - 2x + 3x^2 + x + 2 \longrightarrow -4x^2 + 3x^2 - 2x + x + 2 \longrightarrow -x^2 - x + 2$

Therefore, $-4x^2 - 2x - (-3x^2) + x + 2 = -4x^2 - 2x + 3x^2 + x + 2$
$= -4x^2 + 3x^2 - 2x + x + 2$ (group like terms)
$= -x^2 - x + 2$.

RECALL

Like terms are terms that have the same variable, e.g. $2x$ and $5x$. When two terms are not like terms, they are known as **unlike terms**, e.g. $2x$ and $2y$; x and x^2.

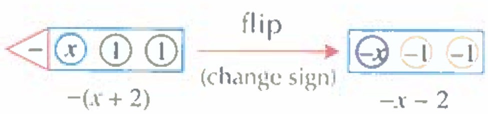
PRACTISE NOW

Simplify each of the following expressions by using algebra discs. Alternatively, you may visit http://www.shinglee.com.sg/StudentResources/ to access the AlgeTools™ software (Go to algedisc/Quadratic Expressions/Activity 1).

(a) $-4x^2 + 2x^2$ **(b)** $-4x^2 + (-2x^2)$
(c) $4x^2 - 2x^2$ **(d)** $4x^2 - (-2x^2)$
(e) $2x^2 - 3 - x^2 + 1$ **(f)** $5x^2 + (-x) + 2 - (-2x^2) - 3x - 4$

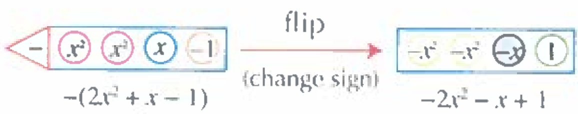
Negative of a Quadratic Expression

In Book 1, we have learnt how to find the negative of a linear expression, e.g. $-(x + 2) = -x - 2$, using algebra discs:



Similarly, to find the negative of a quadratic expression, we flip all the discs.

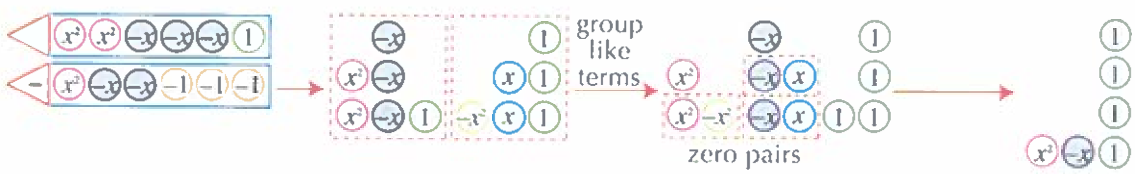
Example: $-(2x^2 + x - 1)$



Therefore, $-(2x^2 + x - 1) = -2x^2 - x + 1$.

We can also simplify quadratic expressions involving the negative of a quadratic expression using algebra discs.

Example: $2x^2 - 3x + 1 - (x^2 - 2x - 3)$



$(2x^2 - 3x + 1) - (x^2 - 2x - 3) \rightarrow 2x^2 - 3x + 1 - x^2 + 2x + 3 \rightarrow 2x^2 - x^2 - 3x + 2x + 1 + 3 \rightarrow x^2 - x + 4$

Therefore, $2x^2 - 3x + 1 \boxed{- (x^2 - 2x - 3)} = 2x^2 - 3x + 1 - x^2 + 2x + 3$ (simplify to get negative of expression)

$= 2x^2 - x^2 - 3x + 2x + 1 + 3$ (group like terms)

$= x^2 - x + 4$.

PRACTISE NOW

Simplify each of the following expressions by using algebra discs. Alternatively, you may visit http://www.shinglee.com.sg/StudentResources/ to access the AlgeTools™ software (Go to algedisc/Quadratic Expressions/Activity 1).

(a) $-(2x^2 + x + 1)$

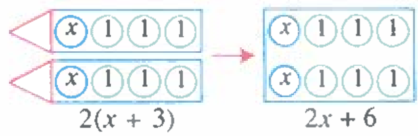
(b) $-(-2x^2 - x + 1)$

(c) $x^2 + 2x + 1 - (3x^2 + 5x - 2)$

(d) $-(-x^2 - 4) + 2x^2 - 7x + 3$

Expansion and Simplification of Simple Quadratic Expressions

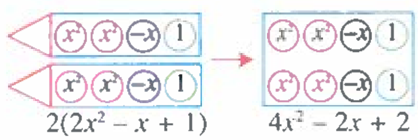
In Book 1, we have learnt how to expand the linear expression $2(x + 3)$ by representing it as '2 groups of $(x + 3)$' using algebra discs:



Therefore, $2(x + 3) = 2x + 6$.

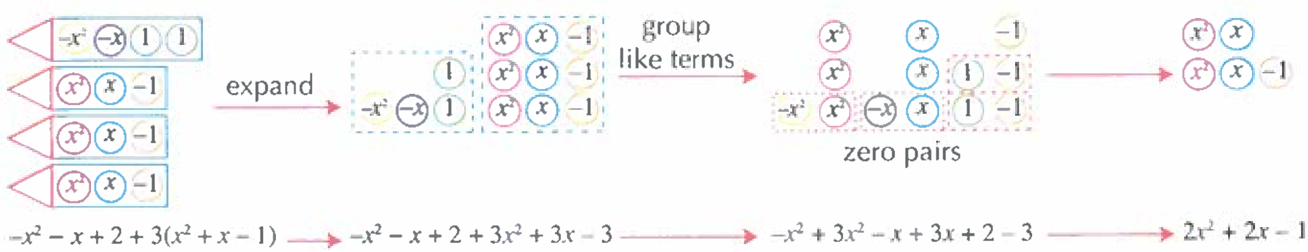
We can **expand** quadratic expressions of similar forms using algebra discs.

Example: $2(2x^2 - x + 1)$ can be represented by '2 groups of $(2x^2 - x + 1)$':



We can also simplify quadratic expressions involving the expansion of a quadratic expression using algebra discs.

Example: $-x^2 - x + 2 + 3(x^2 + x - 1)$, where $3(x^2 + x - 1)$ can be represented by '3 groups of $(x^2 + x - 1)$':



Therefore, $-x^2 - x + 2 + 3(x^2 + x - 1) = -x^2 - x + 2 + 3x^2 + 3x - 3$ (expand)

$= -x^2 + 3x^2 - x + 3x + 2 - 3$ (group like terms)

$= 2x^2 + 2x - 1$.

PRACTISE NOW

Expand and/or simplify each of the following expressions by using algebra discs. Alternatively, you may visit http://www.shinglee.com.sg/StudentResources/ to access the AlgeTools™ software (Go to algedisc/Quadratic Expressions/Activity 1).

(a) $2(-2x^2 + x - 1)$ **(b)** $3(x^2 - 2x + 3)$

(c) $4x^2 + (-3x) + (-1) + 3(x^2 - 4)$ **(d)** $2(x^2 + 4x - 5) - (6 + x^2)$

Worked Example 1

(Expanding and Simplifying Simple Quadratic Expressions)

Expand and/or simplify each of the following expressions.

(a) $3x^2 + 4x - x^2 - x$

(b) $2x^2 - 5x + 7 + (-4x^2) - (-2x) - 3$

(c) $5x^2 + 6x + 1 - (4x^2 - 5x + 1)$

(d) $-(1 - x^2) + 2(4x^2 - 3)$

Solution:

(a) $3x^2 + 4x - x^2 - x = 3x^2 - x^2 + 4x - x$ (group like terms)

$= 2x^2 + 3x$

(b) $2x^2 - 5x + 7 \boxed{+ (-4x^2)} \boxed{- (-2x)} - 3 = 2x^2 - 5x + 7 - 4x^2 + 2x - 3$

$= 2x^2 - 4x^2 - 5x + 2x + 7 - 3$ (group like terms)

$= -2x^2 - 3x + 4$

(c) $5x^2 + 6x + 1 \boxed{- (4x^2 - 5x + 1)} = 5x^2 + 6x + 1 - 4x^2 + 5x - 1$ (simplify to get negative of expression)

$= 5x^2 - 4x^2 + 6x + 5x + 1 - 1$ (group like terms)

$= x^2 + 11x$

(d) $\boxed{-(1 - x^2)} + 2(4x^2 - 3) = -1 + x^2 + 8x^2 - 6$ (simplify to get negative of expression; '2 groups of $(4x^2 - 3)$')

$= x^2 + 8x^2 - 1 - 6$ (group like terms)

$= 9x^2 - 7$

PRACTISE NOW 1

SIMILAR QUESTIONS

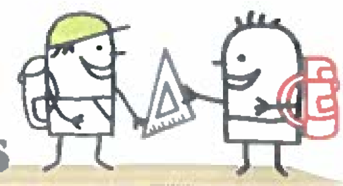
Exercise 3A Questions 1(a)–(i)

Expand and/or simplify each of the following expressions.

(a) $7x^2 - 4x + 6x^2 - x$ **(b)** $-(-5x^2) + 3x + (-6) + 2(3x^2 - 8x + 4)$

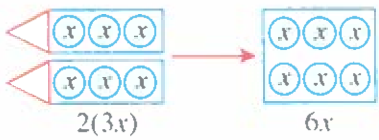
(c) $4x^2 - 1 - (7x^2 + 13x - 2)$ **(d)** $-(3x^2 + 5x - 8) + x^2 + 6x + 5$

3.2 Expansion and Simplification of Quadratic Expressions



Product of Algebraic Terms in One Variable

In Book 1, we have learnt that $2(3x)$ can be represented by '2 groups of $3x$':



Therefore, $2(3x) = 6x$.

Notice that the '6' in '$6x$' is obtained by multiplying 3 by 2.

When we multiply an algebraic term by another algebraic term containing the same variable, the coefficients and the variable are multiplied separately, e.g. $2x \times 3x = (2 \times 3) \times (x \times x) = 6x^2$.

What do we get when we multiply $4x$ by $-5x$?

RECALL

$x \times x = x^2$

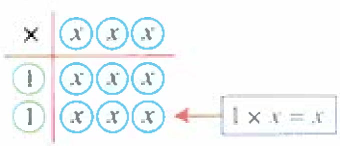
SIMILAR QUESTIONS

Exercise 3A Questions 2(a)–(d)

Expansion and Simplification of Quadratic Expressions

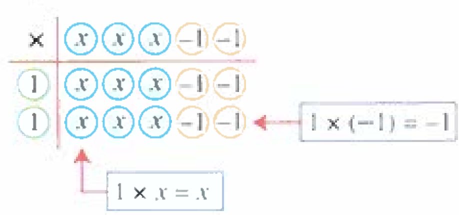
We shall now learn how to expand quadratic expressions of the form $a(b + c)$ using algebra discs and a **multiplication frame**. The following examples illustrate the multiplication process.

Example: $2(3x)$
We multiply each disc in $3x$ by 2.



Therefore, $2(3x) = 6x$.

Example: $2(3x - 2)$
We multiply each disc in $3x - 2$ by 2.



Therefore, $2(3x - 2) = 2(3x) + 2(-2)$
$= 6x - 4$.

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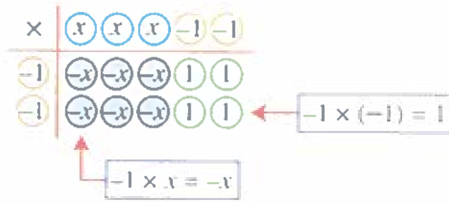
Example: $-2(3x - 2)$

We multiply each disc in $3x - 2$ by 2 before changing its sign.

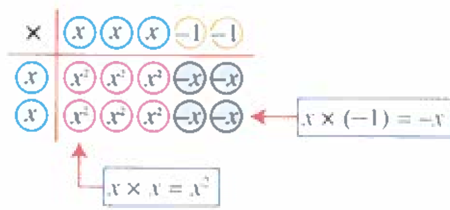


Therefore, $-2(3x - 2) = (-2)(3x) + (-2)(-2)$

$= -6x + 4.$

Example: $2x(3x - 2)$

We multiply each disc in $3x - 2$ by $2x$.

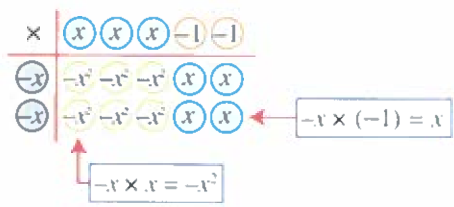


Therefore, $2x(3x - 2) = 2x(3x) + 2x(-2)$

$= 6x^2 - 4x.$

Example: $-2x(3x - 2)$

We multiply each disc in $3x - 2$ by $2x$ before changing its sign.



Therefore, $-2x(3x - 2) = (-2x)(3x) + (-2x)(-2)$

$= -6x^2 + 4x.$

PRACTISE NOW

Expand each of the following expressions by using algebra discs. Alternatively, you may visit http://www.shinglee.com.sg/StudentResources/ to access the AlgeTools™ software (Go to algedisc/Quadratic Expressions/Activity 2).

(a) $3(2x + 1)$ **(b)** $-3(2x - 1)$

(c) $x(-2x + 3)$ **(d)** $-2x(x - 3)$

In general, we have:

$$a(b + c) = ab + ac$$

Recall from Book 1 that this is known as the **Distributive Law**, where 'a groups of b and c' is the same as 'a groups of b' and 'a groups of c', i.e. 'a times of b' and 'a times of c'.

Worked Example 2

(Expanding Linear and Quadratic Expressions of the Form $a(b + c)$)

Expand each of the following expressions.

(a) $2(x - 2)$ **(b)** $-2(3 + 4x)$

(c) $2x(x + 5)$ **(d)** $-3x(3 - 4x)$

Solution:

(a) $2(x - 2) = 2x - 4$ (Distributive Law; '2 groups of $(x - 2)$')

(b) $-2(3 + 4x) = -6 - 8x$ (Distributive Law; 'the negative of 2 groups of $(3 + 4x)$')

(c) $2x(x + 5) = 2x^2 + 10x$ (Distributive Law)

(d) $-3x(3 - 4x) = -9x + 12x^2$ (Distributive Law)

ATTENTION

For $x \neq 0$ and 2, $x \times x \neq 2x$.

PRACTISE NOW 2

Expand each of the following expressions.

(a) $3(4x + 1)$ **(b)** $7(5x - 2)$

(c) $5x(2x - 3)$ **(d)** $-2x(8x - 3)$

SIMILAR QUESTIONS

Exercise 3A Questions 3(a)–(h)

Worked Example 3

(Expanding and Simplifying Linear and Quadratic Expressions of the Form $a(b + c)$)

Expand and simplify each of the following expressions.

(a) $2(2x + 3) - 2(x - 5)$ **(b)** $x(2x + 1) + 2x(x + 4)$

Solution:

(a) $2(2x + 3) - 2(x - 5) = 4x + 6 - 2x + 10$ (Distributive Law; '2 groups of $(2x + 3)$', 'the negative of 2 groups of $(x - 5)$')

$= 4x - 2x + 6 + 10$ (group like terms)

$= 2x + 16$

(b) $x(2x + 1) + 2x(x + 4) = 2x^2 + x + 2x^2 + 8x$ (Distributive Law)

$= 2x^2 + 2x^2 + x + 8x$ (group like terms)

$= 4x^2 + 9x$

RECALL

The rules by which operations are performed when an algebraic expression involves brackets are as follows:

- Simplify the expression *within* the brackets first.
- Use the **Distributive Law** when an expression within a pair of brackets is multiplied by an algebraic term.
- When an expression contains more than one pair of brackets, simplify the expression within the *innermost* pair of brackets first.

PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 3A Questions 4(a)–(d), 6(a)–(f)

Expand and simplify each of the following expressions.

(a) $5(x - 4) - 3(2x + 4)$ **(b)** $2x(2x + 3) - x(2 - 5x)$

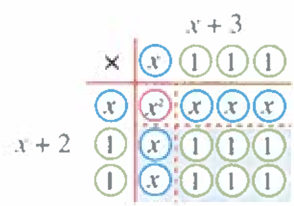
Further Expansion and Simplification of Quadratic Expressions

Now, let us learn how to expand quadratic expressions of the form $(a + b)(c + d)$ using algebra discs and a multiplication frame. The following examples illustrate the multiplication process.

Example: $(x + 2)(x + 3)$

First, we multiply each disc in $x + 3$ by x.

Next, we multiply each disc in $x + 3$ by 2.



×	x	3
x	x^2	$3x$
2	$2x$	6

Therefore, $(x + 2)(x + 3) = x(x + 3) + 2(x + 3)$

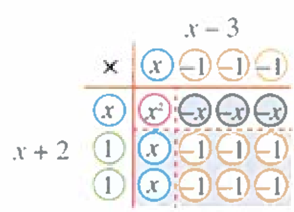
$= x^2 + 3x + 2x + 6$

$= x^2 + 5x + 6.$

Example: $(x + 2)(x - 3)$

First, we multiply each disc in $x - 3$ by x.

Next, we multiply each disc in $x - 3$ by 2.



×	x	-3
x	x^2	$-3x$
2	$2x$	-6

Therefore, $(x + 2)(x - 3) = x(x - 3) + 2(x - 3)$

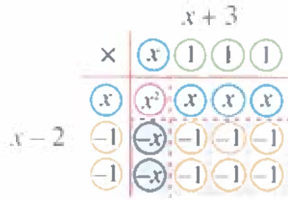
$= x^2 - 3x + 2x - 6$

$= x^2 - x - 6.$

Example: $(x-2)(x+3)$

First, we multiply each disc in $x+3$ by x.

Next, we multiply each disc in $x+3$ by 2 before changing its sign.



×	x	3
x	x^2	$3x$
-2	$-2x$	-6

Therefore, $(x-2)(x+3) = x(x+3) - 2(x+3)$

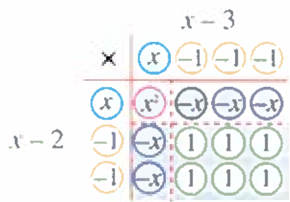
$= x^2 + 3x - 2x - 6$

$= x^2 + x - 6.$

Example: $(x-2)(x-3)$

First, we multiply each disc in $x-3$ by x.

Next, we multiply each disc in $x-3$ by 2 before changing its sign.



×	x	-3
x	x^2	$-3x$
-2	$-2x$	6

Therefore, $(x-2)(x-3) = x(x-3) - 2(x-3)$

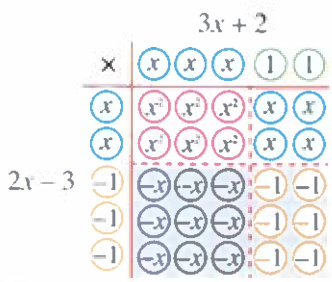
$= x^2 - 3x - 2x + 6$

$= x^2 - 5x + 6.$

Example: $(2x-3)(3x+2)$

First, we multiply each disc in $3x+2$ by $2x$.

Next, we multiply each disc in $3x+2$ by 3 before changing its sign.



×	$3x$	2
$2x$	$6x^2$	$4x$
-3	$-9x$	-6

Therefore, $(2x-3)(3x+2) = 2x(3x+2) - 3(3x+2)$

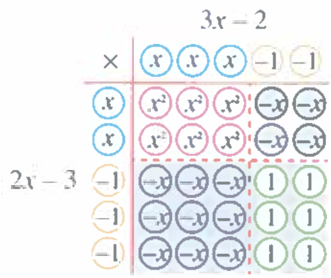
$= 6x^2 + 4x - 9x - 6$

$= 6x^2 - 5x - 6.$

Example: $(2x - 3)(3x - 2)$

First, we multiply each disc in $3x - 2$ by $2x$.

Next, we multiply each disc in $3x - 2$ by 3 before changing its sign.



×	$3x$	-2
$2x$	$6x^2$	$-4x$
-3	$-9x$	6

$$\begin{aligned}\text{Therefore, } (2x - 3)(3x - 2) &= 2x(3x - 2) - 3(3x - 2)\\ &= 6x^2 - 4x - 9x + 6\\ &= 6x^2 - 13x + 6.\end{aligned}$$

When we expand quadratic expressions of the form $(a + b)(c + d)$ using algebra discs, we obtain a rectangular array. Notice that there are four distinct regions in the rectangular array. The top-left region contains the x^2-discs, the bottom-right region contains the 1-discs and the other two regions contain the x-discs.

Class Discussion

Expansion of Quadratic Expressions of the Form $(a + b)(c + d)$

Work in pairs.

1. Expand each of the following expressions by using algebra discs. Alternatively, you may visit http://www.shinglee.com.sg/StudentResources/ to access the AlgeTools™ software (Go to algedisc/Quadratic Expressions/Activity 2).

   (a) $(x + 3)(x + 6)$ **(b)** $(x + 3)(x - 6)$

   (c) $(x - 3)(x + 6)$ **(d)** $(x - 3)(x - 6)$

   (e) $(3x + 1)(2x + 3)$ **(f)** $(3x + 1)(2x - 3)$

   (g) $(3x - 1)(2x + 3)$ **(h)** $(3x - 1)(2x - 3)$

2. Expand $(a + b)(c + d)$.

From the class discussion, we have:

$$\begin{aligned}(a + b)(c + d) &= a(c + d) + b(c + d)\\ &= ac + ad + bc + bd\end{aligned}$$

Worked Example 4

(Expanding Quadratic Expressions of the Form $(a + b)(c + d)$)

Expand each of the following expressions.

(a) $(x + 4)(x + 5)$ (b) $(2x + 3)(x + 2)$

(c) $(3x + 8)(5x - 9)$ (d) $(4 - 3x)(5 - 2x)$

Solution:

(a) $(x + 4)(x + 5) = x(x + 5) + 4(x + 5)$ (multiply the expression in the second bracket by each term in the first bracket)

$= x^2 + 5x + 4x + 20$ (Distributive Law)

$= x^2 + 9x + 20$

(b) $(2x + 3)(x + 2) = 2x(x + 2) + 3(x + 2)$ (multiply the expression in the second bracket by each term in the first bracket)

$= 2x^2 + 4x + 3x + 6$ (Distributive Law)

$= 2x^2 + 7x + 6$

(c) $(3x + 8)(5x - 9) = 3x(5x - 9) + 8(5x - 9)$ (multiply the expression in the second bracket by each term in the first bracket)

$= 15x^2 - 27x + 40x - 72$ (Distributive Law)

$= 15x^2 + 13x - 72$

(d) $(4 - 3x)(5 - 2x) = 4(5 - 2x) - 3x(5 - 2x)$ (multiply the expression in the second bracket by each term in the first bracket)

$= 20 - 8x - 15x + 6x^2$ (Distributive Law)

$= 20 - 23x + 6x^2$

PRACTISE NOW 4

Expand each of the following expressions.

(a) $(x + 2)(x + 4)$ **(b)** $(3x - 4)(5x - 6)$

(c) $(5 + x)(2 - 3x)$ **(d)** $(1 - 7x)(11x - 4)$

SIMILAR QUESTIONS

Exercise 3A Questions 5(a)–(b), 7(a)–(f)

Worked Example 5

(Expanding and Simplifying Quadratic Expressions of the Form $(a + b)(c + d)$)

Expand and simplify the expression
$(x + 5)(x - 4) - (x + 2)(x - 7)$.

Solution:

$(x + 5)(x - 4) - (x + 2)(x - 7) = x(x - 4) + 5(x - 4) - [x(x - 7) + 2(x - 7)]$

(multiply the expression in the second bracket by each term in the first bracket; do not forget to include the square brackets for the expansion of $(x + 2)(x - 7)$)

$= x^2 - 4x + 5x - 20 - (x^2 - 7x + 2x - 14)$ (Distributive Law)

$= x^2 + x - 20 \boxed{- (x^2 - 5x - 14)}$

$= x^2 + x - 20 - x^2 + 5x + 14$ (simplify to get negative of expression)

$= x^2 - x^2 + x + 5x - 20 + 14$ (group like terms)

$= 6x - 6$

PRACTISE NOW 5

Expand and simplify the expression $(2x - 1)(x + 5) - 5x(x - 4)$.

SIMILAR QUESTIONS

Exercise 3A Questions 8(a)–(d), 9(a)–(d)

Exercise 3A

BASIC LEVEL

1. Expand and/or simplify each of the following expressions.
   (a) $6x^2 + 19 + 9x^2 - 8$
   (b) $x^2 + 2x - 7 - (-11x^2) - 5x - 1$
   (c) $y + (-3y^2) + 2(y^2 - 6y)$
   (d) $5x^2 - x - (x^2 - 10x)$
   (e) $-(4x^2 + 9x + 2) + 3x^2 - 7x + 2$
   (f) $-(1 - 7y - 8y^2) + 2(y^2 - 3y - 1)$

2. Find each of the following products.
   (a) $12 \times 5x$ (b) $x \times 6x$
   (c) $(-2x) \times 8x$ (d) $(-3x) \times (-10x)$

3. Expand each of the following expressions.
   (a) $4(3x + 4)$ (b) $-6(-7x - 3)$
   (c) $8(-x - 3)$ (d) $-2(5x - 1)$
   (e) $5x(3x - 4)$ (f) $-8x(3x + 5)$
   (g) $-5x(2 - 3x)$ (h) $-x(-x - 1)$

4. Expand and simplify each of the following expressions.
   (a) $4(2a + 3) + 5(a + 3)$
   (b) $9(5 - 2b) + 3(6 - 5b)$
   (c) $c(3c + 1) + 2c(c + 3)$
   (d) $6d(5d - 4) + 2d(3d - 2)$

5. Expand each of the following expressions.
   (a) $(x + 3)(x + 7)$ (b) $(4x + 1)(3x + 5)$

INTERMEDIATE LEVEL

6. Expand and simplify each of the following expressions.
   (a) $7(2a + 1) - 4(8a + 3)$
   (b) $3(2b - 1) - 2(5b - 3)$
   (c) $3c(5 + c) - 2c(3c - 7)$
   (d) $2d(3d - 5) - d(2 - d)$
   (e) $-f(9 - 2f) + 4f(f - 8)$
   (f) $-2h(3 + 4h) - 5h(h - 1)$

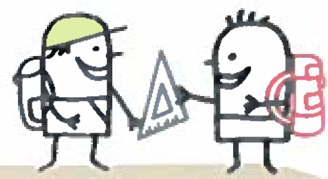
7. Expand each of the following expressions.
(a) $(a + 1)(a - 9)$ (b) $(b - 2)(b + 7)$
(c) $(c - 5)(c - 6)$ (d) $(3d + 1)(5 - 2d)$
(e) $(1 - f)(7f + 6)$ (f) $(4 - 3h)(10 - 9h)$

8. Expand and simplify each of the following expressions.
(a) $5 + (x + 1)(x + 3)$
(b) $3x + (x + 7)(2x - 1)$
(c) $(3x + 2)(x - 9) + 2x(4x + 1)$
(d) $(x - 3)(x - 8) + (x - 4)(2x + 9)$

ADVANCED LEVEL

9. Expand and simplify each of the following expressions.
(a) $4x^2 - (3x - 4)(2x + 1)$
(b) $2x(x - 6) - (2x + 5)(7 - x)$
(c) $(4x - 3)(x + 2) - (3x - 5)(-x - 9)$
(d) $(2x + 3)(5x - 2) - 2(5x - 3)(x + 1)$

3.3 Factorisation of Quadratic Expressions

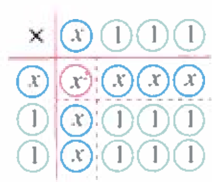


In Book 1, we have learnt that **factorisation** is the process of expressing an algebraic expression as a *product* of two or more algebraic expressions. Factorisation is the *reverse* of expansion.

In Section 3.2, we have learnt how to expand the product of two linear factors to obtain a quadratic expression of the form $ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$. In this section, we shall learn how to carry out the reverse process, i.e. factorisation.

Factorisation of Quadratic Expressions Using Algebra Discs

Recall that in Section 3.2, when we expand the product of the two linear factors, $x + 2$ and $x + 3$, we obtain the quadratic expression, $x^2 + 5x + 6$. Recall also that there are four distinct regions in the rectangular array. The top-left region contains the x^2-discs, the bottom-right region contains the 1-discs and the other two regions contain the x-discs.

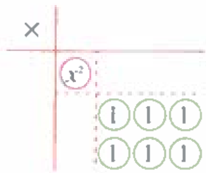


Therefore, $(x + 2)(x + 3) = x(x + 3) + 2(x + 3)$
$= x^2 + 3x + 2x + 6$
$= x^2 + 5x + 6.$

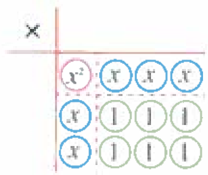
Since factorisation is the reverse of expansion, when we factorise a quadratic expression, we will obtain two linear factors. Hence,

$$\begin{aligned} x^2 + 5x + 6 &= x^2 + 3x + 2x + 6 \\ &= x(x + 3) + 2(x + 3) \\ &= (x + 2)(x + 3). \end{aligned}$$

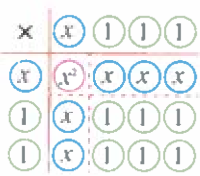
Algebra discs can also be used to factorise quadratic expressions. To factorise $x^2 + 5x + 6$, we first form a rectangular array with the (x^2) disc at the top-left region and the six (1) discs at the bottom-right region.



Next, we put in the five (x) discs to form the rectangle.



Thus we have:



- $x^2 = x \times x$
- $x = x \times 1$

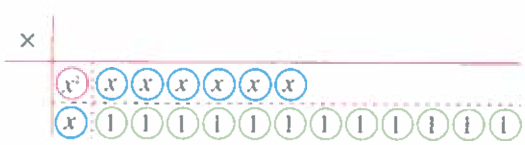
Hence, the two linear factors are $x + 2$ and $x + 3$.

Notice that in this case, the constant term 6 is factorised into 2×3 and the five (x) discs ($5x$) are divided into two groups ($2x$ and $3x$) to complete the rectangle.

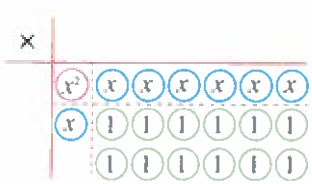
Example: $x^2 + 7x + 12$

We have to form a rectangle with the (x^2) disc at the top-left region and the twelve (1) discs at the bottom-right region. The possible factorisations of 12 are 1×12, 2×6 and 3×4.

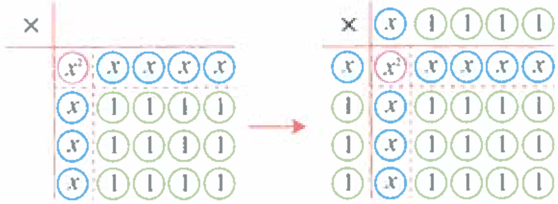
Consider 1×12. We are not able to divide '$7x$' into two groups to complete the rectangle.



Consider 2×6. We are also not able to divide '$7x$' into two groups to complete the rectangle.



Consider 3×4. We are able to divide '$7x$' into '$3x$' and '$4x$' to complete the rectangle.



The factorisation can be simply illustrated using a multiplication frame:

×	x	4
x	x^2	$4x$
3	$3x$	12

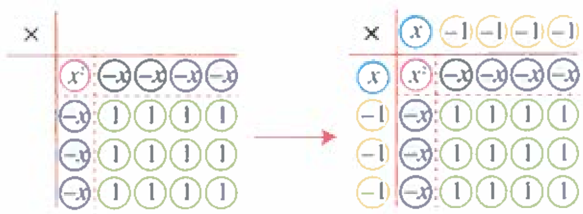
Therefore, $x^2 + 7x + 12 = (x + 3)(x + 4)$.

ATTENTION

- $x \times x = x^2$
- $3 \times 4 = 12$
- $3 \times x = 3x$
- $4 \times x = 4x$
- $3x + 4x = 7x$

Example: $x^2 - 7x + 12$

We have to form a rectangle with the x^2 disc at the top-left region and the twelve 1 discs at the bottom-right region. Then we divide '$-7x$' into '$-3x$' and '$-4x$' to complete the rectangle.



The factorisation can be simply illustrated using a multiplication frame:

×	x	-4
x	x^2	$-4x$
-3	$-3x$	12

Therefore, $x^2 - 7x + 12 = (x - 3)(x - 4)$.

ATTENTION

- $x \times x = x^2$
- $(-3) \times (-4) = 12$
- $(-3) \times x = -3x$
- $(-4) \times x = -4x$
- $(-3x) + (-4x) = -7x$

PRACTISE NOW

Factorise each of the following expressions by using algebra discs. Alternatively, you may visit http://www.shinglee.com.sg/StudentResources/ to access the AlgeTools™ software (Go to algedisc/Quadratic Expressions/Activity 4).

(a) $x^2 + 6x + 5$ **(b)** $x^2 - 6x + 5$

(c) $x^2 + 8x + 12$ **(d)** $x^2 - 8x + 12$

Factorisation of Quadratic Expressions Using a Multiplication Frame

Without using algebra discs, we can use a multiplication frame to help us factorise quadratic expressions.

Consider the expression $x^2 + 8x + 12$.

Step 1

$\times$		
	x^2	
		12

→

Step 2

$\times$	x	6
x	x^2	
2		12

→

Step 3

$\times$	x	6
x	x^2	$6x$
2	$2x$	12

$2x + 6x = 8x$

Step 1: Write x^2 in the top-left corner and 12 in the bottom-right corner of the multiplication frame.

Step 2: Consider the factors of x^2 and 12. Write them in the first column and the first row.

Step 3: Multiply them to complete the multiplication frame and check whether the result matches the given expression.

Therefore, $x^2 + 8x + 12 = (x + 2)(x + 6)$.

Let us apply the above method to factorise another expression $x^2 - 5x + 4$.

Consider the factorisation of x^2 and 4, i.e. $x^2 = x \times x$ and $4 = (-1) \times (-4)$.

$\times$		
	x^2	
		4

→

$\times$	x	-4
x	x^2	
-1		4

→

$\times$	x	-4
x	x^2	$-4x$
-1	$-x$	4

$(-x) + (-4x) = -5x$

Therefore, $x^2 - 5x + 4 = (x - 1)(x - 4)$.

ATTENTION

The possible factorisations of 12 are 1×12, 2×6 and 3×4.

- Consider 1×12.

$\times$	x	12
x	x^2	$12x$
1	x	12

$x + 12x = 13x \neq 8x$

- Consider 3×4.

$\times$	x	4
x	x^2	$4x$
3	$3x$	12

$3x + 4x = 7x \neq 8x$

The above cases are rejected as the term in x does not match that in the given expression.

Worked Example 6

Factorising Quadratic Expressions of the Form $x^2 + bx + c$

Factorise each of the following expressions completely.

(a) $x^2 + 7x + 10$ **(b)** $x^2 - 9x + 14$

(c) $x^2 + x - 20$ **(d)** $x^2 - x - 12$

Problem Solving Tip

For $ax^2 + bx + c$ where a, b and $c > 0$, both of the factors of c must be positive, e.g. in **(a)**, we only have to consider $10 = 1 \times 10$ and 2×5.

Solution:

(a) $x^2 = x \times x$

$10 = 1 \times 10$ or $(-1) \times (-10)$

$= 2 \times 5$ or $(-2) \times (-5)$

$\times$	x	5
x	x^2	$5x$
2	$2x$	10

$2x + 5x = 7x$

2 and 5 are selected as $2x + 5x = 7x$.

$\therefore x^2 + 7x + 10 = (x + 2)(x + 5)$

Problem Solving Tip

It is a good practice to check your answer by expanding the product of the two linear factors to see if it gives the original quadratic expression, e.g. in **(a)**, $(x + 2)(x + 5) = x(x + 5) + 2(x + 5)$ $= x^2 + 5x + 2x + 10 = x^2 + 7x + 10$.

(b) $x^2 = x \times x$

$14 = 1 \times 14$ or $(-1) \times (-14)$

$\quad = 2 \times 7$ or $(-2) \times (-7)$

$\times$	x	-7
x	x^2	$-7x$
-2	$-2x$	14

$(-2x) + (-7x) = -9x$

-2 and -7 are selected as $(-2x) + (-7x) = -9x$.

$\therefore x^2 - 9x + 14 = (x - 2)(x - 7)$

Problem Solving Tip

For $ax^2 + bx + c$ where a and $c > 0$ but $b < 0$, both of the factors of c must be negative, e.g. in **(b)**, we only have to consider $14 = (-1) \times (-14)$ and $(-2) \times (-7)$.

(c) $x^2 = x \times x$

$-20 = 1 \times (-20)$ or $(-1) \times 20$

$\quad = 2 \times (-10)$ or $(-2) \times 10$

$\quad = 4 \times (-5)$ or $(-4) \times 5$

$\times$	x	5
x	x^2	$5x$
-4	$-4x$	-20

$(-4x) + 5x = x$

-4 and 5 are selected as $(-4x) + 5x = x$.

$\therefore x^2 + x - 20 = (x - 4)(x + 5)$

(d) $x^2 = x \times x$

$-12 = 1 \times (-12)$ or $(-1) \times 12$

$\quad = 2 \times (-6)$ or $(-2) \times 6$

$\quad = 3 \times (-4)$ or $(-3) \times 4$

$\times$	x	-4
x	x^2	$-4x$
3	$3x$	-12

$3x + (-4x) = -x$

3 and -4 are selected as $3x + (-4x) = -x$.

$\therefore x^2 - x - 12 = (x + 3)(x - 4)$

PRACTISE NOW 6

Factorise each of the following expressions completely.

(a) $x^2 + 8x + 7$ **(b)** $x^2 - 11x + 28$

(c) $x^2 + x - 2$ **(d)** $x^2 - 7x - 8$

SIMILAR QUESTIONS

Exercise 3B Questions 1(a)–(h), 4

Worked Example 7

Factorising Quadratic Expressions of the Form $ax^2 + bx + c$

Factorise each of the following expressions completely.

(a) $2x^2 + 7x + 3$ **(b)** $3x^2 + 7x - 6$

(c) $-x^2 - 4x + 32$ **(d)** $4x^2 - 6x - 4$

Solution:

(a) $2x^2 = 2x \times x$

$3 = 1 \times 3$ or $(-1) \times (-3)$

$\times$	x	3
$2x$	$2x^2$	$6x$
1	x	3

$x + 6x = 7x$

1 and 3 are selected as $x + 6x = 7x$.

$\therefore 2x^2 + 7x + 3 = (2x + 1)(x + 3)$

INFORMATION

Not all quadratic expressions can be factorised using the multiplication frame, e.g. $x^2 + 2x - 1$.

(b) $3x^2 = 3x \times x$

$-6 = 1 \times (-6)$ or $(-1) \times 6$

$= 2 \times (-3)$ or $(-2) \times 3$

$\times$	x	3
$3x$	$3x^2$	$9x$
-2	$-2x$	-6

$(-2x) + 9x = 7x$

-2 and 3 are selected as $(-2x) + 9x = 7x$.

$\therefore 3x^2 + 7x - 6 = (3x - 2)(x + 3)$

(c) $-x^2 = -x \times x$

$32 = 1 \times 32$ or $(-1) \times (-32)$

$= 2 \times 16$ or $(-2) \times (-16)$

$= 4 \times 8$ or $(-4) \times (-8)$

$\times$	x	8
$-x$	$-x^2$	$-8x$
4	$4x$	32

$4x + (-8x) = -4x$

4 and 8 are selected as $4x + (-8x) = -4x$.

$\therefore -x^2 - 4x + 32 = (-x + 4)(x + 8)$

(d) $4x^2 - 6x - 4 = 2(2x^2 - 3x - 2)$ (extract the common factor 2)

$2x^2 = 2x \times x$

$-2 = 1 \times (-2)$ or $(-1) \times 2$

$\times$	x	-2
$2x$	$2x^2$	$-4x$
1	x	-2

$x + (-4x) = -3x$

1 and -2 are selected as $x + (-4x) = -3x$.

$\therefore 4x^2 - 6x - 4 = 2(2x + 1)(x - 2)$

PRACTISE NOW 7

Factorise each of the following expressions completely.

(a) $2x^2 + 11x + 12$ (b) $5x^2 - 13x + 6$

(c) $-2x^2 + 9x - 9$ (d) $9x^2 - 33x + 24$

SIMILAR QUESTIONS

Exercise 3B Questions 2(a)–(h), 3(a)–(h), 5(a)–(b)

Exercise 3B

BASIC LEVEL

1. Factorise each of the following expressions completely.

(a) $a^2 + 9a + 8$ (b) $b^2 + 8b + 15$

(c) $c^2 - 9c + 20$ (d) $d^2 - 16d + 28$

(e) $f^2 + 6f - 16$ (f) $h^2 + 2h - 120$

(g) $k^2 - 4k - 12$ (h) $m^2 - 20m - 21$

2. Factorise each of the following expressions completely.

(a) $3n^2 + 10n + 7$ (b) $4p^2 + 8p + 3$

(c) $6q^2 - 17q + 12$ (d) $4r^2 - 7r + 3$

(e) $8s^2 + 2s - 15$ (f) $6t^2 + 19t - 20$

(g) $4u^2 - 8u - 21$ (h) $18w^2 - w - 39$

INTERMEDIATE LEVEL

3. Factorise each of the following expressions completely.

(a) $-a^2 + 2a + 35$ (b) $-3b^2 + 76b - 25$

(c) $4c^2 + 10c + 4$ (d) $5d^2 - 145d + 600$

(e) $8f^2 + 4f - 60$ (f) $24h^2 - 15h - 9$

(g) $30 + 14k - 4k^2$ (h) $35m^2n + 5mn - 30n$

4. The area of a rectangle is $(x^2 + 8x + 12)$ cm². If the length of the rectangle is $(x + 6)$ cm, show that its breadth is $(x + 2)$ cm.

ADVANCED LEVEL

5. Factorise each of the following expressions completely.

(a) $\frac{4}{9}p^2 + p - 1$ (b) $0.6r - 0.8qr - 12.8q^2r$

1. The general form of a **quadratic expression** in one variable is $ax^2 + bx + c$, where x is the variable, a, b and c are constants and $a \neq 0$.

2. **Expansion of Quadratic Expressions:** $(a + b)(c + d) = a(c + d) + b(c + d)$

$$= ac + ad + bc + bd$$

3. We can use a **multiplication frame** to **factorise** quadratic expressions.
For example, consider the expression $x^2 - 2x - 8$.

×		
	x^2	
		-8

→

×	x	-4
x	x^2	
2		-8

→

×	x	-4
x	x^2	$-4x$
2	$2x$	-8

$2x + (-4x) = -2x$

Step 1 **Step 2** **Step 3**

Step 1: Write x^2 in the top-left corner and -8 in the bottom-right corner of the multiplication frame.

Step 2: Consider the factors of x^2 and -8. Write them in the first column and the first row.

Step 3: Multiply them to complete the multiplication frame and check whether the result matches the given expression.

Therefore, $x^2 - 2x - 8 = (x + 2)(x - 4)$.

Review Exercise 3

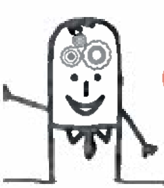
1. Expand each of the following expressions.
   (a) $10a(2a - 7)$ (b) $-3b(7 - 4b)$
   (c) $(c - 4)(c - 11)$ (d) $(3d - 5)(4 - d)$

2. Expand and simplify each of the following expressions.
   (a) $7f(3f - 4) + 4f(3 - 2f)$ (b) $6h^2 + (2h + 3)(h - 1)$
   (c) $(2k - 1)(k - 4) - 3k(k - 7)$ (d) $(m + 2)(m + 1) - (3m + 5)(9 - 5m)$

3. Factorise each of the following expressions completely.
   (a) $a^2 + 13a + 36$ (b) $b^2 - 15b + 56$
   (c) $c^2 + 14c - 51$ (d) $d^2 - 12d - 45$

4. Factorise each of the following expressions completely.
   (a) $9f^2 + 18f - 16$ (b) $3h^2 - 19h - 14$
   (c) $14k^2 + 49k + 21$ (d) $18m^2 - 39m + 18$

5. Factorise the expression $3x^2 - \frac{11}{2}x - 5$ completely.



Challenge Yourself

Determine the integer values of n for which $n^2 - 18n + 45$ is a prime number.

Further Expansion and Factorisation of Algebraic Expressions

Various algebraic properties can be used to quickly simplify and evaluate complicated numerical expressions. What are some examples of such algebraic properties?

Chapter Four

ARNING OBJECTIVES

ne end of this chapter, you should be able to:

xpand and simplify algebraic expressions,

se a multiplication frame to factorise algebraic expressions,

ecognise and apply the three special algebraic identities to
xpand and factorise algebraic expressions,

ctorise algebraic expressions by grouping.

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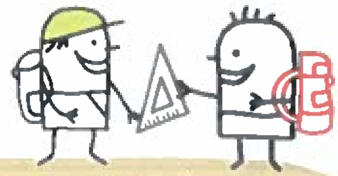
4.1 Expansion and Factorisation of Algebraic Expressions



Product of Algebraic Terms in More Than One Variable

In Chapter 3, we have learnt that when we multiply an algebraic term by another algebraic term containing the same variable, the coefficients and the variable are multiplied separately, e.g. $2x \times 3x = (2 \times 3) \times (x \times x) = 6x^2$.

In general, when we multiply an algebraic term by another algebraic term, the coefficients and the variables are multiplied separately, e.g. $2x \times 3y = (2 \times 3) \times (x \times y) = 6xy$.

Worked Example 1

(Finding the Product of Algebraic Terms involving More Than One Variable)

Find each of the following products.

(a) $4x \times (-7y)$ **(b)** $2xy^2 \times 3xy^2$

Solution:

(a) $4x \times (-7y) = [4 \times (-7)] \times (x \times y)$ (rearrange the coefficients and the variables accordingly)

$= -28xy$

(b) $2xy^2 \times 3xy^2 = (2 \times x \times y \times y) \times (3 \times x \times y \times y)$

$= (2 \times 3) \times (x \times x) \times (y \times y \times y \times y)$ (rearrange the coefficients and the variables accordingly)

$= 6x^2y^4$ (express in index notation)

ATTENTION

- $x^2 = x \times x$
- $x^3 = x \times x \times x$
- $x^4 = x \times x \times x \times x$
- $x^5 = x \times x \times x \times x \times x$

PRACTISE NOW 1

1. Find each of the following products.

   (a) $5x \times 6y$ **(b)** $(-8x) \times 2y$

   (c) $x^2yz \times y^2z$ **(d)** $(-xy) \times (-11x^3y^2)$

2. Find the product of $\frac{1}{2}a$ and $-\frac{8}{3}b$.

SIMILAR QUESTIONS

Exercise 4A Questions 1(a)–(b), 5(a)–(d), 12

Expansion and Simplification of Simple Algebraic Expressions

In Book 1 and in Chapter 3, we have learnt the Distributive Law:

$$a(b + c) = ab + ac,$$

where 'a groups of b and c' is the same as 'a groups of b' and 'a groups of c', i.e. 'a times of b' and 'a times of c'.

Worked Example 2

(Expanding Algebraic Expressions of the Form $a(b + c)$)

Expand each of the following expressions.

(a) $x(y + 3)$ **(b)** $-2x(6x - 5y)$

Solution:

(a) $x(y + 3) = xy + 3x$ (Distributive Law; 'x groups of $(y + 3)$')

(b) $-2x(6x - 5y) = -12x^2 + 10xy$ (Distributive Law)

PRACTISE NOW 2

Expand each of the following expressions.

(a) $-y(5 - 2x)$ **(b)** $2x(7x + 3y)$

SIMILAR QUESTIONS

Exercise 4A Questions 2(a)–(j), 6(a)–(d)

Worked Example 3

(Expanding and Simplifying Algebraic Expressions of the Form $a(b + c)$)

Expand and simplify each of the following expressions.

(a) $-3x(y + z) - 4x(2y - 5z)$ **(b)** $2x(x + 2y) + 3x(2x - 3y)$

Solution:

(a) $-3x(y + z) - 4x(2y - 5z) = -3xy - 3xz - 8xy + 20xz$ (Distributive Law)

$= -3xy - 8xy - 3xz + 20xz$ (group like terms)

$= -11xy + 17xz$

(b) $2x(x + 2y) + 3x(2x - 3y) = 2x^2 + 4xy + 6x^2 - 9xy$ (Distributive Law)

$= 2x^2 + 6x^2 + 4xy - 9xy$ (group like terms)

$= 8x^2 - 5xy$

PRACTISE NOW 3

Expand and simplify each of the following expressions.

(a) $4x(3y - 5z) - 5x(2y - 3z)$ **(b)** $x(2x - y) + 3x(y - 3x)$

SIMILAR QUESTIONS

Exercise 4A Questions 3(a)–(b), 7(a)–(d)

Further Expansion and Simplification of Algebraic Expressions

In Chapter 3, we have learnt that:

$$(a + b)(c + d) = a(c + d) + b(c + d)$$
$$= ac + ad + bc + bd$$

Worked Example 4

(Expanding Algebraic Expressions of the Form $(a + b)(c + d)$)

Expand each of the following expressions.

(a) $(x - 2y)(x + 5y)$ **(b)** $(2x^2 - 1)(5x - 3)$

Solution:

(a) $(x - 2y)(x + 5y) = x(x + 5y) - 2y(x + 5y)$ (multiply the expression in the second bracket by each term in the first bracket)

$= x^2 + 5xy - 2xy - 10y^2$ (Distributive Law)

$= x^2 + 3xy - 10y^2$

(b) $(2x^2 - 1)(5x - 3) = 2x^2(5x - 3) \boxed{- (5x - 3)}$ (multiply the expression in the second bracket by each term in the first bracket)

$= 10x^3 - 6x^2 - 5x + 3$ (Distributive Law; simplify to get negative of expression)

PRACTISE NOW 4

Expand each of the following expressions.

(a) $(x + 9y)(2x - y)$ **(b)** $(x^2 - 3)(6x + 7)$

SIMILAR QUESTIONS

Exercise 4A Questions 4(a)–(b), 8(a)–(d)

Worked Example 5

(Expanding and Simplifying Algebraic Expressions of the Form $(a + b)(c + d)$)

Expand and simplify the expression

$(x - 2y)(2x + y) - (3x + y)(5x - 4y)$.

Solution:

$(x - 2y)(2x + y) - (3x + y)(5x - 4y) = x(2x + y) - 2y(2x + y) - [3x(5x - 4y) + y(5x - 4y)]$

(multiply the expression in the second bracket by each term in the first bracket; do not forget to include the square brackets for the expansion of $(3x + y)(5x - 4y)$)

$= 2x^2 + xy - 4xy - 2y^2 - (15x^2 - 12xy + 5xy - 4y^2)$

(Distributive Law)

$= 2x^2 - 3xy - 2y^2 \boxed{- (15x^2 - 7xy - 4y^2)}$

$= 2x^2 - 3xy - 2y^2 - 15x^2 + 7xy + 4y^2$

(simplify to get negative of expression)

$= 2x^2 - 15x^2 - 3xy + 7xy - 2y^2 + 4y^2$ (group like terms)

$= -13x^2 + 4xy + 2y^2$

PRACTISE NOW 5

Expand and simplify the expression $2x(3x - 4y) - (x - y)(x + 3y)$.

SIMILAR QUESTIONS

Exercise 4A Questions 9(a)–(b), 13(a)–(b)

Thinking Time

What do we get when we expand $(a + b)(c + d + e)$?

From the thinking time, we have:

$$(a + b)(c + d + e) = a(c + d + e) + b(c + d + e)$$
$$= ac + ad + ae + bc + bd + be$$

Worked Example 6

(Expanding Algebraic Expressions of the Form $(a + b)(c + d + e)$)

Expand each of the following expressions.

(a) $(2x - y)(3x + 2y + 1)$ **(b)** $(5x - 4)(2x^2 + 3x + 2)$

Solution:

(a) $(2x - y)(3x + 2y + 1) = 2x(3x + 2y + 1) - y(3x + 2y + 1)$

(multiply the expression in the second bracket by each term in the first bracket)

$= 6x^2 + 4xy + 2x - 3xy - 2y^2 - y$ (Distributive Law)

$= 6x^2 + 4xy - 3xy + 2x - 2y^2 - y$ (group like terms)

$= 6x^2 + xy + 2x - 2y^2 - y$

(b) $(5x - 4)(2x^2 + 3x + 2) = 5x(2x^2 + 3x + 2) - 4(2x^2 + 3x + 2)$

(multiply the expression in the second bracket by each term in the first bracket)

$= 10x^3 + 15x^2 + 10x - 8x^2 - 12x - 8$ (Distributive Law)

$= 10x^3 + 15x^2 - 8x^2 + 10x - 12x - 8$ (group like terms)

$= 10x^3 + 7x^2 - 2x - 8$

PRACTISE NOW 6

Expand each of the following expressions.

(a) $(x - 5y)(x + 4y - 1)$ **(b)** $(x + 3)(x^2 - 7x - 2)$

SIMILAR QUESTIONS

Exercise 4A Questions 10(a)–(b), 14(a)–(d)

Factorisation of Algebraic Expressions Using a Multiplication Frame

In Chapter 3, we have learnt how to use a multiplication frame to factorise quadratic expressions. Worked Example 7 illustrates how we are able to use the multiplication frame to factorise other algebraic expressions.

Worked Example 7

(Factorising Algebraic Expressions of the Form $ax^2 + bxy + cy^2$)

Factorise each of the following expressions completely.

(a) $x^2 + 2xy - 8y^2$ **(b)** $2x^2 - 7xy + 6y^2$

Solution:

(a) $x^2 = x \times x$

$-8y^2 = y \times (-8y)$ or $(-y) \times 8y$

$= 2y \times (-4y)$ or $(-2y) \times 4y$

$\times$	x	$4y$
x	x^2	$4xy$
$-2y$	$-2xy$	$-8y^2$

$(-2xy) + 4xy = 2xy$

$\therefore x^2 + 2xy - 8y^2 = (x - 2y)(x + 4y)$

(b) $2x^2 = 2x \times x$

$6y^2 = y \times 6y$ or $(-y) \times (-6y)$

$= 2y \times 3y$ or $(-2y) \times (-3y)$

$\times$	x	$-2y$
$2x$	$2x^2$	$-4xy$
$-3y$	$-3xy$	$6y^2$

$(-3xy) + (-4xy) = -7xy$

$\therefore 2x^2 - 7xy + 6y^2 = (2x - 3y)(x - 2y)$

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 4A Questions 11(a)–(f), 15(a)–(d)

1. Factorise each of the following expressions completely.

(a) $x^2 - 2xy - 15y^2$ **(b)** $6x^2 + 11xy + 5y^2$

2. Factorise the expression $3x^2y^2 - 14xy + 16$ completely.

Exercise 4A

BASIC LEVEL

1. Find each of the following products.
   (a) $6x \times (-2y)$
   (b) $14x \times \frac{1}{2}y$

2. Expand each of the following expressions.
   (a) $8x(y - 1)$
   (b) $-9x(3y - 2z)$
   (c) $3x(2x + 7y)$
   (d) $3y(x - 11y)$
   (e) $-3a(2a + 3b)$
   (f) $-4c(2c - 5d)$
   (g) $-6h(7k - 3h)$
   (h) $-8m(-12m - 7n)$
   (i) $2p(3p + q + 7r)$
   (j) $-7s(s - 4t - 3u)$

3. Expand and simplify each of the following expressions.
   (a) $7a(3b - 4c) + 4a(3c - 2b)$
   (b) $4d(d - 5f) + 2f(3d + 7f)$

4. Expand each of the following expressions.
   (a) $(x + y)(x + 6y)$
   (b) $(x^2 + 2)(x + 5)$

INTERMEDIATE LEVEL

5. Find each of the following products.
   (a) $-\frac{3}{7}x \times \frac{14}{9}y$
   (b) $9x^3y \times 3x^2y^2$
   (c) $2x^3y \times (-13y^2)$
   (d) $(-4xyz) \times (-2x^2y^3z^4)$

6. Expand each of the following expressions.
   (a) $-3xy(x - 2y)$
   (b) $9x(-3x^2y - 7xz)$
   (c) $-13x^2y(3x - y)$
   (d) $-5x(-6x - 4x^3y - 3y)$

7. Expand and simplify each of the following expressions.
   (a) $a(5b + c) - 2a(3c - b)$
   (b) $-2d(4f - 5h) - f(3d + 7h)$
   (c) $4k(3k + m) - 3k(2k - 5m)$
   (d) $2n(p - 2n) - 4n(n - 2p)$

8. Expand each of the following expressions.
   (a) $(a + 3b)(a - b)$
   (b) $(3c + 7d)(c - 2d)$
   (c) $(3k - 5h)(-h - 7k)$
   (d) $(7m^2 + 2)(m - 4)$

9. Expand and simplify each of the following expressions.
   (a) $5x(x - 6y) + (x + 3y)(3x - 4y)$
   (b) $(7x - 3y)(x - 4y) + (5x - 9y)(y - 2x)$

10. Expand each of the following expressions.
   (a) $(x + 9y)(x + 3y + 1)$
   (b) $(x + 2)(x^2 + x + 1)$

11. Factorise each of the following expressions completely.
   (a) $a^2 + 3ab - 4b^2$
   (b) $c^2 - 4cd - 21d^2$
   (c) $2h^2 + 7hk - 15k^2$
   (d) $3m^2 - 16mn - 12n^2$
   (e) $3p^2 + 15pq + 18q^2$
   (f) $2r^2t - 9rst + 10s^2t$

ADVANCED LEVEL

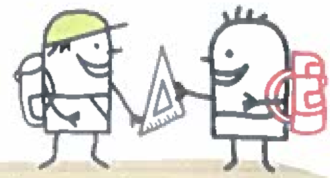
12. Find the product of $\frac{1}{4}x^2y$ and $\frac{16}{5}yz^3$.

13. Expand and simplify each of the following expressions.
   (a) $(8x - y)(x + 3y) - (4x + y)(9y - 2x)$
   (b) $(10x + y)(3x + 2y) - (5x - 4y)(-x - 6y)$

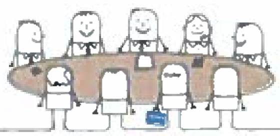
14. Expand each of the following expressions.
   (a) $(2x - 3y)(x + 5y - 2)$
   (b) $(x + 4)(x^2 - 5x + 7)$
   (c) $(x - 1)(x^2 + 2x - 1)$
   (d) $(3x^2 - 3x + 4)(3 - x)$

15. Factorise each of the following expressions completely.
   (a) $x^2y^2 + 2xy - 15$
   (b) $12x^2y^2 - 17xy - 40$
   (c) $4x^2y^2z - 22xyz + 24z$
   (d) $2x^2 + \frac{5}{3}xy - 2y^2$

4.2 Expansion Using Special Algebraic Identities



In this section, we shall learn how to expand certain algebraic expressions using three special algebraic identities.



Class Discussion

Special Algebraic Identities

Copy and complete each of the following.

1. $(a+b)^2 = (a+b)(a+b)$
$= a(a+b) + b(a+b)$
$= a^2 + ab + ab + b^2$
= ______________

2. $(a-b)^2 = (a-b)(a-b)$
$= a(a-b) - b(a-b)$
= ______________
= ______________

3. $(a+b)(a-b) = a(a-b) + b(a-b)$
= ______________
= ______________

4. Verify your answers for Questions 1, 2 and 3 with your classmate.

From the class discussion, we have:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)(a-b) = a^2 - b^2$$

These algebraic identities are useful for *expanding* algebraic expressions which are of similar forms. $(a+b)^2$ and $(a-b)^2$ are known as **perfect squares** while $(a+b)(a-b)$ is called the **difference of two squares**.

Worked Example 8

(Expanding Algebraic Expressions of the Form $(a + b)^2$)

Expand each of the following expressions.

(a) $(x + 3)^2$ **(b)** $(2x + 7)^2$

Solution:

(a) $(x + 3)^2 = x^2 + 2(x)(3) + 3^2$ (apply $(a + b)^2 = a^2 + 2ab + b^2$, where $a = x$ and $b = 3$)

$= x^2 + 6x + 9$

(b) $(2x + 7)^2 = (2x)^2 + 2(2x)(7) + 7^2$ (apply $(a + b)^2 = a^2 + 2ab + b^2$, where $a = 2x$ and $b = 7$)

$= 4x^2 + 28x + 49$

ATTENTION

$(2x)^2 = 2x \times 2x$
$= 4x^2$
$(2x)^2 \neq 2x^2$

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 4B Questions 1(a)–(d), 7(a)–(b)

1. Expand each of the following expressions.
   (a) $(x + 2)^2$ **(b)** $(5x + 3)^2$
2. Expand the expression $\left(\frac{1}{2}x + 3\right)^2$.

Worked Example 9

(Expanding Algebraic Expressions of the Form $(a - b)^2$)

Expand each of the following expressions.

(a) $(6 - x)^2$ **(b)** $(4x - 3y)^2$

Solution:

(a) $(6 - x)^2 = 6^2 - 2(6)(x) + x^2$ (apply $(a - b)^2 = a^2 - 2ab + b^2$, where $a = 6$ and $b = x$)

$= 36 - 12x + x^2$

(b) $(4x - 3y)^2 = (4x)^2 - 2(4x)(3y) + (3y)^2$ (apply $(a - b)^2 = a^2 - 2ab + b^2$, where $a = 4x$ and $b = 3y$)

$= 16x^2 - 24xy + 9y^2$

ATTENTION

- $(4x)^2 = 4x \times 4x$
  $= 16x^2$
  $(4x)^2 \neq 4x^2$
- $(3y)^2 = 3y \times 3y$
  $= 9y^2$
  $(3y)^2 \neq 3y^2$

PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 4B Questions 2(a)–(d), 8(a)–(b)

1. Expand each of the following expressions.
   (a) $(1 - 3x)^2$ **(b)** $(2x - 3y)^2$
2. Expand the expression $\left(x - \frac{1}{3}y\right)^2$.

Worked Example 10

(Expanding Algebraic Expressions of the Form $(a + b)(a - b)$)

Expand each of the following expressions.

(a) $(x + 2)(x - 2)$ **(b)** $(3x + 2y)(3x - 2y)$

Solution:

(a) $(x + 2)(x - 2) = x^2 - 2^2$ (apply $(a + b)(a - b) = a^2 - b^2$, where $a = x$ and $b = 2$)

$= x^2 - 4$

(b) $(3x + 2y)(3x - 2y) = (3x)^2 - (2y)^2$ (apply $(a + b)(a - b) = a^2 - b^2$, where $a = 3x$ and $b = 2y$)

$= 9x^2 - 4y^2$

PRACTISE NOW 10

SIMILAR QUESTIONS

Exercise 4B Questions 3(a)–(d), 9(a)–(d), 10(a)–(b), 13

1. Expand each of the following expressions.

(a) $(5x + 8)(5x - 8)$ **(b)** $(-2x + 7y)(-2x - 7y)$

2. Expand the expression $\left(\frac{x}{4} + y\right)\left(\frac{x}{4} - y\right)$.

Worked Example 11

(Problem involving the Use of Special Algebraic Identities)

Without using a calculator, evaluate each of the following.

(a) 102^2 **(b)** 497^2 **(c)** 201×199

Solution:

(a) $102^2 = (100 + 2)^2$

$= 100^2 + 2(100)(2) + 2^2$ (apply $(a + b)^2 = a^2 + 2ab + b^2$, where $a = 100$ and $b = 2$)

$= 10\,000 + 400 + 4$

$= 10\,404$

(b) $497^2 = (500 - 3)^2$

$= 500^2 - 2(500)(3) + 3^2$ (apply $(a - b)^2 = a^2 - 2ab + b^2$, where $a = 500$ and $b = 3$)

$= 250\,000 - 3000 + 9$

$= 247\,009$

(c) $201 \times 199 = (200 + 1)(200 - 1)$

$= 200^2 - 1^2$ (apply $(a + b)(a - b) = a^2 - b^2$, where $a = 200$ and $b = 1$)

$= 40\,000 - 1$

$= 39\,999$

PRACTISE NOW 11

SIMILAR QUESTIONS

Exercise 4B Questions 4(a)–(c), 14–15

Without using a calculator, evaluate each of the following.

(a) 1001^2 **(b)** 797^2 **(c)** 305×295

Worked Example 12

(Problem involving the Use of a Special Algebraic Identity)

If $(x + y)^2 = 361$ and $xy = -120$, find the value of $x^2 + y^2$.

Solution:

$(x + y)^2 = 361$

$x^2 + 2xy + y^2 = 361$ (apply $(a + b)^2 = a^2 + 2ab + b^2$, where $a = x$ and $b = y$)

Since $xy = -120$,

$\therefore x^2 + 2(-120) + y^2 = 361$

$x^2 - 240 + y^2 = 361$

$\therefore x^2 + y^2 = 601$

PRACTISE NOW 12

If $(x - y)^2 = 441$ and $xy = 46$, find the value of $x^2 + y^2$.

SIMILAR QUESTIONS

Exercise 4B Questions 5–6, 11–12

Exercise 4B

BASIC LEVEL

1. Expand each of the following expressions.

(a) $(a + 4)^2$ **(b)** $(3b + 2)^2$

(c) $(c + 4d)^2$ **(d)** $(9h + 2k)^2$

2. Expand each of the following expressions.

(a) $(m - 9)^2$ **(b)** $(5n - 4)^2$

(c) $(9 - 5p)^2$ **(d)** $(3q - 8r)^2$

3. Expand each of the following expressions.

(a) $(s - 5)(s + 5)$ **(b)** $(2t + 11)(2t - 11)$

(c) $(7 + 2u)(7 - 2u)$ **(d)** $(w - 10x)(w + 10x)$

4. Without using a calculator, evaluate each of the following.

(a) 1203^2 **(b)** 892^2

(c) 1998×2002

5. If $x^2 + y^2 = 80$ and $xy = 12$, find the value of $(x - y)^2$.

6. If $x + y = 10$ and $x - y = 4$, find the value of $x^2 - y^2$.

INTERMEDIATE LEVEL

7. Expand each of the following expressions.

(a) $\left(\frac{1}{5}a + 3b\right)^2$ **(b)** $\left(\frac{1}{2}c + \frac{2}{3}d\right)^2$

8. Expand each of the following expressions.

(a) $\left(\frac{3}{2}h - 5k\right)^2$ **(b)** $\left(-\frac{6}{5}m - 3n\right)^2$

9. Expand each of the following expressions.

(a) $(6p + 5)(5 - 6p)$ **(b)** $\left(9r - \frac{4}{5}q\right)\left(9r + \frac{4}{5}q\right)$

(c) $\left(\frac{s}{2} + \frac{t}{3}\right)\left(\frac{t}{3} - \frac{s}{2}\right)$ **(d)** $(u + 2)(u - 2)(u^2 + 4)$

10. Expand and simplify each of the following expressions.

(a) $4(x + 3)^2 - 3(x + 4)(x - 4)$

(b) $(5x - 7y)(5x + 7y) - 2(x - 2y)^2$

11. If $x^2 + y^2 = 14$ and $xy = 5$, find the value of $\left(\frac{1}{2}x + \frac{1}{2}y\right)^2$.

12. If $2x^2 - 2y^2 = 125$ and $x - y = 2.5$, find the value of $x + y$.

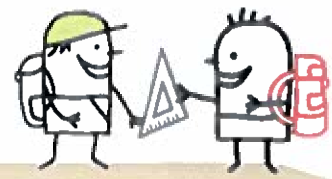
ADVANCED LEVEL

13. Expand the expression
$\left(\frac{1}{16}x^2 + \frac{1}{25}y^2\right)\left(\frac{1}{4}x + \frac{1}{5}y\right)\left(\frac{1}{4}x - \frac{1}{5}y\right)$.

14. (i) Expand and simplify the expression
$(p - 2q)^2 - p(p - 4q)$.
(ii) Hence, by substituting a suitable value of p and of q, find the value of $5310^2 - 5330 \times 5290$.

15. (i) Expand and simplify the expression
$n^2 - (n - a)(n + a)$.
(ii) Hence, by substituting a suitable value of n and of a, find the value of
$16\,947^2 - 16\,944 \times 16\,950$.

4.3 Factorisation Using Special Algebraic Identities



In Section 4.2, we have learnt how to expand certain algebraic expressions using the three special algebraic identities:

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(a + b)(a - b) = a^2 - b^2$$

Since factorisation is the reverse of expansion, we have:

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$
$$a^2 - b^2 = (a + b)(a - b)$$

We can *factorise* an expression using these special algebraic identities if the expression can be expressed in one of the forms on the left-hand side of the equations.

Worked Example 13

Factorising Algebraic Expressions of the Form $a^2 + 2ab + b^2$

Factorise each of the following expressions completely.

(a) $x^2 + 18x + 81$ **(b)** $9x^2 + 24x + 16$

Solution:

(a) $x^2 + 18x + 81 = x^2 + 2(x)(9) + 9^2$
$= (x + 9)^2$ (apply $a^2 + 2ab + b^2 = (a + b)^2$, where $a = x$ and $b = 9$)

(b) $9x^2 + 24x + 16 = (3x)^2 + 2(3x)(4) + 4^2$
$= (3x + 4)^2$ (apply $a^2 + 2ab + b^2 = (a + b)^2$, where $a = 3x$ and $b = 4$)

RECALL

$0^2 = 0$
$1^2 = 1$
$2^2 = 4$
$3^2 = 9$
$4^2 = 16$
$5^2 = 25$
⋮
0, 1, 4, 9, 16, 25, … are called **perfect squares**.

PRACTISE NOW 13

SIMILAR QUESTIONS

Exercise 4C Questions 1(a)–(d), 5(a)–(d), 9

1. Factorise each of the following expressions completely.
   (a) $x^2 + 12x + 36$ (b) $4x^2 + 20x + 25$

2. Factorise the expression $4x^2 + 2x + \frac{1}{4}$ completely.

Worked Example 14

(Factorising Algebraic Expressions of the Form $a^2 - 2ab + b^2$)

Factorise each of the following expressions completely.

(a) $49 - 84x + 36x^2$ (b) $9x^2 - 30xy + 25y^2$

Solution:

(a) $49 - 84x + 36x^2 = 7^2 - 2(7)(6x) + (6x)^2$
$= (7 - 6x)^2$ (apply $a^2 - 2ab + b^2 = (a - b)^2$, where $a = 7$ and $b = 6x$)

(b) $9x^2 - 30xy + 25y^2 = (3x)^2 - 2(3x)(5y) + (5y)^2$
$= (3x - 5y)^2$ (apply $a^2 - 2ab + b^2 = (a - b)^2$, where $a = 3x$ and $b = 5y$)

PRACTISE NOW 14

SIMILAR QUESTIONS

Exercise 4C Questions 2(a)–(d), 6(a)–(d)

1. Factorise each of the following expressions completely.
   (a) $4 - 36x + 81x^2$ (b) $25x^2 - 10xy + y^2$

2. Factorise the expression $36x^2 - 4xy + \frac{1}{9}y^2$ completely.

Worked Example 15

(Factorising Algebraic Expressions of the Form $a^2 - b^2$)

Factorise each of the following expressions completely.

(a) $4x^2 - 25y^2$ (b) $8x^2 - 18y^2$

Solution:

(a) $4x^2 - 25y^2 = (2x)^2 - (5y)^2$
$= (2x + 5y)(2x - 5y)$ (apply $a^2 - b^2 = (a + b)(a - b)$, where $a = 2x$ and $b = 5y$)

(b) $8x^2 - 18y^2 = 2(4x^2 - 9y^2)$ (extract the common factor 2)
$= 2[(2x)^2 - (3y)^2]$
$= 2(2x + 3y)(2x - 3y)$ (apply $a^2 - b^2 = (a + b)(a - b)$, where $a = 2x$ and $b = 3y$)

PRACTISE NOW 15

SIMILAR QUESTIONS

Exercise 4C Questions 3(a)–(d), 7(a)–(d), 8(a)–(f), 10(a)–(d)

1. Factorise each of the following expressions completely.
   (a) $36x^2 - 121y^2$ (b) $-4x^2 + 81$

2. Factorise the expression $4x^2 - \frac{9}{25}y^2$ completely.

3. Factorise the expression $4(x + 1)^2 - 49$ completely.

Worked Example 16

(Problem involving the Use of the Special Algebraic Identity $a^2 - b^2 = (a + b)(a - b)$)

Without using a calculator, evaluate $103^2 - 9$.

Solution:

$103^2 - 9 = 103^2 - 3^2$

$= (103 + 3)(103 - 3)$ (apply $a^2 - b^2 = (a + b)(a - b)$, where $a = 103$ and $b = 3$)

$= 106 \times 100$

$= 10\ 600$

PRACTISE NOW 16

Without using a calculator, evaluate $256^2 - 156^2$.

SIMILAR QUESTIONS

Exercise 4C Questions 4(a)–(b)

Exercise 4C

BASIC LEVEL

1. Factorise each of the following expressions completely.

(a) $a^2 + 14a + 49$ **(b)** $4b^2 + 4b + 1$

(c) $c^2 + 2cd + d^2$ **(d)** $4h^2 + 20hk + 25k^2$

2. Factorise each of the following expressions completely.

(a) $m^2 - 10m + 25$ **(b)** $169n^2 - 52n + 4$

(c) $81 - 180p + 100p^2$ **(d)** $49q^2 - 42qr + 9r^2$

3. Factorise each of the following expressions completely.

(a) $s^2 - 144$ **(b)** $36t^2 - 25$

(c) $225 - 49u^2$ **(d)** $49w^2 - 81x^2$

4. Without using a calculator, evaluate each of the following.

(a) $59^2 - 41^2$ **(b)** $7.7^2 - 2.3^2$

INTERMEDIATE LEVEL

5. Factorise each of the following expressions completely.

(a) $3a^2 + 12a + 12$ **(b)** $25b^2 + 5bc + \frac{1}{4}c^2$

(c) $\frac{16}{49}d^2 + \frac{8}{35}df + \frac{1}{25}f^2$ **(d)** $h^4 + 2h^2k + k^2$

6. Factorise each of the following expressions completely.

(a) $36m^2 - 48mn + 16n^2$ **(b)** $\frac{1}{3}p^2 - \frac{2}{3}pq + \frac{1}{3}q^2$

(c) $16r^2 - rs + \frac{1}{64}s^2$ **(d)** $25 - 10tu + t^2u^2$

7. Factorise each of the following expressions completely.

(a) $32a^2 - 98b^2$ **(b)** $c^2 - \frac{1}{4}d^2$

(c) $\frac{9h^2}{100} - 16k^2$ **(d)** $m^2 - 64n^4$

8. Factorise each of the following expressions completely.

(a) $(a + 3)^2 - 9$ **(b)** $16 - 25(b + 3)^2$

(c) $c^2 - (d + 2)^2$ **(d)** $(2h - 1)^2 - 4k^2$

(e) $25m^2 - (n - 1)^2$ **(f)** $(p + 1)^2 - (p - 1)^2$

9. The surface area of each face of a cube is $(x^2 + 4x + 4)$ cm². Find

(i) the length,

(ii) the volume,

of the cube.

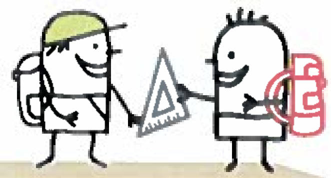
ADVANCED LEVEL

10. Factorise each of the following expressions completely.

(a) $4(x-1)^2 - 81(x+1)^2$ (b) $16x^2 + 8x + 1 - 9y^2$

(c) $4x^2 - y^2 + 4y - 4$ (d) $13x^2 + 26xy + 13y^2 - 13$

4.4 Factorisation by Grouping



Factorisation of Algebraic Expressions of the Form $ax + ay$

In Book 1, we have learnt that to factorise algebraic expressions of the form $ax + ay$, we will need to identify the common factors, i.e. common numbers or common variables of the terms. For example, to factorise $4ay - 24az$ completely, we extract the common factors 4 and a to get $4a(y - 6z)$.

Worked Example 17

(Factorising Algebraic Expressions of the Form $ax + ay$)

Factorise each of the following expressions completely.

(a) $3x^2 + 9xy$ (b) $2\pi r^2 + 2\pi rh$

(c) $a^2b - a^2b^2$ (d) $c^2d^3 + c^3d^2 - c^2d^2$

Solution:

(a) $3x^2 + 9xy = 3x(x + 3y)$

(b) $2\pi r^2 + 2\pi rh = 2\pi r(r + h)$

(c) $a^2b - a^2b^2 = a^2b(1 - b)$

(d) $c^2d^3 + c^3d^2 - c^2d^2 = c^2d^2(d + c - 1)$

ATTENTION

π is a constant.

PRACTISE NOW 17

Factorise each of the following expressions completely.

(a) $8x^2y + 4x$ (b) $\pi r^2 + \pi rl$

(c) $-a^3by + a^2y$ (d) $3c^2d + 6c^2d^2 + 3c^3$

SIMILAR QUESTIONS

Exercise 4D Questions 1(a)–(d), 7

Factorisation of Algebraic Expressions of the Form $ax + bx + kay + kby$

If we are given the expression $ax - bx + 2ay - 2by$, how do we factorise it completely? We shall now learn how to factorise algebraic expressions of the form $ax + bx + kay + kby$.

Sometimes, it is possible to identify the common factors by grouping the terms of an algebraic expression.

For example, $ax - ay + bx - by = (ax - ay) + (bx - by)$
$= a(x - y) + b(x - y)$
$= (x - y)(a + b)$ (extract the common factor $(x - y)$).

It may be necessary to regroup the terms of an algebraic expression before we are able to identify the common factors.

For example, $cx + dy + dx + cy = (cx + cy) + (dx + dy)$
$= c(x + y) + d(x + y)$
$= (x + y)(c + d)$.

Alternatively, $cx + dy + dx + cy = (cx + dx) + (cy + dy)$
$= x(c + d) + y(c + d)$
$= (x + y)(c + d)$.

We may need to change the sign of the factor in a group before we can factorise an algebraic expression.

For example, $h(x - y) + k(y - x) = h(x - y) - k(x - y)$
$= (x - y)(h - k)$.

ATTENTION

$y - x = -(x - y)$

Worked Example 18

Factorising Algebraic Expressions of the Form $a(x + y) + b(x + y)$

Factorise each of the following expressions completely.

(a) $a(2x + 3) + 2(3 + 2x)$ **(b)** $2b(5x + 2) - (5x + 2)$
(c) $3c(x - y) - 3d(x - y)$ **(d)** $h(x - 2) + k(2 - x)$

Solution:

(a) $a(2x + 3) + 2(3 + 2x) = (2x + 3)(a + 2)$
(b) $2b(5x + 2) - (5x + 2) = (5x + 2)(2b - 1)$
(c) $3c(x - y) - 3d(x - y) = 3[c(x - y) - d(x - y)]$
$= 3(x - y)(c - d)$
(d) $h(x - 2) + k(2 - x) = h(x - 2) - k(x - 2)$
$= (x - 2)(h - k)$

ATTENTION

- In **(a)**, $2x + 3 = 3 + 2x$.
- In **(b)**, $-(5x + 2) = (-1)(5x + 2)$.
- In **(d)**, $k(2 - x) = -k(x - 2)$.

PRACTISE NOW 18

Factorise each of the following expressions completely.

(a) $2(x + 1) + a(1 + x)$ **(b)** $9(x + 2) - b(x + 2)$
(c) $3c(2x - 3) - 6d(2x - 3)$ **(d)** $7h(4 - x) - (x - 4)$

SIMILAR QUESTIONS

Exercise 4D Questions 2(a)–(d), 4(a)–(d), 6(a)–(b)

Worked Example 19

Factorising Algebraic Expressions of the Form $ax + bx + kay + kby$

Factorise each of the following expressions completely.

(a) $ax - bx + 2ay - 2by$ (b) $6ax + 12by + 9bx + 8ay$

(c) $x^2 + xy - 3x - 3y$ (d) $6xy - 15y + 10 - 4x$

Solution:

(a) $ax - bx + 2ay - 2by = (ax - bx) + (2ay - 2by)$ (arrange the terms into two groups)

$= x(a - b) + 2y(a - b)$ (factorise each group)

$= (a - b)(x + 2y)$ (factorise the two groups)

(b) $6ax + 12by + 9bx + 8ay = (6ax + 9bx) + (12by + 8ay)$

(rearrange the terms into two groups)

$= 3x(2a + 3b) + 4y(3b + 2a)$ (factorise each group)

$= (2a + 3b)(3x + 4y)$ (factorise the two groups)

(c) $x^2 + xy - 3x - 3y = (x^2 + xy) - (3x + 3y)$ (arrange the terms into two groups)

$= x(x + y) - 3(x + y)$ (factorise each group)

$= (x + y)(x - 3)$ (factorise the two groups)

(d) $6xy - 15y + 10 - 4x = (6xy - 15y) + (10 - 4x)$ (arrange the terms into two groups)

$= 3y(2x - 5) + 2(5 - 2x)$ (factorise each group)

$= 3y(2x - 5) - 2(2x - 5)$

(change the sign of the factor in the second group)

$= (2x - 5)(3y - 2)$ (factorise the two groups)

PRACTISE NOW 19

Factorise each of the following expressions completely.

(a) $xy + 4x + 3y + 12$ (b) $3by + 4ax + 12ay + bx$

(c) $x^3 - x^2 - 1 + x$ (d) $6xy - 4x - 2z + 3yz$

SIMILAR QUESTIONS

Exercise 4D Questions 3(a)–(d), 5(a)–(h), 8

Thinking Time

How are we able to apply the concept of factorisation by grouping in factorising a quadratic expression such as $5x^2 - 12x - 9$?

Class Discussion

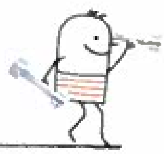
Equivalent Expressions

Work in pairs.

Some algebraic expressions, which consist of a few sets of equivalent expressions, are given in Table 4.1. An example of a set of equivalent expressions is $4ay - 24az$ and $4a(y - 6z)$ as $4ay - 24az = 4a(y - 6z)$. Match and justify each set of equivalent expressions. If your classmate does not obtain the correct answer, explain to him what he has done wrong.

A $(x - y)^2$	**B** $(x + y)(x + y)$	**C** $x^2 + y^2$	**D** $(2w - x)(z - 3y)$	**E** $-5x^2 + 28x - 24$
F $2wz - 6wy + 3xy - xz$	**G** $(x + y)^2$	**H** $(2w + x)(z - 3y)$	**I** $(x - y)(x - y)$	**J** $x^2 - y^2$
K $(x + y)(x - y)$	**L** $2x - (x - 4)(5x - 6)$	**M** $x^2 - 2xy + y^2$	**N** $-5x^2 - 24x + 24$	**O** $x^2 + 2xy + y^2$

Table 4.1



Exercise 4D

BASIC LEVEL

1. Factorise each of the following expressions completely.

(a) $45x^2 - 81xy$ (b) $39xy - 15x^2z$

(c) $xy^2z^2 - x^2y^3$ (d) $-15\pi x^3y - 10\pi x^3$

2. Factorise each of the following expressions completely.

(a) $6a(x - 2y) + 5(x - 2y)$

(b) $2b(x + 3y) - c(3y + x)$

(c) $3d(5x - y) - 4f(5x - y)$

(d) $5h(x + 3y) + 10k(x + 3y)$

3. Factorise each of the following expressions completely.

(a) $ax - 5a + 4x - 20$ (b) $ax + bx + ay + by$

(c) $x + xy + 2y + 2y^2$ (d) $x^2 - 3x + 2xy - 6y$

INTERMEDIATE LEVEL

4. Factorise each of the following expressions completely.

(a) $(x + y)(a + b) - (y + z)(a + b)$

(b) $(c + 2d)^2 - (c + 2d)(3c - 7d)$

(c) $x(2h - k) + 3y(k - 2h)$

(d) $6x(4m - n) - 2y(n - 4m)$

5. Factorise each of the following expressions completely.

(a) $3ax + 28by + 4ay + 21bx$

(b) $12cy + 20c - 15 - 9y$

(c) $dy + fy - fz - dz$

(d) $3x^2 + 6xy - 4xz - 8yz$

(e) $2xy - 8x + 12 - 3y$

(f) $5xy - 25x^2 + 50x - 10y$

(g) $x^2y^2 - 5x^2y - 5xy^2 + xy^3$

(h) $kx + hy - hx - ky$

ADVANCED LEVEL

6. Factorise each of the following expressions completely.
   (a) $144p(y - 5x^2) - 12q(10x^2 - 2y)$
   (b) $2(5x + 10y)(2y - x)^2 - 4(6y + 3x)(x - 2y)$

7. (i) Factorise the expression $\frac{1}{3}p^2q + \frac{4}{3}p^2r$ completely.
   (ii) Hence, by substituting suitable values of p, q and r, find the value of $\frac{1}{3} \times 1.2^2 \times 36 + \frac{4}{3} \times 1.2^2 \times 16$.

8. (i) Factorise the expression $x^3 + 3x - x^2 - 3$ completely.
   (ii) Hence, express $(x^2 - 3)^3 - (2 - x^2)^2 + 3(x^2 - 3)$ in the form $(x^4 + Ax^2 + B)(x^2 + C)$, where A, B and C are integers.

1. **Expansion of Algebraic Expressions:**

$$(a + b)(c + d + e) = a(c + d + e) + b(c + d + e)$$
$$= ac + ad + ae + bc + bd + be$$

2. **Special Algebraic Identities:**

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(a + b)(a - b) = a^2 - b^2$$

ATTENTION

- $(a + b)^2 \neq a^2 + b^2$
- $(a - b)^2 \neq a^2 - b^2$

These algebraic identities are useful for *expanding* algebraic expressions which are of similar forms. $(a + b)^2$ and $(a - b)^2$ are known as **perfect squares** while $(a + b)(a - b)$ is called the **difference of two squares**.

Since **factorisation** is the reverse of expansion, we have:

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$
$$a^2 - b^2 = (a + b)(a - b)$$

3. **Factorisation by Grouping:**

$$ax + bx + kay + kby = x(a + b) + ky(a + b)$$
$$= (a + b)(x + ky)$$

Review Exercise 4

1. Expand each of the following expressions.
   (a) $-2a(a - 5b + 7)$ (b) $(2c + 3d)(3c + 4d)$
   (c) $(k + 3h)(5h - 4k)$ (d) $(2m + 1)(m^2 + 3m - 1)$

2. Expand and simplify each of the following expressions.
   (a) $2p(3p - 5q) - q(2q - 3p)$ (b) $-4s(3s + 4r) - 2r(2r - 5s)$
   (c) $(8t - u)(t + 9u) - t(2u - 7t)$ (d) $(2w + 3x)(w - 5x) - (3w + 7x)(w - 7x)$

3. Factorise each of the following expressions completely.
   (a) $x^2 + 2xy - 63y^2$ (b) $2x^2 + 5xy + 3y^2$
   (c) $6x^2y^2 - 5xy - 4$ (d) $3z - 8xyz + 4x^2y^2z$

4. Expand each of the following expressions.
   (a) $(-x + 5y)^2$ (b) $(x^2 + y)(x^2 - y)$
   (c) $\left(3x + \frac{4}{5}y\right)^2$ (d) $\left(-\frac{1}{4}x - \frac{1}{6}y\right)^2$
   (e) $\left(5x - \frac{7}{4}y\right)\left(5x + \frac{7}{4}y\right)$ (f) $\left(\frac{3}{4}xy + \frac{1}{3}z\right)\left(\frac{3}{4}xy - \frac{1}{3}z\right)$

5. Factorise each of the following expressions completely.
   (a) $1 - 121x^2$ (b) $x^2 + 6xy + 9y^2$
   (c) $25x^2 - 100xy + 100y^2$ (d) $36y^2 - 49(x + 1)^2$

6. Factorise each of the following expressions completely.
   (a) $-14xy - 21y^2$ (b) $9xy^2 - 36x^2y$
   (c) $(2x - 3y)(a + b) + (x - y)(b + a)$ (d) $5(x - 2y) - (x - 2y)^2$
   (e) $x^2 + 3xy + 2x + 6y$ (f) $3x^3 - 2x^2 + 3x - 2$
   (g) $4cx - 6cy - 8dx + 12dy$ (h) $5xy - 10x - 12y + 6y^2$

7. Factorise the expression $x^3 + x^2 - 4x - 4$ completely.

8. Without using a calculator, evaluate each of the following.
   (a) 899^2 (b) $659^2 - 341^2$

9. If $2(x - y)^2 = 116$ and $xy = 24$, find the value of $x^2 + y^2$.

10. (i) Expand and simplify the expression $(f + 3)^2$.

(ii) Hence, expand and simplify the expression $[(2h + k) + 3]^2$.

Challenge Yourself

1. If $(a + b)^2 = a^2 + b^2$ is true for some real numbers a and b, find the value of $\sqrt{ab}$.

2. If $h^2 + k^2 - m^2 - n^2 = 15$ and $(h^2 + k^2)^2 + (m^2 + n^2)^2 = 240.5$, find the value of $h^2 + k^2 + m^2 + n^2$.

A1 Revision Exercise

1. The variables x and y are connected by the equation $y = k\sqrt{x-3}$, where k is a constant. Some values of x and the corresponding values of y are given in the table.

x	39	67	q
y	p	2	2.75

Find the values of k, p and q.

2. The resistance, R ohms (Ω) of a copper wire of fixed length is inversely proportional to the square of its diameter, d mm. Given that the resistance of a wire 2.5 mm in diameter is 20 Ω, find the diameter of a wire with a resistance of 31.25 Ω.

3. Solve the simultaneous equations

$$8x + 3y = 14,$$
$$2x + y = 4.$$

4. Two iron atoms and three oxygen atoms contain 76 protons. One iron atom and one oxygen atom contain 34 protons. How many protons are there in each atom?

5. Expand and simplify each of the following expressions.

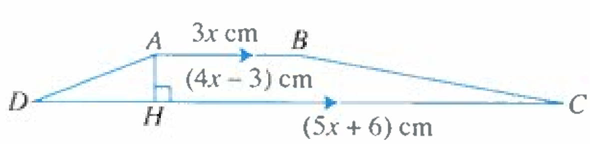
(a) $(2a + 5b)^2 - (a + 3b)(a - 6b)$ **(b)** $(4c + d)(4c - d) - \left(2c - \frac{1}{5}d\right)^2$

6. Factorise each of the following expressions completely.

(a) $4f^2 - 10f + 6$ **(b)** $1 - 12hk + 36h^2k^2$

(c) $5m^2n - 15mn^2 - 25mn$ **(d)** $2px + 3qy - 2py - 3qx$

7. The figure shows a trapezium $ABCD$ where $AB = 3x$ cm and $DC = (5x + 6)$ cm. Given that the perpendicular distance between AB and DC, $AH = (4x - 3)$ cm, show that the area of the trapezium can be expressed as $(16x^2 - 9)$ cm^2.



A2 Revision Exercise

1. The energy stored in a spring is directly proportional to the square of its extension. When a spring is stretched by d cm, the energy stored in it is E joules (J). If the extension of the spring is decreased by 25%, find the percentage decrease in the energy stored.

2. If y is inversely proportional to $2x^2 + 5$ and $y = 7$ when $x = 2$,
   (i) find an equation connecting x and y,
   (ii) find the value of y when $x = 8$,
   (iii) calculate the values of x when $y = 5.2$.

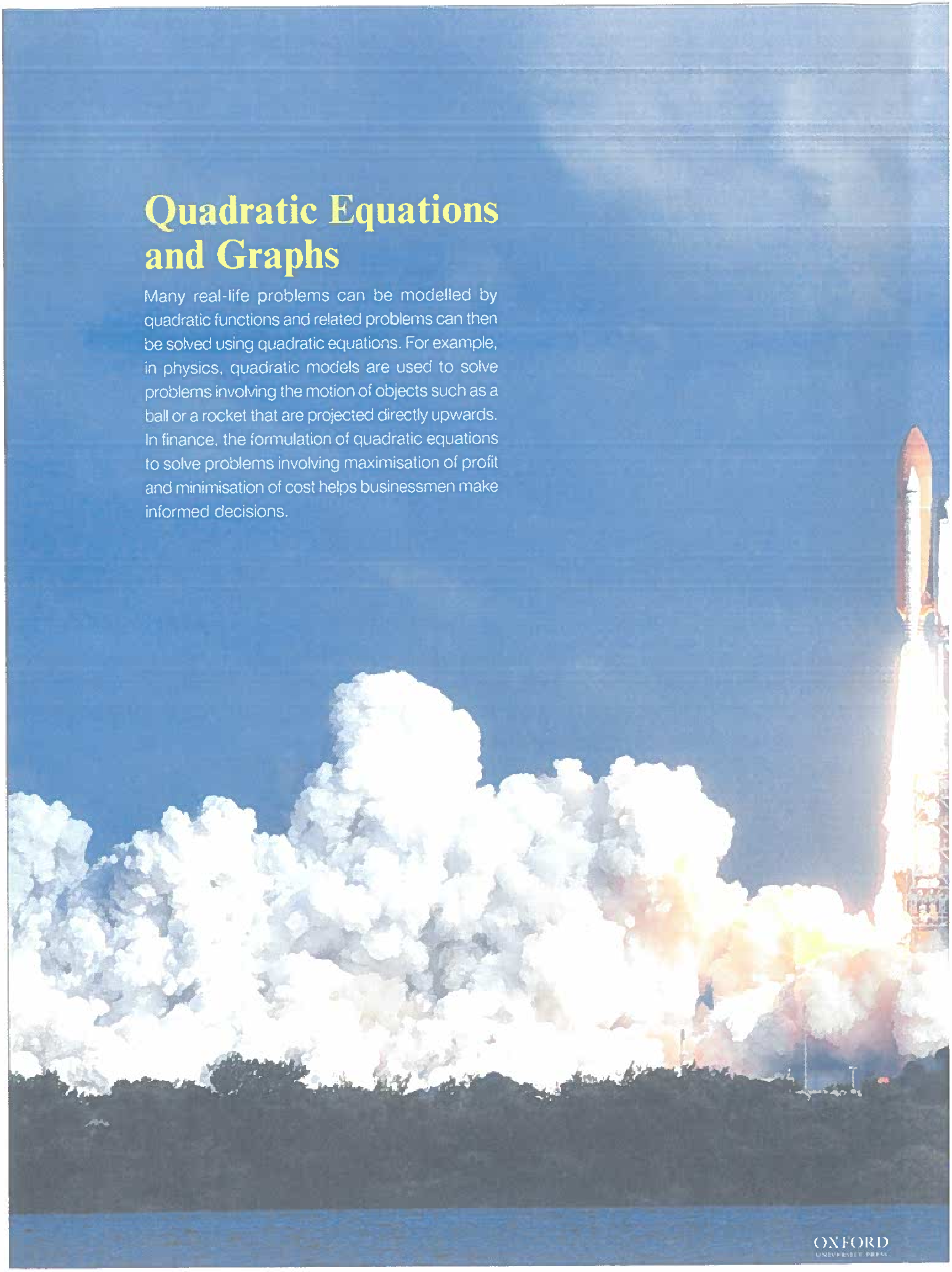
3. The coordinates of the point of intersection of the lines $px + y = 3$ and $x + 2y = q$ are $(2, -3)$. Find the value of p and of q.

4. There are x chickens and y rabbits on a farm. Given that the animals have a total of 70 heads and 196 legs, formulate a pair of simultaneous equations involving x and y. By solving the simultaneous equations, find the number of chickens and rabbits on the farm.

5. Expand and simplify each of the following expressions.
   (a) $(a^2 - 7a + 6)(3a - 2) - a(2a^2 - 7)$
   (b) $\left(b^2 + \frac{1}{9}\right)\left(b + \frac{1}{3}\right)\left(b - \frac{1}{3}\right)$

6. Factorise each of the following expressions completely.
   (a) $2c^2d^2 + 5cd - 12$
   (b) $25h^2k^2 + 10hk + 1$
   (c) $16 - 4(m + 2)^2$
   (d) $3pr - ps + 6qr - 2qs$

7. Without using a calculator, evaluate each of the following.
   (a) 805^2
   (b) $903^2 - 97^2$



Quadratic Equations and Graphs

Many real-life problems can be modelled by quadratic functions and related problems can then be solved using quadratic equations. For example, in physics, quadratic models are used to solve problems involving the motion of objects such as a ball or a rocket that are projected directly upwards. In finance, the formulation of quadratic equations to solve problems involving maximisation of profit and minimisation of cost helps businessmen make informed decisions.

Chapter Five

RNING OBJECTIVES

e end of this chapter, you should be able to:
lve quadratic equations by factorisation,
aw graphs of quadratic functions,
ate the properties of the graphs of quadratic functions,
lve mathematical and real-life problems involving
adratic equations algebraically and graphically.

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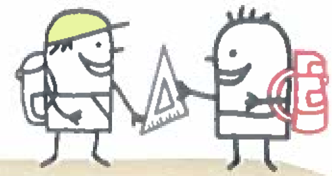
5.1 Solving Quadratic Equations by Factorisation



In arithmetic, we have learnt that the product of any number and zero is equal to zero. For example, $2 \times 0 = 0$, $0 \times 8 = 0$, $-6 \times 0 = 0$, $0 \times (-7) = 0$, etc.

Similarly, in algebra, if two factors P and Q are such that $P \times Q = 0$, then either $P = 0$ or $Q = 0$ or both P and Q are equal to 0. We shall use this principle to solve quadratic equations of the form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$.

ATTENTION

If two factors P and Q are such that $P \times Q = R$, where $R \neq 0$, we cannot conclude that either $P = R$ or $Q = R$ or both P and Q are equal to R.

Worked Example 1

Solving Quadratic Equations of the Form [illegible]

Solve each of the following equations.

(a) $x(x - 1) = 0$ **(b)** $2x(x + 1) = 0$

(c) $(x - 2)(x + 3) = 0$ **(d)** $(2x - 3)(3x + 5) = 0$

Solution:

(a) $x(x - 1) = 0$

$x = 0$ or $x - 1 = 0$

$\therefore x = 0$ or $x = 1$

(b) $2x(x + 1) = 0$

$2x = 0$ or $x + 1 = 0$

$\therefore x = 0$ or $x = -1$

(c) $(x - 2)(x + 3) = 0$

$x - 2 = 0$ or $x + 3 = 0$

$\therefore x = 2$ or $x = -3$

(d) $(2x - 3)(3x + 5) = 0$

$2x - 3 = 0$ or $3x + 5 = 0$

$\therefore x = 1\frac{1}{2}$ or $x = -1\frac{2}{3}$

PRACTISE NOW 1

Solve each of the following equations.

(a) $x(x + 2) = 0$ **(b)** $3x(x - 1) = 0$

(c) $(x + 5)(x - 7) = 0$ **(d)** $(3x + 2)(4x - 5) = 0$

SIMILAR QUESTIONS

Exercise 5A Questions 1(a)–(f), 2(a)–(h)

Worked Example 2

(Solving Quadratic Equations of the Form $ax^2 + bx + c = 0$)

Solve each of the following equations.

(a) $2x^2 + 6x = 0$ (b) $9x^2 - 4 = 0$

(c) $x^2 - 3x - 28 = 0$ (d) $2x^2 + 5x - 12 = 0$

Solution:

(a) $2x^2 + 6x = 0$

$2x(x + 3) = 0$ (factorise by extracting the common factor $2x$)

$2x = 0$ or $x + 3 = 0$

$\therefore x = 0$ or $x = -3$

(b) $9x^2 - 4 = 0$

$(3x)^2 - 2^2 = 0$

$(3x + 2)(3x - 2) = 0$ (factorise by using $a^2 - b^2 = (a + b)(a - b)$)

$3x + 2 = 0$ or $3x - 2 = 0$

$\therefore x = -\frac{2}{3}$ or $x = \frac{2}{3}$

(c) $x^2 - 3x - 28 = 0$

$(x + 4)(x - 7) = 0$ (factorise by using the multiplication frame)

$x + 4 = 0$ or $x - 7 = 0$

$\therefore x = -4$ or $x = 7$

(d) $2x^2 + 5x - 12 = 0$

$(2x - 3)(x + 4) = 0$ (factorise by using the multiplication frame)

$2x - 3 = 0$ or $x + 4 = 0$

$\therefore x = 1\frac{1}{2}$ or $x = -4$

PRACTISE NOW 2

SIMILAR QUESTIONS

Exercise 5A Questions 3(a)–(f), 4(a)–(f), 5(a)–(h), 6, 7(a)–(d), 8(a)–(b), 10, 11(a)–(b), 12, 14

1. Solve each of the following equations.

   (a) $3x^2 - 15x = 0$ (b) $x^2 + 8x + 16 = 0$

   (c) $x^2 + 5x + 4 = 0$ (d) $3x^2 - 17x + 10 = 0$

2. (i) Solve the equation $3x^2 - 10x + 8 = 0$.

   (ii) Hence, solve the equation $3(y + 1)^2 - 10(y + 1) + 8 = 0$.

3. (i) If $x = -2$ is a solution of the equation $x^2 + px + 8 = 0$, find the value of p.

   (ii) Hence, find the other solution of the equation.

Worked Example 3

(Solving Equations Reducible to the Form $ax^2 + bx + c = 0$)

Solve each of the following equations.

(a) $x(x + 6) = -5$ **(b)** $2y(8y + 3) = 1$

ATTENTION

$x(x + 6) = -5$
$\nRightarrow x = -5$ or $x + 6 = -5$

Solution:

(a)

$$x(x + 6) = -5$$
$$x^2 + 6x = -5$$
$$x^2 + 6x + 5 = 0$$
$$(x + 1)(x + 5) = 0 \text{ (factorise by using the multiplication frame)}$$
$$x + 1 = 0 \quad \text{or} \quad x + 5 = 0$$
$$\therefore x = -1 \quad \text{or} \quad x = -5$$

(b)

$$2y(8y + 3) = 1$$
$$16y^2 + 6y = 1$$
$$16y^2 + 6y - 1 = 0$$
$$(8y - 1)(2y + 1) = 0 \text{ (factorise by using the multiplication frame)}$$
$$8y - 1 = 0 \quad \text{or} \quad 2y + 1 = 0$$
$$\therefore y = \frac{1}{8} \quad \text{or} \quad y = -\frac{1}{2}$$

PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 5A Questions 9(a)–(h), 13

Solve each of the following equations.

(a) $x(x + 1) = 6$ **(b)** $9y(1 - y) = 2$

Thinking Time

Assume $x = y$.

Multiply by x on both sides: $x^2 = xy$

Subtract y^2 from both sides: $x^2 - y^2 = xy - y^2$

Factorise both sides: $(x + y)(x - y) = y(x - y)$

Divide by $x - y$ on both sides: $x + y = y$

Since $x = y$, then $2y = y$.

Divide by y on both sides: $2 = 1$

Since $2 \neq 1$, what went wrong in the steps above?

Exercise 5A

BASIC LEVEL

1. Solve each of the following equations.
   (a) $a(a - 9) = 0$ (b) $b(b + 7) = 0$
   (c) $5c(c + 1) = 0$ (d) $2d(d - 6) = 0$
   (e) $-f(3f - 7) = 0$ (f) $-\frac{1}{2}h(2h + 3) = 0$

2. Solve each of the following equations.
   (a) $(k - 4)(k - 9) = 0$ (b) $(m - 3)(m + 5) = 0$
   (c) $(n + 4)(n - 11) = 0$ (d) $(p + 1)(p + 2) = 0$
   (e) $(7q - 6)(4q - 5) = 0$ (f) $(3r - 5)(2r + 1) = 0$
   (g) $(5s + 3)(2 - s) = 0$ (h) $(-2t - 5)(8t - 5) = 0$

3. Solve each of the following equations.
   (a) $a^2 + 9a = 0$ (b) $b^2 - 7b = 0$
   (c) $5c^2 + 25c = 0$ (d) $3d^2 - 4d = 0$
   (e) $3f - 81f^2 = 0$ (f) $-4h^2 - 16h = 0$

4. Solve each of the following equations.
   (a) $k^2 + 12k + 36 = 0$ (b) $m^2 - 16m + 64 = 0$
   (c) $n^2 - 16 = 0$ (d) $25p^2 + 70p + 49 = 0$
   (e) $4q^2 - 12q + 9 = 0$ (f) $4r^2 - 100 = 0$

5. Solve each of the following equations.
   (a) $s^2 + 10s + 21 = 0$ (b) $t^2 - 16t + 63 = 0$
   (c) $u^2 + 6u - 27 = 0$ (d) $v^2 - 5v - 24 = 0$
   (e) $3w^2 + 49w + 60 = 0$ (f) $6x^2 - 29x + 20 = 0$
   (g) $3y^2 + 7y - 6 = 0$ (h) $2z^2 - 3z - 14 = 0$

INTERMEDIATE LEVEL

6. Solve the equation $7x^3 + 21x^2 = 0$.

7. Solve each of the following equations.
   (a) $121 - a^2 = 0$ (b) $128 - 2b^2 = 0$
   (c) $c^2 - \frac{1}{4} = 0$ (d) $\frac{4}{9} - \frac{d^2}{25} = 0$

8. Solve each of the following equations.
   (a) $7f + f^2 = 60$ (b) $15 = 8h^2 - 2h$

9. Solve each of the following equations.
   (a) $k(2k + 5) = 3$ (b) $2m(m - 5) = 5m - 18$
   (c) $(n - 2)(n + 4) = 27$ (d) $(p - 1)(p - 6) = 126$
   (e) $(2q - 3)(q - 4) = 18$ (f) $3r^2 - 5(r + 1) = 7r + 58$
   (g) $(3s + 1)(s - 4) = -5(s - 1)$ (h) $(t + 2)(t - 3) = t + 2$

10. (i) Solve the equation $6x^2 - x - 15 = 0$.
    (ii) Hence, solve the equation
    $6(y - 3)^2 - (y - 3) - 15 = 0$.

ADVANCED LEVEL

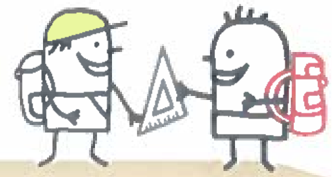
11. Solve each of the following equations.
    (a) $\frac{1}{2}x^2 - \frac{11}{4}x + \frac{5}{4} = 0$ (b) $2 - 3.5y - 9.75y^2 = 0$

12. Solve the equation $9x^2y^2 - 12xy + 4 = 0$, expressing y in terms of x.

13. Solve the equation $x - (2x - 3)^2 = -6(x^2 + x - 2)$.

14. (i) If $x = 5$ is a solution of the equation
    $x^2 - qx + 10 = 0$, find the value of q.
    (ii) Hence, find the other solution of the equation.

5.2 Applications of Quadratic Equations in Real-World Contexts



In this section, we will take a look at how mathematical and real-life problems can be solved using quadratic equations.

Worked Example 4

(Finding Two Numbers Given the Sum)

Two consecutive positive odd numbers are such that the sum of their squares is 130. Find the two numbers.

Solution:

Let the smaller number be x.

Then the next consecutive odd number is $x + 2$.

$$\therefore x^2 + (x + 2)^2 = 130$$
$$x^2 + x^2 + 4x + 4 = 130$$
$$2x^2 + 4x - 126 = 0$$
$$x^2 + 2x - 63 = 0$$
$$(x - 7)(x + 9) = 0$$
$$x - 7 = 0 \quad \text{or} \quad x + 9 = 0$$
$$\therefore x = 7 \quad \text{or} \quad x = -9 \text{ (rejected since } x > 0)$$

When $x = 7$,

$$x + 2 = 7 + 2$$
$$= 9$$

The two consecutive positive odd numbers are 7 and 9.

PRACTISE NOW 4

SIMILAR QUESTIONS
Exercise 5B Questions 1–5, 13

1. Two consecutive positive even numbers are such that the sum of their squares is 164. Find the two numbers.

2. The difference between two positive numbers is 5 and the square of their sum is 169. Find the two numbers.

Worked Example 5

(Finding the Dimensions of a Rectangular Garden)

The perimeter of a rectangular garden is 50 m and its area is 150 m². Calculate the length and the breadth of the garden.

Solution:

Let the length of the rectangular garden be x m.

Then the breadth of the rectangular garden is $\left(\frac{50-2x}{2}\right)$ m $= (25 - x)$ m.

$$\therefore x(25 - x) = 150$$
$$25x - x^2 = 150$$
$$x^2 - 25x + 150 = 0$$
$$(x - 10)(x - 15) = 0$$
$$x - 10 = 0 \quad \text{or} \quad x - 15 = 0$$
$$\therefore x = 10 \quad \text{or} \quad x = 15$$

When $x = 10$,

Breadth of garden $= 25 - 10$

$= 15$ m

When $x = 15$,

Breadth of garden $= 25 - 15$

$= 10$ m

Since the length of a rectangle usually refers to the longer side,

Length of garden = 15 m

Breadth of garden = 10 m

PRACTISE NOW 5

The perimeter of a rectangle is 20 cm and its area is 24 cm². Find the length and the breadth of the rectangle.

SIMILAR QUESTIONS

Exercise 5B Questions 6–11

Worked Example 6

(Real-Life Problem Modelled by a Quadratic Equation)

The height, y metres, of an object projected directly upwards from the ground can be modelled by $y = 45t - 5t^2$, where t is the time in seconds after it leaves the ground.

(i) Calculate the height of the object 5 seconds after it leaves the ground.

(ii) At what time will the object strike the ground again?

ATTENTION

The height, y metres, of the object can be modelled by $y = 45t - 5t^2$ only when $y \geq 0$

Solution:

(i) When $t = 5$,

$$y = 45(5) - 5(5^2)$$
$$= 100$$

$\therefore$ Height of object 5 seconds after it leaves the ground = 100 m

(ii) Let $y = 0$.

$$45t - 5t^2 = 0$$
$$5t(9 - t) = 0$$
$$5t = 0 \quad \text{or} \quad 9 - t = 0$$
$$\therefore t = 0 \quad \text{or} \quad t = 9$$

$\therefore$ The object will strike the ground again 9 seconds after it leaves the ground.

PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 5B Questions 12, 14–15

1. The height, y metres, of a model rocket launched directly upwards from the ground can be modelled by $y = 96t - 4t^2$, where t is the time in seconds after it leaves the ground.
   (i) Find the height of the rocket 12 seconds after it leaves the ground.
   (ii) At what time will the rocket strike the ground again?

2. A ball is thrown vertically upwards from the top of a school building. Its height, h metres, above the ground, can be modelled by $h = 18 + 6t - 4t^2$, where t is the time in seconds after it leaves the top of the building.
   (i) How tall is the school building?
   (ii) At what time will the ball strike the ground?

Exercise 5B

BASIC LEVEL

1. The sum of a whole number and twice the square of the number is 10. Find the number.

2. If four times a whole number is subtracted from three times the square of the number, the result 15 is obtained. Find the number.

3. Two consecutive positive numbers are such that the sum of their squares is 113. Find the two numbers.

4. The difference between two positive numbers is 7 and the square of their sum is 289. Find the two numbers.

5. The difference between two numbers is 9 and the product of the numbers is 162. Find the two numbers.

INTERMEDIATE LEVEL

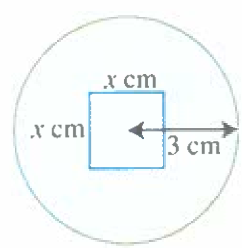
6. The perimeter of a rectangular campsite is 64 m and its area is 207 m². Find the length and the breadth of the campsite.

7. A rectangular field, 70 m long and 50 m wide, is surrounded by a concrete path of uniform width. Given that the area of the path is 1024 m², find the width of the path.

8. The length of a side and the corresponding height of a triangle are $(x + 3)$ cm and $(2x - 5)$ cm respectively. Given that the area of the triangle is 20 cm², find the value of x.

9. A piece of wire 44 cm long is cut into two parts. Each part is bent to form a square. Given that the total area of the two squares is 65 cm², find the perimeter of each square.

10. The figure shows an ancient coin which was once used in China. The coin is in the shape of a circle of radius 3 cm with a square of sides x cm removed from its centre. The area of each face of the coin is 7π cm².
    (i) Form an equation in x and show that it reduces to $2\pi - x^2 = 0$.
    (ii) Solve the equation $2\pi - x^2 = 0$.
    (iii) Find the perimeter of the square.



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11. Amirah walks at an average speed of $(x + 1)$ km/h for x hours and cycles at an average speed of $(2x + 5)$ km/h for $(x - 1)$ hours. She covers a total distance of 90 km.
   (i) Form an equation in x and show that it reduces to $3x^2 + 4x - 95 = 0$.
   (ii) Solve the equation $3x^2 + 4x - 95 = 0$.
   (iii) Find the time taken for her entire journey.

12. A hawk drops its prey from a certain height above the ground. The height, h metres, of the prey can be modelled by $h = 4 + 11t - 3t^2$, where t is the time in seconds after it is dropped by the hawk.
   (i) At what height above the ground does the hawk drop its prey?
   (ii) At what time will the prey fall onto the ground?

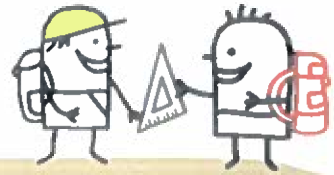
ADVANCED LEVEL

13. When $(x + 1)^2$ is divided by $x - 2$, the quotient is 16 and the remainder is $x - 3$. Find the values of x.

14. The height, y metres, of an object projected directly upwards from the ground can be modelled by $y = 17t - 5t^2$, where t is the time in seconds after it leaves the ground.
   (i) Find the height of the object 2.5 seconds after it leaves the ground.
   (ii) At what time will the object strike the ground again?
   (iii) 1.5 seconds after the object has been projected, a second object is also projected directly upwards from the ground. Given that the equations of motion of the two objects are the same, what is the distance between the two objects one second after the second object has been projected?

15. The height, h metres, of a ball projected directly upwards from the ground can be modelled by $h = 56t - 7t^2$, where t is the time in seconds after it leaves the ground.
   (i) Find the height of the ball 3.5 seconds after it leaves the ground.
   (ii) At what time will the ball strike the ground again?
   (iii) When will the ball be 49 m above the ground? Briefly explain why there are two possible answers.

5.3 Graphs of Quadratic Functions



Quadratic Functions

Investigation

Relationship Between the Area of a Square and the Length of its Side

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry software template 'Area of Square'.

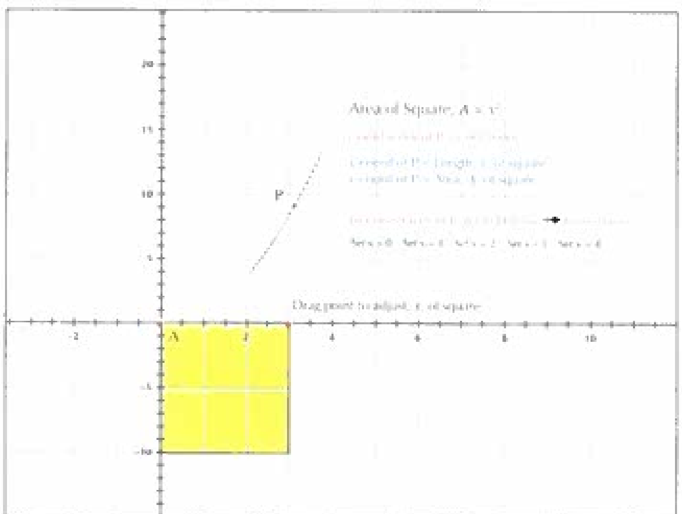


Fig. 5.1

In the template, the x-coordinate of the point P represents the length, x units, of a square while the y-coordinate of P represents its area, A units^2. Hence, P will trace out the graph of $A = x^2$.

Click on each of the buttons 'Set $x = 0$', 'Set $x = 1$', etc, to obtain the graph of $A = x^2$ and answer the following questions.

1. For each value of x, how many corresponding values of A are there? Is $A = x^2$ the equation of a function?
2. What do you notice about the shape of the graph? Is it linear or non-linear? Explain your answer.

Recall from Book 1 that a function is a relationship between two variables x and y such that every input x produces *exactly one* output y. In the investigation, since for each value of x, there is exactly one corresponding value of A, then $A = x^2$ is the equation of a function.

Recall also from Book 1 and Chapter 2 that the graphs of linear functions are straight lines. Since the graph of $A = x^2$ is not a straight line, then the graph of the function $A = x^2$ is non-linear, i.e. $A = x^2$ is the equation of a *non-linear function*. In fact, $A = x^2$ is the equation of a **quadratic function**.

In this section, we shall take a look at the graphs of quadratic functions. Unlike linear graphs which are straight lines, quadratic graphs are a family of curves called **parabolas**.

Internet Resources

Search on the Internet for examples of the applications of parabolas in real life and in the sciences.

Graphs of $y = ax^2 + bx + c$ $(a \neq 0)$

Investigation

Graphs of $y = x^2$ and $y = -x^2$

1. Using a graphing software, draw each of the following graphs.
   (a) $y = x^2$ **(b)** $y = -x^2$

2. Study the graphs and answer each of the following questions.
   (a) Both graphs pass through a particular point on the coordinate axes. What are the coordinates of this point?
   (b) State the lowest or highest point of each graph.
   (c) Both graphs are symmetrical about one of the axes. Name the axis. Hence, state the equation of the line of symmetry of the graphs.

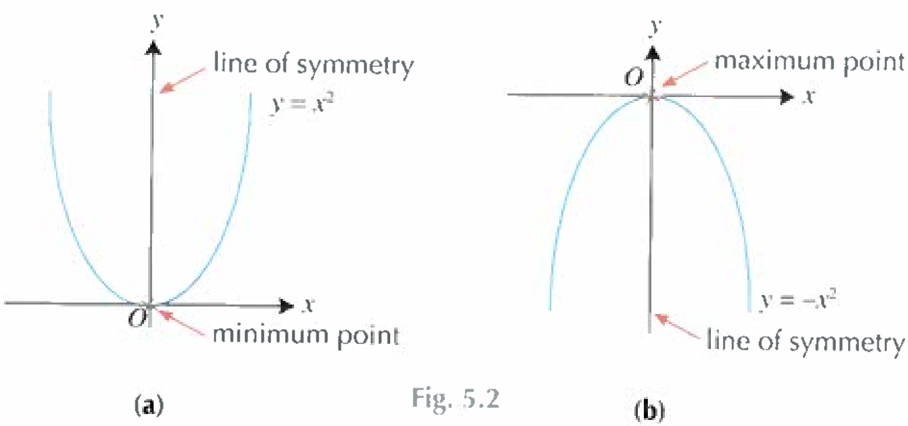


Fig. 5.2

From the investigation, we can conclude that:

- The graphs of $y = x^2$ and $y = -x^2$ pass through the origin.
- The graph of $y = x^2$ opens *upwards* indefinitely and has a *lowest point* known as the **minimum point** (see Fig. 5.2**(a)**).
  The graph of $y = -x^2$ opens *downwards* indefinitely and has a *highest point* known as the **maximum point** (see Fig. 5.2**(b)**).
- The graphs of $y = x^2$ and $y = -x^2$ are symmetrical about the y-axis, i.e. the equation of the **line of symmetry** of the graphs is $x = 0$.

The general form of the equation of a quadratic function is given by:

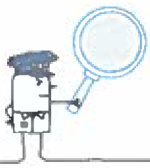
$$y = ax^2 + bx + c,$$

where a, b and c are constants and $a \neq 0$.

When $a = 1$, $b = 0$ and $c = 0$, we have a quadratic function which has an equation $y = x^2$.

When $a = -1$, $b = 0$ and $c = 0$, we have a quadratic function which has an equation $y = -x^2$.

Do all graphs of quadratic functions have the same shapes and properties as the graphs of $y = x^2$ and $y = -x^2$? We shall now take a look at the graphs of other quadratic functions.



Investigation

Graphs of $y = ax^2 + bx + c$, where $a \neq 0$

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry software template 'Graphs of Quadratic Functions'.

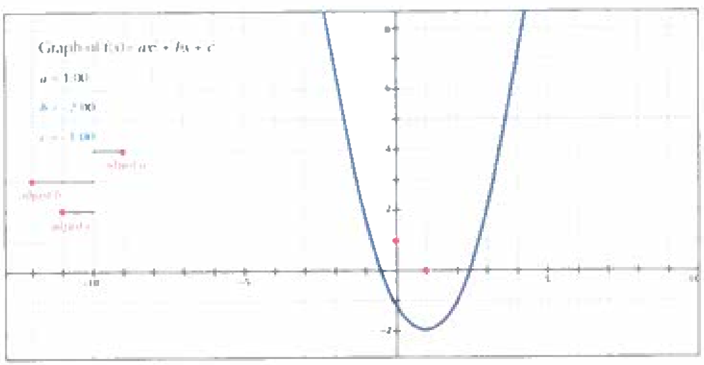


Fig. 5.3

Part I: Effect of Value of a

For this part of the investigation, adjust the value of b and of c to 0.

1. Increase the value of a. What do you notice about the shape of the graph?
2. Decrease the value of a but ensure that it is positive. What do you notice about the shape of the graph?
3. Decrease the value of a until it becomes negative. What do you notice about the shape of the graph?
4. How does the value of a affect the shape of the graph? What happens when a is positive and when a is negative?

Part II: Effect of Value of c

For this part of the investigation, adjust the value of a and of b to 1.

5. Increase the value of c. What do you notice about the position of the graph?
6. Decrease the value of c. What do you notice about the position of the graph?
7. How does the value of c affect the position of the graph?

Part III: Effect of Value of b

8. Adjust the value of a to 1, b to –2 and c to 0 before increasing the value of b. What do you notice about the shape and the position of the graph?

9. Adjust the value of a to –1, b to –2 and c to 0 before increasing the value of b. What do you notice about the shape and the position of the graph?

10. How does the value of b affect the graph?

Part IV: Quadratic Graphs

11. Change the values of a, b and c to obtain each of the quadratic graphs in Table 5.1. Complete the table. The first one has been done for you.

Quadratic Graph	Coefficient of x^2	Opens upwards / downwards	Coordinates of minimum / maximum point	Equation of line of symmetry	x-intercept(s)	y-intercept
$y = x^2 - 4x + 3$	1	Opens upwards	(2, –1)	$x = 2$	1, 3	3
$y = -x^2 - 2x + 3$						
$y = x^2 - 4x + 4$						
$y = -4x^2 + 12x - 9$						
$y = 2x^2 + 2x + 1$						
$y = -3x^2 + x - 4$						

Table 5.1

Note: An x-intercept refers to the x-coordinate of a point of intersection of a graph with the x-axis.

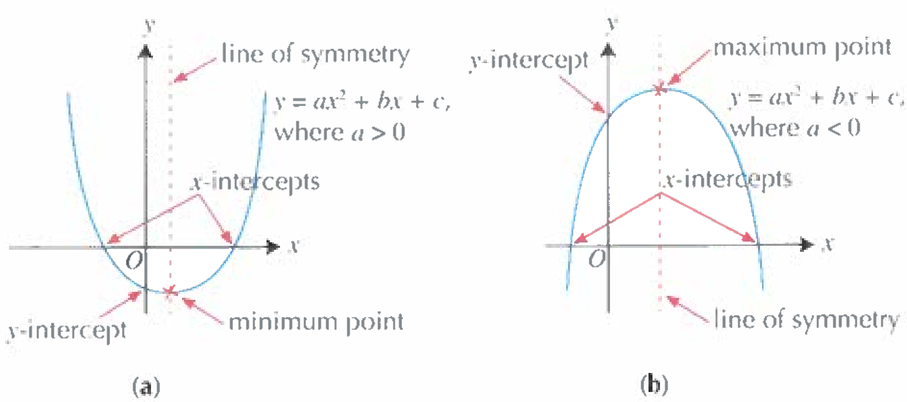


Fig. 5.4

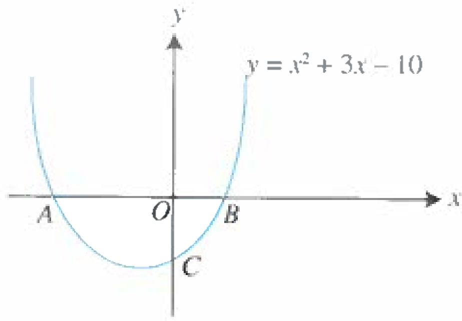
From the investigation, for the graph of $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$, we can conclude that:

- For $a > 0$, the graph opens *upwards* indefinitely and has a **minimum point** (see Fig. 5.4(**a**)).
  For $a < 0$, the graph opens *downwards* indefinitely and has a **maximum point** (see Fig. 5.4(**b**)).
- The *smaller* the absolute value of a, the *wider* the graph opens.
- The **line of symmetry** of the graph passes through its minimum / maximum point.
- The graph may have 0, 1 or 2 x-intercept(s) but it has only 1 y-intercept.

Worked Example 7

Finding the Coordinates of Points Given the Sketch of a Curve

In the figure, the curve $y = x^2 + 3x - 10$ cuts the x-axis at two points A and B, and the y-axis at the point C. Calculate the coordinates of A, B and C.



Solution:

Let $y = 0$.

$x^2 + 3x - 10 = 0$

$(x - 2)(x + 5) = 0$

$x - 2 = 0$ or $x + 5 = 0$

$\therefore x = 2$ or $x = -5$

$\therefore$ The coordinates of A and B are $(-5, 0)$ and $(2, 0)$ respectively.

When $x = 0$,

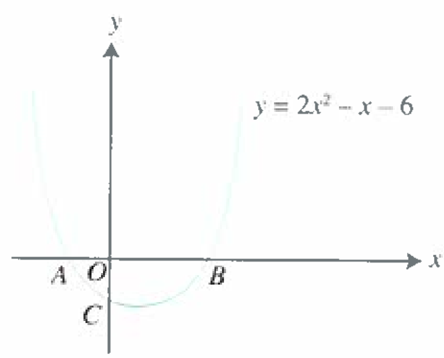
$y = 0^2 + 3(0) - 10$

$= -10$

$\therefore$ The coordinates of C are $(0, -10)$.

PRACTISE NOW 7

In the figure, the curve $y = 2x^2 - x - 6$ cuts the x-axis at two points A and B, and the y-axis at the point C. Find the coordinates of A, B and C.



SIMILAR QUESTIONS

Exercise 5C Question 3

Worked Example 8

(Drawing the Graph of $y = ax^2 + bx + c$, where $a > 0$)

The variables x and y are connected by the equation $y = x^2 + 2x + 2$. Some values of x and the corresponding values of y are given in the table.

x	–3	–2	–1	0	1	2	3
y	p	2	1	2	5	10	17

(a) Calculate the value of p.

(b) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = x^2 + 2x + 2$ for $-3 \leqslant x \leqslant 3$.

(c) Use your graph in **(b)** to find

(i) the value of y when $x = 2.3$,

(ii) the values of x when $y = 3$,

(iii) the minimum value of y and the value of x at which this occurs.

Solution:

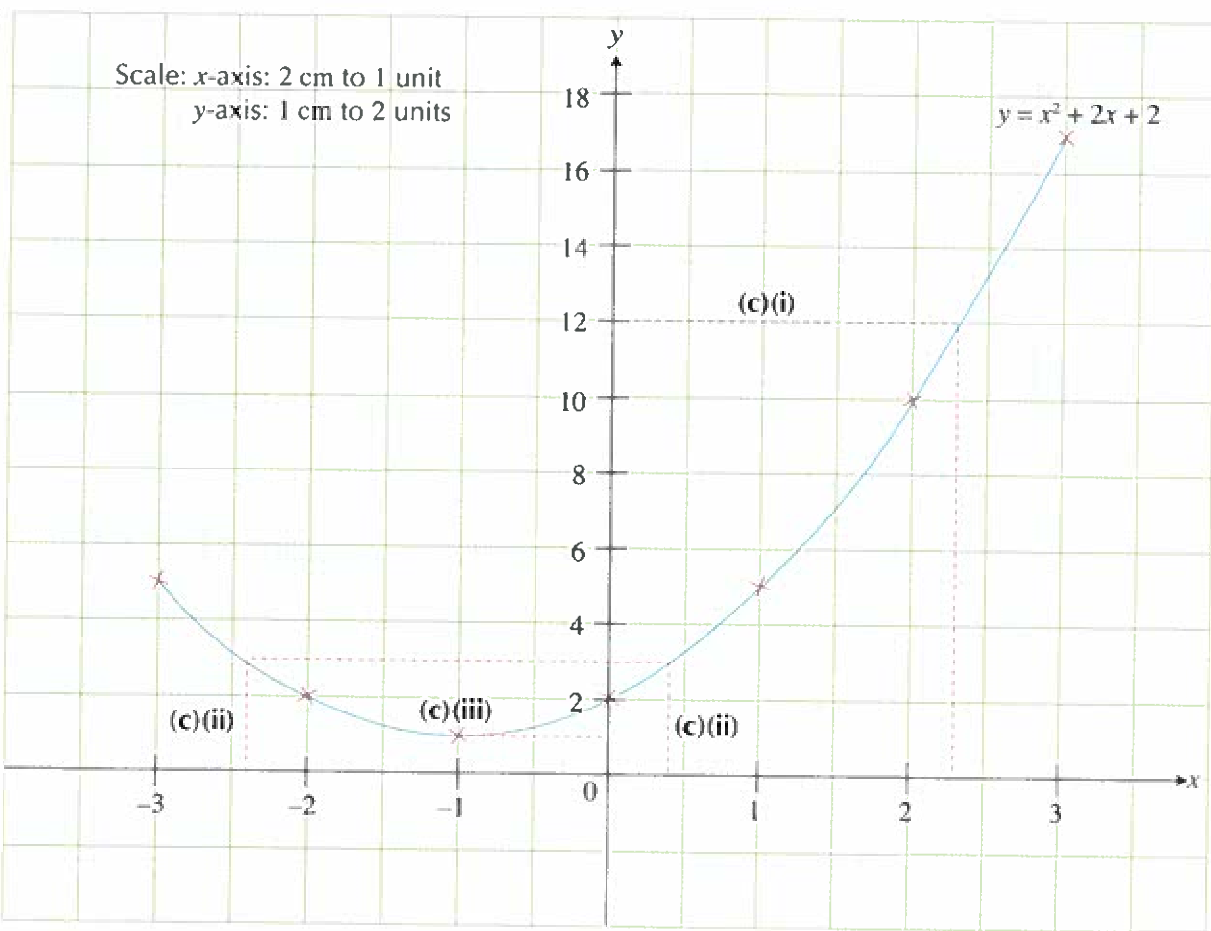
(a) When $x = -3$,

$y = (-3)^2 + 2(-3) + 2$

$= 5$

$\therefore p = 5$

(b)



(c) **(i)** When $x = 2.3$,
$y \approx 12$

(ii) When $y = 3$,
$x \approx -2.4$ or 0.4

(iii) Minimum value of $y = 1$
Minimum value of y occurs when $x = -1$

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 5C Questions 1, 7

The variables x and y are connected by the equation $y = 2x^2 - 8x + 11$. Some values of x and the corresponding values of y are given in the table.

x	−1	0	1	2	3	4	5
y	21	11	5	3	q	11	21

(a) Find the value of q.

(b) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 2x^2 - 8x + 11$ for $-1 \leqslant x \leqslant 5$.

(c) Use your graph in **(b)** to find
- **(i)** the value of y when $x = 1.5$,
- **(ii)** the values of x when $y = 8$,
- **(iii)** the minimum value of y and the value of x at which this occurs.

Worked Example 9

(Drawing the Graph of $y = ax^2 + bx + c$, where $a < 0$)

The variables x and y are connected by the equation $y = 3 + 2x - x^2$. Some values of x and the corresponding values of y are given in the table.

x	−3	−2	−1	0	1	2	3	4	5
y	−12	−5	0	3	4	3	0	−5	−12

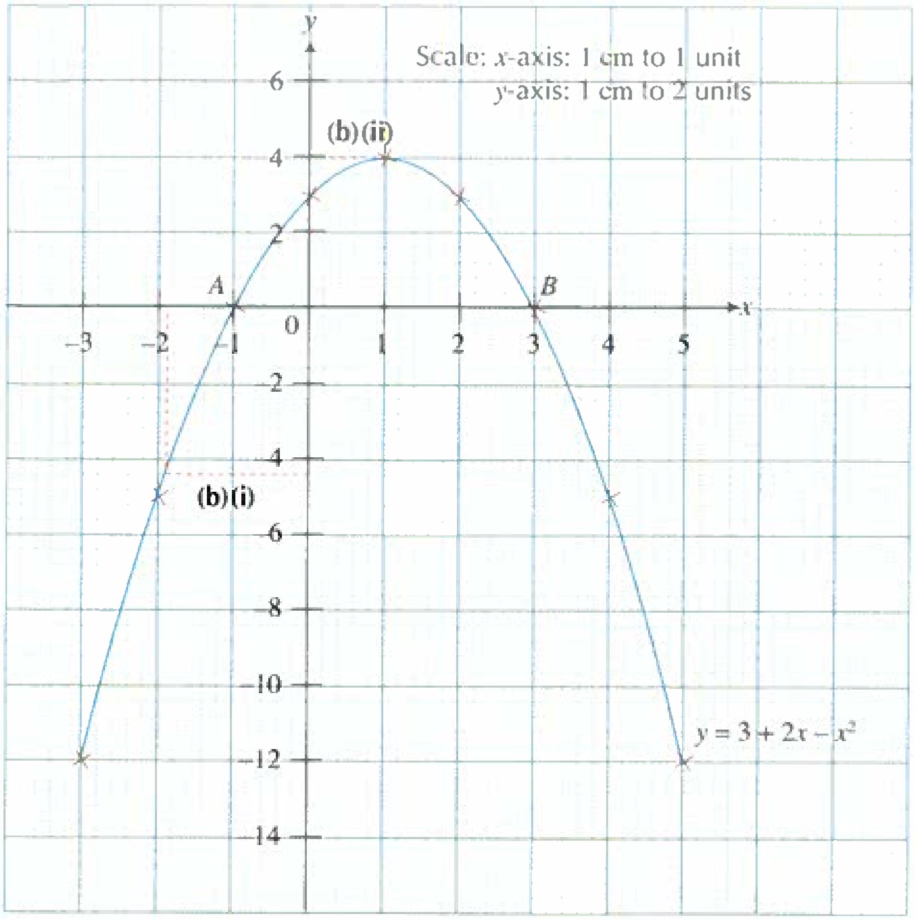
(a) On a sheet of graph paper, using a scale of 1 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = 3 + 2x - x^2$ for $-3 \leqslant x \leqslant 5$.

(b) Use your graph in **(a)** to find
- **(i)** the value of y when $x = -1.9$,
- **(ii)** the maximum value of y.

(c) State the equation of the line of symmetry of the graph.

Solution:

(a)



(b) **(i)** When $x = -1.9$,

$y \approx -4.4$

(ii) Maximum value of $y = 4$

(c) The equation of the line of symmetry of the graph is $x = 1$.

ATTENTION

The line of symmetry of the graph, i.e. $x = 1$, passes through the midpoint of AB.

PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 5C Questions 2, 4

The variables x and y are connected by the equation $y = 5 - x^2$. Some values of x and the corresponding values of y are given in the table.

x	−3	−2	−1	0	1	2	3
y	−4	1	4	5	4	1	−4

(a) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on each axis, draw the graph of $y = 5 - x^2$ for $-3 \leqslant x \leqslant 3$.

(b) Use your graph in **(a)** to find

(i) the values of x when $y = 2.75$,

(ii) the maximum value of y.

(c) State the equation of the line of symmetry of the graph.

Applications of Graphs of Quadratic Functions in Real-World Contexts

Now, we will take a look at mathematical and real-life problems that involve the graphs of quadratic functions.

Worked Example 10

(Real-Life Problem involving Graphs of Quadratic Functions)

Vishal kicks a soccer ball vertically upwards. The height, h metres, of the ball can be modelled by $h = 27t - 6t^2$, where t is the time in seconds after it leaves the ground.

(a) On a sheet of graph paper, using a scale of 2 cm to represent 1 second on the t-axis and 1 cm to represent 5 m on the h-axis, draw the graph of $h = 27t - 6t^2$ for $0 \leqslant t \leqslant 5$.

(b) Use your graph in **(a)** to find

(i) the maximum height of the ball above the ground and the time at which this occurs,

(ii) the time at which the ball will strike the ground again.

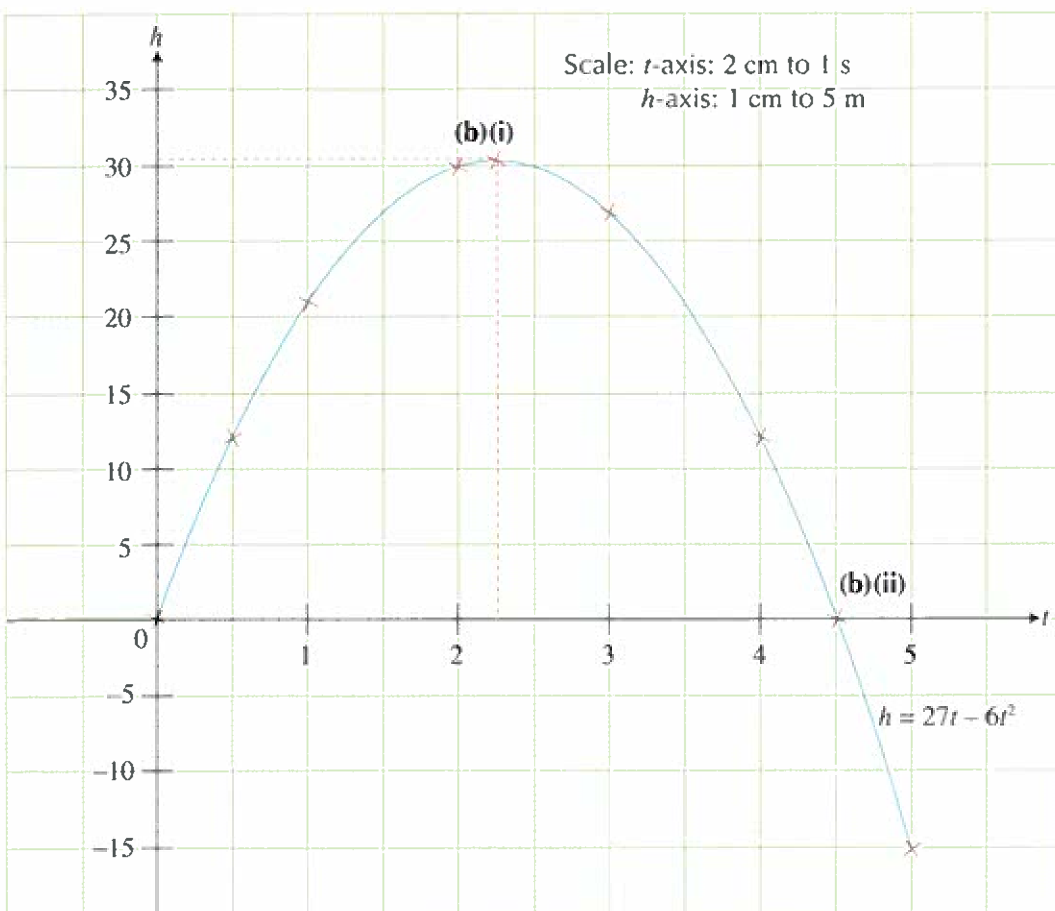
ATTENTION

The height, h metres, of the ball can be modelled by $h = 27t - 6t^2$ only when $h \geqslant 0$.

Solution:

(a)

t	0	0.5	1	2	3	4	5
h	0	12	21	30	27	12	−15



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(b) (i) Maximum height of ball ≈ 30.5 m
Time at which maximum height of ball occurs ≈ 2.25 s

(ii) The ball will strike the ground again approximately 4.5 seconds after it leaves the ground.

PRACTISE NOW 10

SIMILAR QUESTIONS

Exercise 5C Questions 5–6, 8

1. A stone is thrown vertically upwards from the top of a cliff. Its height, h metres, above level ground, can be modelled by $h = 28 + 42t - 12t^2$, where t is the time in seconds after the stone has been thrown.

(a) How high is the cliff?

(b) On a sheet of graph paper, using a scale of 2 cm to represent 1 second on the t-axis and 1 cm to represent 5 m on the h-axis, draw the graph of $h = 28 + 42t - 12t^2$ for $0 \leqslant t \leqslant 4.5$.

(c) Use your graph in **(b)** to find

(i) the maximum height of the stone above level ground and the time at which this occurs,

(ii) the time at which the stone will strike level ground.

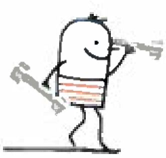
2. An object slides down from the top of a slope. Its distance travelled, s metres, can be modelled by $s = 4t + t^2$, where t is the time in seconds after it leaves the top of the slope.

(a) On a sheet of graph paper, using a scale of 2 cm to represent 1 second on the t-axis and 2 cm to represent 5 m on the s-axis, draw the graph of $s = 4t + t^2$ for $0 \leqslant t \leqslant 5$.

(b) Use your graph in **(a)** to find

(i) the distance the object travels after 2.6 seconds,

(ii) the time taken by the object to travel 30 m.



Exercise 5C

BASIC LEVEL

1. The variables x and y are connected by the equation $y = x^2 + 2x - 8$. Some values of x and the corresponding values of y are given in the table.

x	−5	−4	−3	−2	−1	1	2	3
y	7	a	−5	−8	−9	b	0	7

(a) Find the value of a and of b.

(b) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = x^2 + 2x - 8$ for $-5 \leqslant x \leqslant 3$.

(c) Use your graph in **(b)** to find

(i) the values of x when $y = 3$,

(ii) the minimum value of y.

(d) State the equation of the line of symmetry of the graph.

2. The variables x and y are connected by the equation $y = 2 - 3x - 2x^2$. Some values of x and the corresponding values of y are given in the table.

x	−4	−3	−2	−1	0	1	2
y	−18	p	0	3	2	q	−12

(a) Find the value of p and of q.

(b) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 2 - 3x - 2x^2$ for $-4 \leqslant x \leqslant 2$.

(c) Use your graph in **(b)** to find

(i) the value of y when $x = -3.5$,

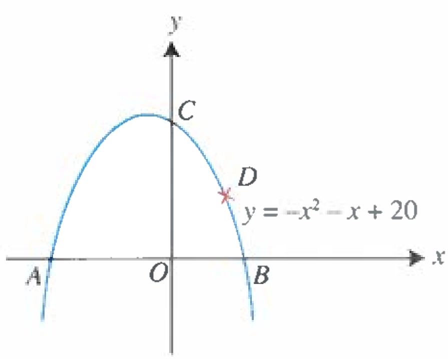
(ii) the maximum value of y and the value of x at which this occurs.

INTERMEDIATE LEVEL

3. The figure shows the curve $y = -x^2 - x + 20$.

(i) The curve cuts the x-axis at two points A and B, and the y-axis at the point C. Find the coordinates of A, B and C.

(ii) The point $D(3, h)$ lies on the curve. Find the value of h.



4. The variables x and y are connected by the equation $y = 10 - x - x^2$. Some values of x and the corresponding values of y are given in the table.

x	−4	−3	−2	−1	0	1	2	3
y	−2	4	8	10	10	8	4	−2

(a) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 10 - x - x^2$ for $-4 \leqslant x \leqslant 3$.

(b) Use your graph in **(a)** to find

(i) the value of y when $x = -1.5$,

(ii) the maximum value of y and the value of x at which this occurs.

(c) By drawing an appropriate line in your graph in **(a)**, solve the equation $10 - x - x^2 = 1.6$.

5. If a store prices each book at $\$x$, $(64 - 8x)$ books will be sold.

(i) The total amount of money earned from the sale of the books is $\$y$. Express y in terms of x.

(ii) On a sheet of graph paper, using a scale of 2 cm to represent \$1 on the x-axis and 1 cm to represent \$10 on the y-axis, draw the graph of y for $0 \leqslant x \leqslant 8$.

(iii) Use your graph in **(ii)** to find the amount at which the store should price each book such that the total amount earned is at a maximum.

6. The length of a side and the corresponding height of a triangle are $(x + 2)$ cm and $(7 - x)$ cm respectively.

(i) Write down a formula for the area, A, of the triangle in terms of x, and show that $A = 7 + \frac{5}{2}x - \frac{1}{2}x^2$.

(ii) On a sheet of graph paper, using a scale of 1 cm to represent 1 cm on the x-axis and 1 cm to represent 1 cm² on the A-axis, draw the graph of $A = 7 + \frac{5}{2}x - \frac{1}{2}x^2$ for $0 \leqslant x \leqslant 8$.

(iii) Use your graph in **(ii)** to find the length of the side of the triangle and its corresponding height that will result in its maximum area.

ADVANCED LEVEL

7. The variables x and y are connected by the equation $y = x^2 - 2x$. Some values of x and the corresponding values of y are given in the table.

x	−2	−1	0	1	2	3	4
y	8	3	0	−1	0	3	8

(a) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on each axis, draw the graph of $y = x^2 - 2x$ for $-2 \leqslant x \leqslant 4$.

(b) Use your graph in (a) to find

(i) the values of x when $y = 1$,

(ii) the minimum value of y.

(c) State the equation of the line of symmetry of the graph.

(d) By drawing an appropriate line in your graph in (a), solve the equation $x^2 - 2x = x$.

8. The value of an asset can be modelled by $y = 2x^2 - 4x + 7$, where y is the value of the asset in thousands of dollars and x is the time in years.

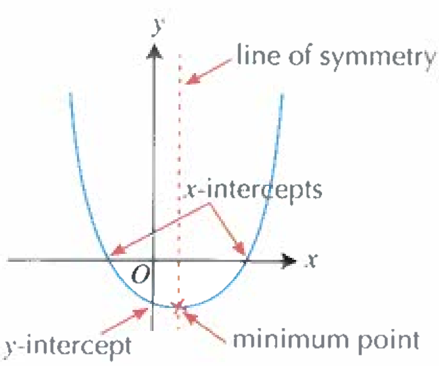
(i) On a sheet of graph paper, using a scale of 1 cm to represent 1 year on the x-axis and 1 cm to represent \$1000 on the y-axis, draw the graph of $y = 2x^2 - 4x + 7$ for $0 \leqslant x \leqslant 3$.

(ii) Use your graph in (i) to find the minimum value of the asset and the time at which it occurs.

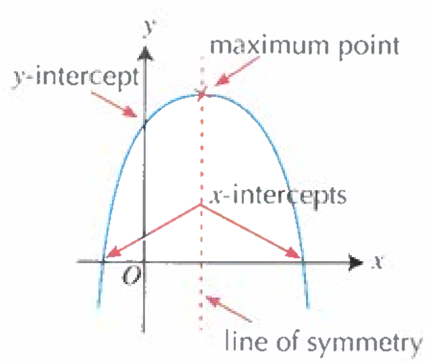
Summary

1. If two factors P and Q are such that $P \times Q = 0$, then either $P = 0$ or $Q = 0$ or both P and Q are equal to 0. This principle can be used to solve quadratic equations of the form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$.

2. The general form of the equation of a quadratic function is $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.



$a > 0$



$a < 0$

- For $a > 0$, the graph opens *upwards* indefinitely and has a **minimum point**.
- For $a < 0$, the graph opens *downwards* indefinitely and has a **maximum point**.
- The **line of symmetry** of the graph passes through its minimum / maximum point.
- The graph may have 0, 1 or 2 x-intercept(s) but it has only 1 y-intercept.

Review Exercise 5

1. Solve each of the following equations.
   (a) $-6a(-5-2a)=0$
   (b) $(4b+11)(3b-7)=0$
   (c) $4c^2=7c$
   (d) $9+6d+d^2=0$
   (e) $49f^2-140f+100=0$
   (f) $147-3h^2=0$

2. Solve each of the following equations.
   (a) $6k^2+11k-10=0$
   (b) $3m(8-3m)=16$
   (c) $(3n-1)^2=12n+8$
   (d) $(5p+1)(p+4)=2(7p+5)$

3. (i) Solve the equation $2x^2-11x-21=0$.
   (ii) Hence, solve the equation $2(y+2)^2-11(y+2)-21=0$.

4. (i) If $x=3$ is a solution of the equation $2x^2-5x+k=0$, find the value of k.
   (ii) Hence, find the other solution of the equation.

5. The difference between two numbers is 3. If the square of the smaller number is equal to four times the larger number, find the two numbers.

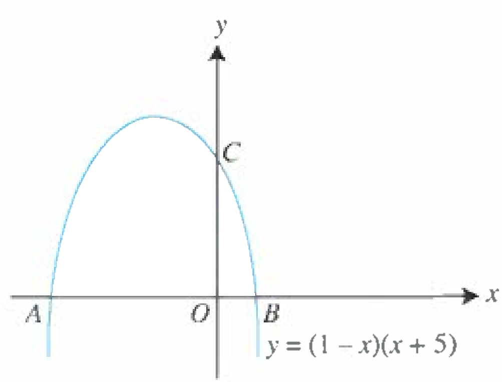
6. Ethan's father is x^2 years old while Ethan is x years old. In $4x$ years' time, Ethan's father will be twice as old as him.
   (i) Form an equation in x and show that it reduces to $x^2-6x=0$.
   (ii) Solve the equation $x^2-6x=0$.
   (iii) Find the age of Ethan's father when Ethan was born.

7. The length and the breadth of a rectangle are $(2x+5)$ cm and $(2x-1)$ cm respectively. The area of the rectangle is three times the area of a square of sides $(x+1)$ cm.
   (i) Form an equation in x and show that it reduces to $x^2+2x-8=0$.
   (ii) Solve the equation $x^2+2x-8=0$.
   (iii) Find the perimeter of the rectangle.

8. If each student in a class sends a New Year greeting card to each of his classmates, a total of 870 cards will be sent. Find the number of students in the class.

9. The height, y metres, of an object projected directly upwards from the ground can be modelled by $y=20t-5t^2$, where t is the time in seconds after it leaves the ground.
   (i) Find the height of the object 2 seconds after it leaves the ground.
   (ii) When will the object be 15 m above the ground?

10. In the figure, the curve $y=(1-x)(x+5)$ cuts the x-axis at two points A and B, and the y-axis at the point C. Find the coordinates of A, B and C.



11. The variables x and y are connected by the equation $y = x^2 + 3x - 4$. Some values of x and the corresponding values of y are given in the table.

x	−5	−4	−3	−2	−1	0	1	2
y	p	0	−4	−6	−6	−4	0	6

(a) Find the value of p.

(b) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = x^2 + 3x - 4$ for $-5 \leqslant x \leqslant 2$.

(c) Use your graph in **(b)** to find

(i) the value of y when $x = -2.7$,

(ii) the values of x when $y = 5$,

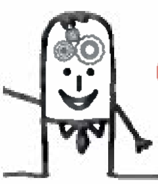
(iii) the minimum value of y.

(d) State the equation of the line of symmetry of the graph.

12. A golfer strikes a golf ball into the air from the ground. The height, h metres, of the ball can be modelled by $h = 32t - 4t^2$, where t is the time in seconds after it leaves the ground.

(i) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the t-axis and 1 cm to represent 5 units on the h-axis, draw the graph of $h = 32t - 4t^2$ for $0 \leqslant t \leqslant 8$.

(ii) Use your graph in **(i)** to find the maximum height of the ball above the ground and the time at which this occurs.



Challenge Yourself

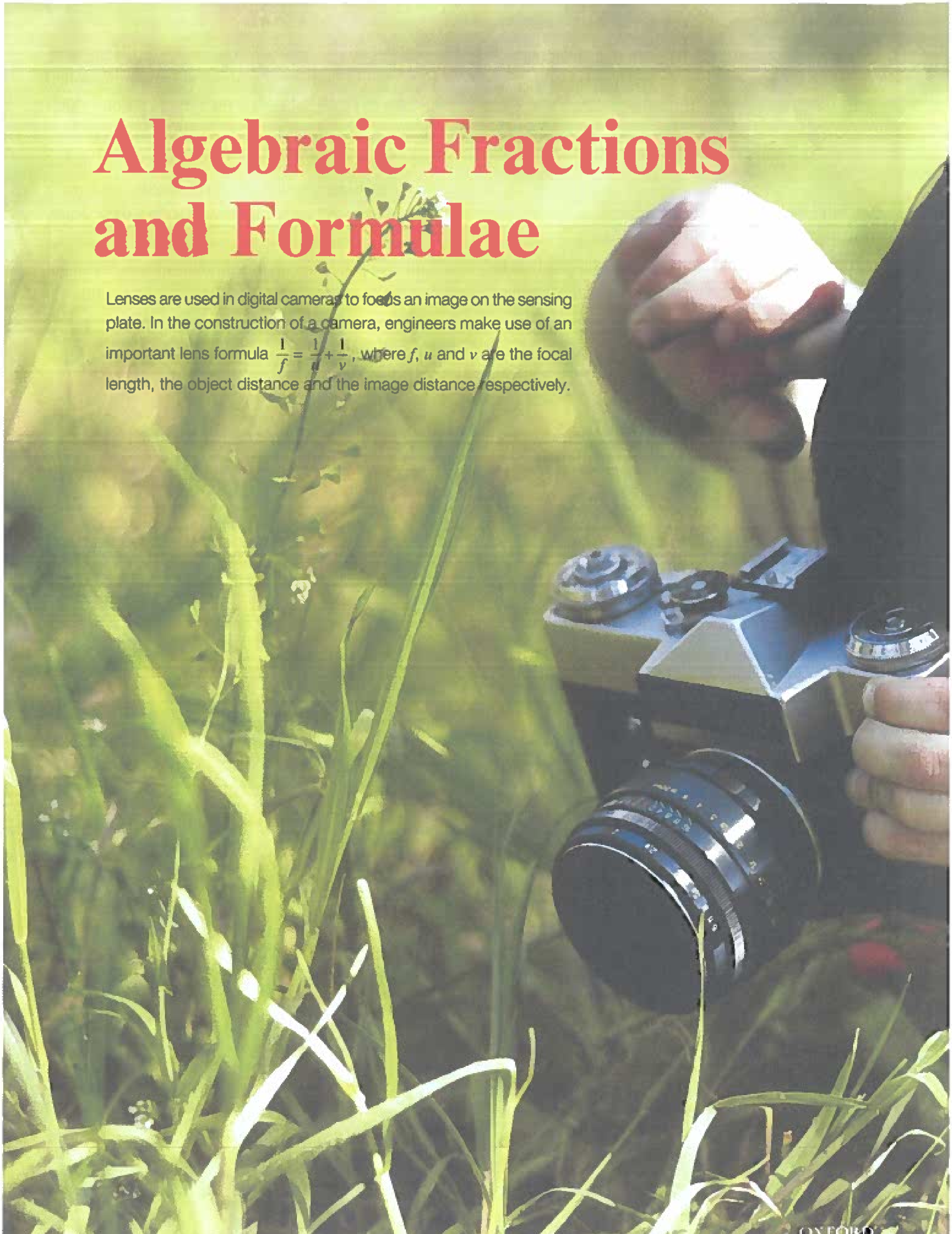
1. Given that the difference between the solutions of the equation $2x^2 - 6x - k = 0$ is 5, find the value of k.

2. **(i)** If α and β are the solutions of the quadratic equation $ax^2 + bx + c = 0$, show that $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

(ii) Hence, if p and q are the solutions of the equation $2x^2 + 10x + 7 = 0$, where $p > q$, find the value of $p - q$.

Algebraic Fractions and Formulae

Lenses are used in digital cameras to focus an image on the sensing plate. In the construction of a camera, engineers make use of an important lens formula $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$, where f, u and v are the focal length, the object distance and the image distance respectively.



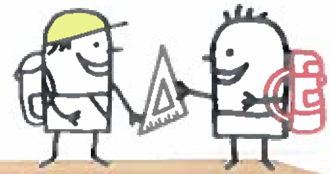
Chapter Six

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- multiply and divide simple algebraic fractions,
- add and subtract algebraic fractions with linear or quadratic denominators,
- change the subject of a formula,
- find the value of an unknown in a formula.

6.1 Algebraic Fractions



Recap (Numerical Fractions)

In Book 1, we have learnt about numerical fractions of the form $\frac{a}{b}$, where a and b are integers, and $b \neq 0$. Examples of numerical fractions are $\frac{3}{5}$, $\frac{12}{7}$ and $-\frac{5}{8}$.

In this section, we will learn about **algebraic fractions** of the form $\frac{A}{B}$, where A and/or B are algebraic expressions, and $B \neq 0$. Examples of algebraic fractions are $\frac{2a}{5}$, $\frac{16}{4b+c}$, $\frac{7d}{(2d+1)(3d-1)}$ and $\frac{3f^2}{f^2-1}$.

The rules for performing operations on algebraic fractions are the same as those for numerical fractions. One important rule is as follows:

> The value of a fraction remains unchanged if both its numerator and denominator are multiplied or divided by the same non-zero number or expression,
>
> i.e. $\frac{a}{b} = \frac{a \times c}{b \times c}$ and $\frac{a}{b} = \frac{a \div c}{b \div c}$,
>
> where $b, c \neq 0$.

Worked Example 1

(Simplification by Dividing by Common Factors)

Simplify each of the following.

(a) $\frac{3xy^3}{9x^4y^2}$ (b) $\frac{28x^3(x+y)^2}{63xy^2(x+y)^4}$

Solution:

(a) $\frac{3xy^3}{9x^4y^2} = \frac{y}{3x^3}$

Division of y^3 and y^2 by y^2

Division of x and x^4 by x

Division of 3 and 9 by 3

(b) $\frac{28x^3(x+y)^2}{63xy^2(x+y)^4} = \frac{4x^2}{9y^2(x+y)^2}$

Problem Solving Tip

We obtain the results shown in Worked Example 1 by dividing the numerators and the denominators by their common factors. The final answers should be in the simplest forms, i.e. the numerators and the denominators have no common factors except 1.

ATTENTION

In (a), $\frac{x}{x^4} = \frac{x^1}{x \times x \times x \times x} = \frac{1}{x^3}$.

PRACTISE NOW 1

Simplify each of the following.

(a) $\frac{8x^5y}{12x^3y^4}$ (b) $\frac{9x^4(x-y)^3}{27x^2y^3(x-y)}$

SIMILAR QUESTIONS

Exercise 6A Questions 1(a)–(f), 5(a)–(d)

Worked Example 2

(Simplification involving Factorisation)

Simplify each of the following.

(a) $\dfrac{a^2 + 4ab^2}{3ab}$ (b) $\dfrac{3t}{t^2 - 2t}$ (c) $\dfrac{x^2 - 3x}{3x - 9}$

Solution:

(a) $\dfrac{a^2 + 4ab^2}{3ab} = \dfrac{a(a + 4b^2)}{3ab}$ (extract the common factor a from the numerator and divide the numerator and the denominator by a)

$= \dfrac{a + 4b^2}{3b}$

(b) $\dfrac{3t}{t^2 - 2t} = \dfrac{3t}{t(t - 2)}$ (extract the common factor t from the denominator and divide the numerator and the denominator by t)

$= \dfrac{3}{t - 2}$

(c) $\dfrac{x^2 - 3x}{3x - 9} = \dfrac{x(x - 3)}{3(x - 3)}$ (extract the common factor x from the numerator and the common factor 3 from the denominator, and divide the numerator and the denominator by $(x - 3)$)

$= \dfrac{x}{3}$

ATTENTION

Divisions are usually carried out after both the numerators and the denominators have been completely factorised. **Do not** divide individual terms of the numerators and the denominators by their common factors. For example, $\dfrac{x^2 - 3x}{3x - 9} = -\dfrac{x^2}{9}$ is wrong.

PRACTISE NOW 2

Simplify each of the following.

(a) $\dfrac{h^2 + 7hk}{5hk}$ (b) $\dfrac{15p}{10p^2 - 5p}$ (c) $\dfrac{z^2 - 4z}{4z - 16}$

SIMILAR QUESTIONS

Exercise 6A Questions 2(a)–(f)

Worked Example 3

(Simplification involving Factorisation)

Simplify each of the following.

(a) $\dfrac{2m^2 - 4m}{m^2 - 4}$ (b) $\dfrac{x^2 - 3xy - 4y^2}{3x^2 - 12xy}$

(c) $\dfrac{a^2 - ac + ab - bc}{2ab + ac - 2bc - c^2}$

Solution:

(a) $\dfrac{2m^2 - 4m}{m^2 - 4} = \dfrac{2m(m - 2)}{(m + 2)(m - 2)}$ (extract the common factor $2m$ from the numerator and factorise the denominator by using $a^2 - b^2 = (a + b)(a - b)$)

$= \dfrac{2m}{m + 2}$

(b) $\dfrac{x^2 - 3xy - 4y^2}{3x^2 - 12xy} = \dfrac{(x - 4y)(x + y)}{3x(x - 4y)}$ (factorise the numerator by using the multiplication frame and extract the common factor $3x$ from the denominator)

$= \dfrac{x + y}{3x}$

(c) $\dfrac{a^2 - ac + ab - bc}{2ab + ac - 2bc - c^2} = \dfrac{a(a - c) + b(a - c)}{a(2b + c) - c(2b + c)}$ (factorise each group in the numerator and the denominator respectively)

$= \dfrac{(a - c)(a + b)}{(2b + c)(a - c)}$ (factorise the two groups in the numerator and the denominator respectively)

$= \dfrac{a + b}{2b + c}$

PRACTISE NOW 3

SIMILAR QUESTIONS

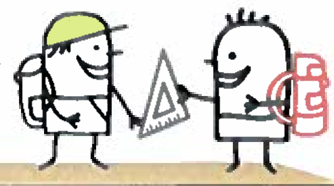
Exercise 6A Questions 3(a)–(f), 5(e)–(m), 7

1. Simplify each of the following.

(a) $\dfrac{3v^2 - 9v}{v^2 - 9}$ (b) $\dfrac{p^2 - 7pq + 12q^2}{5p^2 - 20pq}$ (c) $\dfrac{x^2 - 3xy + 2xz - 6yz}{xy - 2xz - 3y^2 + 6yz}$

2. Simplify $\dfrac{n^4 - 5n^2 + 6}{n^4 - 9}$.

6.2 Multiplication and Division of Algebraic Fractions



The procedure for the multiplication and division of algebraic fractions is similar to that of the multiplication and division of numerical fractions, except that now we have to consider the variables.

In primary school, we have learnt that $\dfrac{1}{2} \times \dfrac{3}{5} = \dfrac{1 \times 3}{2 \times 5}$.

In general, when we multiply $\dfrac{c}{d}$ by $\dfrac{a}{b}$, we have:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} = \frac{ac}{bd}$$

In primary school, we have learnt that $\dfrac{2}{3} \div \dfrac{4}{5} = \dfrac{2}{3} \times \dfrac{5}{4} = \dfrac{2 \times 5}{3 \times 4}$.

In general, when we divide $\dfrac{a}{b}$ by $\dfrac{c}{d}$, we have:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c} = \frac{ad}{bc}$$

RECALL

Dividing one fraction by another fraction is the same as multiplying the first fraction by the reciprocal of the second fraction. The reciprocal of a fraction is obtained by interchanging the numerator and the denominator of the fraction, e.g. the reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$.

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Worked Example 4

(Multiplication and Division of Algebraic Fractions)

Simplify each of the following.

(a) $\dfrac{ab}{c^2} \times \dfrac{4c}{6a^2b}$

(b) $\dfrac{p}{q} \times \dfrac{p^2r}{q^2} \div \dfrac{pr^2}{2q}$

(c) $\dfrac{4x-16}{x+y} \div \dfrac{8}{5x+5y}$

(d) $\dfrac{h}{h^2-2h+1} \times \dfrac{(h-1)^3}{2h+1}$

Solution:

(a) $\dfrac{ab}{c^2} \times \dfrac{4c}{6a^2b} = \dfrac{2}{3ac}$

(b) $\dfrac{p}{q} \times \dfrac{p^2r}{q^2} \div \dfrac{pr^2}{2q} = \dfrac{p}{q} \times \dfrac{p^2r}{q^2} \times \dfrac{2q}{pr^2}$

$= \dfrac{2p^2}{q^2r}$

(c) $\dfrac{4x-16}{x+y} \div \dfrac{8}{5x+5y} = \dfrac{4(x-4)}{x+y} \times \dfrac{5(x+y)}{8}$

$= \dfrac{5(x-4)}{2}$

(d) $\dfrac{h}{h^2-2h+1} \times \dfrac{(h-1)^3}{2h+1} = \dfrac{h}{(h-1)^2} \times \dfrac{(h-1)^3}{2h+1}$ (factorise $h^2 - 2h + 1$ by using $a^2 - 2ab + b^2 = (a - b)^2$)

$= \dfrac{h(h-1)}{2h+1}$

PRACTISE NOW 4

SIMILAR QUESTIONS

Exercise 6A Questions 4(a)–(d), 6(a)–(l)

1. Simplify each of the following.

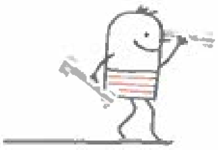
(a) $\dfrac{2a^2}{5c^3} \times \dfrac{15c}{8a^4}$

(b) $\dfrac{3p^2}{15q^3} \times \dfrac{35qr^2}{12pr} \div \dfrac{5r^4}{6p^3q^2}$

(c) $\dfrac{2x-6}{5x+5y} \div \dfrac{3}{7y+7x}$

(d) $\dfrac{h^2-6h+9}{h^2-2h} \times \dfrac{h-2}{h-3}$

2. Simplify $\dfrac{3m-n}{n+m} \div \dfrac{2n-6m}{m+n}$.



Exercise 6A

BASIC LEVEL

1. Simplify each of the following.

(a) $\frac{4x^4}{12x^5y}$ (b) $\frac{16a^3b^4}{24a^5b^2}$

(c) $\frac{23q^3r}{69qr^3s}$ (d) $\frac{3mn^2p^3}{18m^3np^6}$

(e) $\frac{15ac^3}{75a^2b^4c}$ (f) $\frac{16xy^3z}{64x^2y^4z^3}$

2. Simplify each of the following.

(a) $\frac{xy+3y}{4x+12}$ (b) $\frac{8a+4b}{bc+2ac}$

(c) $\frac{a^2+2ab}{6a}$ (d) $\frac{c^2}{c^2-cd}$

(e) $\frac{(m-n)^2}{m^2-mn}$ (f) $\frac{5pq}{15p-10pq}$

3. Simplify each of the following.

(a) $\frac{2a+b}{4a^2-b^2}$ (b) $\frac{c^2+2cd-15d^2}{4c^2+20cd}$

(c) $\frac{3a-6}{a^2+a-6}$ (d) $\frac{x^2+6x-7}{x^2-x}$

(e) $\frac{k^2-9}{k^2-7k+12}$ (f) $\frac{mk+8k}{m^2+4m-32}$

4. Simplify each of the following.

(a) $\frac{15a^2}{8ab^3c} \times \frac{4c}{5ab}$ (b) $\frac{3(c+d)}{c-d} \times \frac{2c-2d}{8c+8d}$

(c) $\frac{a-2b}{16} + \frac{4a-8b}{24}$ (d) $\frac{8c^3}{6(c+d)} + \frac{2c^2}{3c+3d}$

INTERMEDIATE LEVEL

5. Simplify each of the following.

(a) $\frac{9x(a-b)^2}{27x^3(a-b)^3}$ (b) $\frac{7a^3(a-3b)^4}{21a^2b(a-3b)^2}$

(c) $\frac{8ab^3(2a+3b)^2}{32a^2b(3b+2a)}$ (d) $\frac{8an^3(b+c)}{96a^2n(c+b)^2}$

(e) $\frac{y^2-2y-15}{y^2-3y-10}$ (f) $\frac{8-2m-m^2}{2m^2-3m-2}$

(g) $\frac{9x^2-y^2}{y^2-2xy-3x^2}$ (h) $\frac{3x^2+5xy-2y^2}{4x^2+7xy-2y^2}$

(i) $\frac{b^2-a^2}{2a^2+ab-3b^2}$ (j) $\frac{y^2-6y-7}{2y^2-17y+21}$

(k) $\frac{3x-3y}{ax-ay-x+y}$ (l) $\frac{a^2-ab-ac+bc}{a^2+ab-ac-bc}$

(m) $\frac{a^2+am-an-mn}{a^2+am+an+mn}$

6. Simplify each of the following.

(a) $\frac{5a}{9b} \times \frac{ac^2}{2b} \times \frac{c^3}{8b^4}$ (b) $\frac{6df}{9f^3} \times \frac{3f}{16d^2} + \frac{8d^3f^2}{27d}$

(c) $2y + \frac{4y}{5xy} \times \frac{64xy}{100x^3y^4}$ (d) $\frac{3ps}{4pqr} + \frac{3pq^2}{12p^3q} \times \frac{14s^3}{7qr}$

(e) $\frac{3w-7}{5w^3} + \frac{21-9w}{27w}$ (f) $\frac{6x^2y}{16y-8x} \times \frac{12x-24y}{4xy^2}$

(g) $\frac{h^2-h-6}{h^2-9} \times \frac{h^2}{h^2+2h}$ (h) $\frac{c^2-d^2}{c^2-2cd+d^2} + \frac{1}{cd+d^2}$

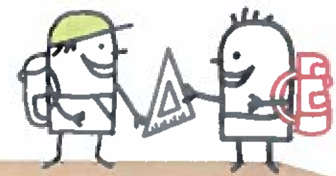
(i) $\frac{m^2-4}{m^2-3m+2} + \frac{m}{m-1}$ (j) $\frac{z^2}{z^2-4} + \frac{3z-z^2}{z^2-5z+6}$

(k) $\left(a^2-4b^2\right) + \frac{a^2+2ab}{ab}$ (l) $\frac{y^2-4y+4}{2-6y} \times \frac{2y+4}{3y^2-12}$

ADVANCED LEVEL

7. Simplify $\frac{x^2+y^2-z^2+2xy}{x^2-y^2-z^2-2yz}$.

6.3 Addition and Subtraction of Algebraic Fractions



Recap (Simplification of Linear Expressions with Fractional Coefficients)

In Book 1, we have learnt how to simplify linear expressions with fractional coefficients. For example,

$$\frac{x}{3}+\frac{2x-5}{7} = \frac{7x}{21}+\frac{3(2x-5)}{21} \quad \text{(convert to like fractions: Lowest Common Multiple (LCM) of 3 and 7 is 21)}$$

$$= \frac{7x+3(2x-5)}{21} \quad \text{(combine into a single fraction)}$$

$$= \frac{7x+6x-15}{21} \quad \text{(Distributive Law)}$$

$$= \frac{13x-15}{21}.$$

In this section, we will learn how to add and subtract algebraic fractions.

Addition and Subtraction of Algebraic Fractions

Worked Example 5

(Addition and Subtraction of Algebraic Fractions)

Express each of the following as a fraction in its simplest form.

(a) $\frac{2}{3a}+\frac{3}{5a}$ **(b)** $\frac{3}{2b-4c}+\frac{2}{3b-6c}$

(c) $\frac{2h}{h-2k}-\frac{3k}{2k-h}$

Solution:

(a) $\frac{2}{3a}+\frac{3}{5a} = \frac{10}{15a}+\frac{9}{15a}$ (convert to like fractions: LCM of $3a$ and $5a$ is $15a$)

$= \frac{10+9}{15a}$ (combine into a single fraction)

$= \frac{19}{15a}$

(b) $\frac{3}{2b-4c}+\frac{2}{3b-6c} = \frac{3}{2(b-2c)}+\frac{2}{3(b-2c)}$ (factorise the denominators)

$= \frac{9}{6(b-2c)}+\frac{4}{6(b-2c)}$ (convert to like fractions: LCM of $2(b-2c)$ and $3(b-2c)$ is $6(b-2c)$)

$= \frac{9+4}{6(b-2c)}$ (combine into a single fraction)

$= \frac{13}{6(b-2c)}$

(c) $\frac{2h}{h-2k} - \frac{3k}{2k-h} = \frac{2h}{h-2k} - \frac{3k}{-(h-2k)}$

$= \frac{2h}{h-2k} + \frac{3k}{h-2k}$

$= \frac{2h+3k}{h-2k}$ (combine into a single fraction)

ATTENTION

$-\frac{3k}{-(h-2k)} = \frac{3k}{h-2k}$

PRACTISE NOW 5

SIMILAR QUESTIONS

Exercise 6B Questions 1(a)–(f), 3(a)–(h)

1. Express each of the following as a fraction in its simplest form.

(a) $\frac{6}{5a} + \frac{3}{8a}$ (b) $\frac{4}{2b+3c} - \frac{7}{6b+9c}$ (c) $\frac{h}{2-3k} - \frac{3h}{3k-2}$

2. Express each of the following as a fraction in its simplest form.

(a) $\frac{2m+3n}{3m} - \frac{m-n}{2n}$ (b) $\frac{3p}{4p-4q} - \frac{5p-2q}{3p-3q}$ (c) $\frac{5x}{4x-3y} - \frac{7y}{6y-8x}$

Worked Example 6

(Addition and Subtraction of Algebraic Fractions with No Common Factors in the Denominators)

Express each of the following as a fraction in its simplest form.

(a) $\frac{3}{x+5} + \frac{1}{x-3}$ (b) $\frac{5y}{y^2-4} - \frac{2}{y-2}$

Solution:

(a) $\frac{3}{x+5} + \frac{1}{x-3} = \frac{3(x-3)}{(x+5)(x-3)} + \frac{x+5}{(x+5)(x-3)}$ (convert to like fractions: LCM of $x+5$ and $x-3$ is $(x+5)(x-3)$)

$= \frac{3(x-3)+x+5}{(x+5)(x-3)}$ (combine into a single fraction)

$= \frac{3x-9+x+5}{(x+5)(x-3)}$ (Distributive Law)

$= \frac{4x-4}{(x+5)(x-3)}$

(b) $\frac{5y}{y^2-4} - \frac{2}{y-2} = \frac{5y}{(y+2)(y-2)} - \frac{2}{y-2}$ (factorise the denominator y^2-4 by using $a^2-b^2=(a+b)(a-b)$)

$= \frac{5y}{(y+2)(y-2)} - \frac{2(y+2)}{(y+2)(y-2)}$ (convert to like fractions: LCM of $(y+2)(y-2)$ and $y-2$ is $(y+2)(y-2)$)

$= \frac{5y-2(y+2)}{(y+2)(y-2)}$ (combine into a single fraction)

$= \frac{5y-2y-4}{(y+2)(y-2)}$ (Distributive Law)

$= \frac{3y-4}{(y+2)(y-2)}$

PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 6B Questions 2(a)–(h), 4(a)–(f), 5(a)–(d), 6

Express each of the following as a fraction in its simplest form.

(a) $\frac{2}{x+1} - \frac{3}{2x-5}$ (b) $\frac{2}{y^2-9} - \frac{y}{y-3}$ (c) $\frac{1}{z+5} - \frac{1}{z-5} + \frac{2z}{z^2-25}$

Exercise 6B

BASIC LEVEL

1. Express each of the following as a fraction in its simplest form.

   (a) $\frac{7}{6a}+\frac{4}{9a}$ (b) $\frac{3}{2b}+\frac{1}{3b}-\frac{5}{6b}$

   (c) $\frac{1}{3c}-\frac{1}{3d}$ (d) $\frac{f-4h}{3k}-\frac{2f-5h}{8k}$

   (e) $\frac{4a}{x-3y}+\frac{3a}{3x-9y}$ (f) $\frac{p+3}{2z}+\frac{p-1}{6z}-\frac{2p+1}{3z}$

2. Express each of the following as a fraction in its simplest form.

   (a) $\frac{5}{a}+\frac{3}{a+4}$ (b) $\frac{1}{2b}-\frac{3}{b+c}$

   (c) $\frac{4}{d-5}+\frac{2}{2d+3}$ (d) $\frac{2}{f+5}-\frac{3}{f-1}$

   (e) $\frac{11}{3h-7}+\frac{2}{6-5h}$ (f) $\frac{3}{k^2-1}+\frac{2}{k-1}$

   (g) $\frac{3}{4m^2-1}-\frac{5}{2m+1}$ (h) $\frac{2}{n-2}+\frac{3}{(n-2)^2}$

INTERMEDIATE LEVEL

3. Express each of the following as a fraction in its simplest form.

   (a) $\frac{5}{2(a-b)}+\frac{4}{3(b-a)}$ (b) $\frac{c-1}{3c-7}-\frac{1}{14-6c}$

   (c) $\frac{4f}{10f-5d}+\frac{2d}{6f-3d}$ (d) $\frac{u+1}{2u-8}-\frac{u+2}{12-3u}$

   (e) $\frac{2m-5}{9n-6}-\frac{m+3}{4-6n}$ (f) $\frac{h+k}{p-q}+\frac{3h+k}{8q-8p}$

   (g) $\frac{5x^2}{6x-6y}-\frac{2x^2}{3y-3x}$ (h) $\frac{3x}{4y-2z}-\frac{2x}{z-2y}+\frac{5}{3z-6y}$

4. Express each of the following as a fraction in its simplest form.

   (a) $\frac{3a}{3a-5}+\frac{4a}{4a-1}$ (b) $\frac{5}{2b+1}-\frac{2b}{(2b+1)^2}$

   (c) $\frac{h+5}{h^2-6h}-\frac{3}{h-6}$ (d) $\frac{1}{m}+\frac{2}{m-4}+\frac{3}{m-3}$

   (e) $\frac{x+y}{x-y}+\frac{x^2-4y^2}{x^2-y^2}-\frac{x-3y}{x+y}$

   (f) $\frac{1}{2z-3}-\frac{2}{3-2z}+\frac{18}{9-4z^2}$

5. Express each of the following as a fraction in its simplest form.

   (a) $\frac{2}{a+3}+\frac{3}{a^2+4a+3}$

   (b) $\frac{1}{b^2-5b-6}-\frac{b}{b-6}$

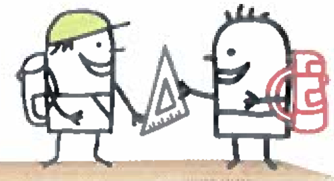
   (c) $\frac{1}{2p^2-8p-10}+\frac{2p}{p-5}$

   (d) $\frac{x}{x+y}+\frac{4}{x^2+3xy+2y^2}-\frac{3x}{x+2y}$

ADVANCED LEVEL

6. Express $\dfrac{\frac{1}{3x}+\frac{2}{y}}{\frac{2}{x}}$ as a fraction in its simplest form.

6.4 Manipulation of Algebraic Formulae



Changing the Subject of a Formula

In Book 1, we have learnt that in general, a formula expresses a rule in algebraic terms by making use of variables to write instructions for performing a calculation. For example, the perimeter P of a rectangle is given by $P = 2l + 2b$, where l and b represent the length and the breadth of the rectangle respectively. Are we able to find an expression for the length, l, of the rectangle in terms of P and b?

Worked Example 7

(Changing the Subject of a Formula)

(i) Make l the subject of the formula $P = 2l + 2b$.

(ii) Hence, calculate the value of l when $P = 132$ and $b = 30$.

Solution:

(i)

$$P = 2l + 2b$$
$$P - 2b = 2l + 2b - 2b \text{ (subtract } 2b \text{ from both sides)}$$
$$P - 2b = 2l$$
$$\frac{P - 2b}{2} = \frac{2l}{2} \text{ (divide by 2 on both sides)}$$
$$\therefore l = \frac{P - 2b}{2}$$

(ii) When $P = 132$, $b = 30$,

$$l = \frac{132 - 2(30)}{2}$$
$$= \frac{132 - 60}{2}$$
$$= \frac{72}{2}$$
$$= 36$$

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 6C Questions 1(a)–(d), 7(a)–(d)

1. (i) Make a the subject of the formula $v = u + at$.
   (ii) Hence, find the value of a when $t = 4$, $u = 10$ and $v = 50$.
2. The simple interest \$$I$ payable on an investment is given by $I = \frac{PRT}{100}$, where \$$P$ is the principal, R% is the interest rate on the investment per annum and T is the number of years that the investment is held.
   (i) Make T the subject of the formula $I = \frac{PRT}{100}$.
   (ii) Hence, find the number of years that an initial investment of \$50 000 must be held in a bank that pays simple interest at a rate of 2% per annum to earn an interest of \$4000.

Worked Example 8

(Changing the Subject of a Formula)

(i) Make x the subject of the formula $y = \dfrac{2-x}{3+2x}$.

(ii) Hence, calculate the value of x when $y = -2$.

Solution:

(i)

$$y = \frac{2-x}{3+2x}$$

$$(3+2x) \times y = (3+2x) \times \frac{2-x}{3+2x} \text{ (multiply by } 3+2x \text{ on both sides)}$$

$$y(3+2x) = 2-x$$

$$3y + 2xy = 2 - x \text{ (Distributive Law)}$$

$$3y + 2xy + x = 2 - x + x \text{ (add } x \text{ to both sides)}$$

$$3y + 2xy + x = 2$$

$$3y - 3y + 2xy + x = 2 - 3y \text{ (subtract } 3y \text{ from both sides)}$$

$$2xy + x = 2 - 3y$$

$$x(2y+1) = 2 - 3y$$

$$\frac{x(2y+1)}{2y+1} = \frac{2-3y}{2y+1} \text{ (divide by } 2y+1 \text{ on both sides)}$$

$$\therefore x = \frac{2-3y}{2y+1}$$

(ii) When $y = -2$,

$$x = \frac{2-3(-2)}{2(-2)+1}$$

$$= \frac{2+6}{-4+1}$$

$$= \frac{8}{-3}$$

$$= -2\frac{2}{3}$$

Problem Solving Tip

To change the subject of a formula involving algebraic fractions, we carry out the steps as follows:

Step 1: Eliminate the fractions.

Step 2: Manipulate the equation such that all the terms with the unknown which we need to express as the subject are on the left-hand side (LHS) of the equation.

Step 3: Factorise the expression on the LHS of the equation, if necessary.

Step 4: Divide both sides of the equation such that only the subject of the formula remains on the LHS.

PRACTISE NOW 8

1. **(i)** Make x the subject of the formula $y = \dfrac{2x+5}{3x-7}$.

   (ii) Hence, find the value of x when $y = -3$.

2. **(i)** Make k the subject of the formula $p = a + \dfrac{bx^2}{3k}$.

   (ii) Hence, find the value of k when $a = 1$, $b = -2$, $p = 3$ and $x = 9$.

SIMILAR QUESTIONS

Exercise 6C Questions 2(a)–(d), 8(a)–(d), 10, 15

Worked Example 9

(Changing the Subject of a Formula)

(i) Make x the subject of the formula $\sqrt[3]{ax+b} = k$.

(ii) Hence, calculate the value of x when $a = 4$, $b = 3$ and $k = -1$.

Solution:

(i)

$$\sqrt[3]{ax+b} = k$$

$$ax + b = k^3 \text{ (take the cube on both sides)}$$

$$ax + b - b = k^3 - b \text{ (subtract } b \text{ from both sides)}$$

$$ax = k^3 - b$$

$$\frac{ax}{a} = \frac{k^3-b}{a} \text{ (divide by } a \text{ on both sides)}$$

$$\therefore x = \frac{k^3-b}{a}$$

(ii) When $a = 4$, $b = 3$, $k = -1$,

$$x = \frac{(-1)^3 - 3}{4}$$
$$= \frac{-1-3}{4}$$
$$= \frac{-4}{4}$$
$$= -1$$

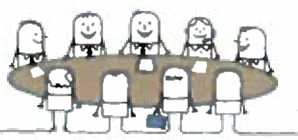
PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 6C Questions 3(a)–(d), 9(a)–(d), 11, 16–17, 20

1. (i) Make x the subject of the formula $3y = \sqrt{b^2 - 4ax}$.
   (ii) Hence, find the value of x when $a = -5$, $b = 4$ and $y = 2$.
2. (i) Make x the subject of the formula $p = a + \frac{bx^2}{3k}$.
   (ii) Hence, find the value of x when $a = -1$, $b = 2$, $k = 1$ and $p = 5$.

Finding the Value of an Unknown in a Formula



Class Discussion

Finding the Value of an Unknown in a Formula

In solving problems, we often need to change the subject of a formula to make it easier to find the value of an unknown.

For example, if the volume V of a cylinder is 780 cm^3 and its height h is 10 cm, how can we find the base radius r of the cylinder?

We first make r the subject of the formula $V = \pi r^2 h$ before substituting in the values of V and h.

1. Can we substitute the values of V and h into the formula $V = \pi r^2 h$ before solving the equation to find the value of r?
2. Are the values of r obtained the same? Explain your answer.

From the class discussion, we observe that to find the value of an unknown, we are able to substitute the given values into a formula without changing the subject of the formula.

Worked Example 10

(Finding the Value of an Unknown in a Formula Without Changing the Subject of the Formula)

Given that $y = \sqrt{\dfrac{64}{3x+1}}$, calculate

(a) the value of y when $x = 1$,
(b) the value of x when $y = 2$.

Solution:

(a) When $x = 1$,

$$y = \sqrt{\frac{64}{3(1)+1}}$$
$$= \sqrt{\frac{64}{3+1}}$$
$$= \sqrt{\frac{64}{4}}$$
$$= \sqrt{16}$$
$$= 4$$

(b) When $y = 2$,

$$2 = \sqrt{\frac{64}{3x+1}}$$
$$4 = \frac{64}{3x+1} \quad \text{(take the square on both sides)}$$
$$4(3x+1) = 64$$
$$3x + 1 = 16$$
$$3x = 15$$
$$\therefore x = 5$$

PRACTISE NOW 10

SIMILAR QUESTIONS

Exercise 6C Questions 4–5, 12–13, 19

1. Given that $y = \sqrt{\dfrac{x+7}{x-2}}$, find

   (a) the value of y when $x = 5$,
   (b) the value of x when $y = 4$.

2. Given that $a = \sqrt[3]{\dfrac{5b+16}{2b-23}}$, find the value of b when $a = 3$.

3. Given that $\sqrt[3]{\dfrac{x+y}{x-y}} = z$, find the value of x when $y = 4$ and $z = 3$.

Equations involving Algebraic Fractions

We shall learn how to solve equations involving algebraic fractions using the method of changing the subject of a formula.

Worked Example 11

(Solving Equations involving Algebraic Fractions)

Solve each of the following equations.

(a) $\frac{a-2}{5}+\frac{a-1}{3}=1$

(b) $\frac{6}{2b-5}-\frac{4}{b-3}=0$

Solution:

(a) $\frac{a-2}{5}+\frac{a-1}{3}=1$

$3(a-2)+5(a-1)=15$ (multiply by the LCM of 5 and 3, i.e. 15)

$3a-6+5a-5=15$

$8a-11=15$

$8a=26$

$\therefore a=3\frac{1}{4}$

(b) $\frac{6}{2b-5}-\frac{4}{b-3}=0$

$6(b-3)-4(2b-5)=0$ (multiply by the LCM of $2b-5$ and $b-3$, i.e. $(2b-5)(b-3)$)

$6b-18-8b+20=0$

$-2b+2=0$

$-2b=-2$

$2b=2$

$\therefore b=1$

PRACTISE NOW 11

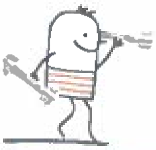
Solve each of the following equations.

(a) $\frac{a-3}{5}+\frac{2a-1}{3}=4$

(b) $\frac{3}{2b+3}-\frac{5}{3b-4}=0$

SIMILAR QUESTIONS

Exercise 6C Questions 6(a)–(g), 14(a)–(e), 18



Exercise 6C

BASIC LEVEL

1. In each of the following cases, make the letter in the brackets the subject of the formula.

(a) $ax + by = k$ $[y]$

(b) $PV = nRT$ $[n]$

(c) $5b - 2d = 3c$ $[d]$

(d) $R = m(a + g)$ $[a]$

2. In each of the following cases, make the letter in the brackets the subject of the formula.

(a) $\frac{a}{m} = b + c$ $[a]$

(b) $5q - r = \frac{2p}{3}$ $[p]$

(c) $\frac{k + a}{5} = 3k$ $[k]$

(d) $A = \frac{1}{2}(a + b)h$ $[b]$

3. In each of the following cases, make the letter in the brackets the subject of the formula.

(a) $\sqrt[3]{h - k} = m$ $[h]$

(b) $b = \sqrt{D + 4ac}$ $[D]$

(c) $P = \frac{V^2}{R}$ $[V]$

(d) $A = \frac{\theta}{360} \times \pi r^2$ $[\theta]$

4. Given that $\sqrt{ax^2 - b} = c$, find the values of x when $a = 2$, $b = 7$ and $c = 5$.

5. Given that $a = \sqrt{\frac{3b + c}{b - c}}$, find

(a) the value of a when $b = 7$ and $c = 2$,

(b) the value of c when $a = 4$ and $b = 9$.

6. Solve each of the following equations.

(a) $\frac{a}{a + 2} = \frac{3}{5}$

(b) $\frac{1}{b - 2} = \frac{2}{b - 1}$

(c) $\frac{4}{c + 3} - \frac{3}{c + 2} = 0$

(d) $\frac{5}{d + 4} - \frac{2}{d - 2} = 0$

(e) $\frac{6}{f} - \frac{10}{3f} = 2$

(f) $\frac{5}{6h} + \frac{6}{7h} - \frac{9}{14h} = 4$

(g) $\frac{3}{k + 1} - \frac{1}{2k + 2} = 5$

INTERMEDIATE LEVEL

7. In each of the following cases, make the letter in the brackets the subject of the formula.

(a) $F = \frac{9}{5}C + 32$ $[C]$

(b) $A = 2\pi r^2 + \pi rl$ $[l]$

(c) $s = ut + \frac{1}{2}at^2$ $[u]$

(d) $S = \frac{n}{2}[2a + (n - 1)d]$ $[d]$

8. In each of the following cases, make the letter in the brackets the subject of the formula.

(a) $\frac{1}{h + 1} + 2 = k$ $[h]$

(b) $z = \frac{y(z - y)}{x}$ $[z]$

(c) $\frac{px}{q} = p + q$ $[p]$

(d) $\frac{1}{a} + \frac{1}{b} = 1$ $[b]$

9. In each of the following cases, make the letter in the brackets the subject of the formula.

(a) $V = \frac{4}{3}\pi r^3$ $[r]$

(b) $v^2 = u^2 + 2as$ $[u]$

(c) $y = (x - p)^2 + q$ $[x]$

(d) $t = \sqrt{\frac{4z}{m - 3}}$ $[z]$

10. Given that $V = \pi r^2 h + \frac{2}{3}\pi r^3$,

(i) make h the subject of the formula,

(ii) find the value of h when $V = 245$ and $r = 7$.

11. Given that $a = \sqrt{\frac{3b + c}{b - c}}$,

(i) make b the subject of the formula,

(ii) find the value of b when $a = 2$ and $c = 5$.

12. Given that $\frac{m(nx - y^2)}{p} = 3n$, find

(a) the value of p when $m = 5$, $n = 7$, $x = 4$ and $y = -2$,

(b) the value of n when $m = 14$, $p = 9$, $x = 2$ and $y = 3$,

(c) the values of y when $m = 5$, $n = 4$, $p = 15$ and $x = 42$.

13. Given that $A = \frac{1}{3}\pi r^2 h + \frac{4}{3}\pi r^3$, find

(a) the value of A when $\pi = 3.142$, $h = 15$ and $r = 7$,

(b) the value of h when $\pi = 3.142$, $A = 15\,400$ and $r = 14$.

14. Solve each of the following equations.

(a) $2 - \frac{5}{x+2} = 1\frac{3}{5}$

(b) $\frac{7x-4}{15} + \frac{x-1}{3} = \frac{3x-1}{5} - \frac{7+x}{10}$

(c) $\frac{x+1}{5x-1} + \frac{1}{2(1-5x)} = \frac{1}{4}$

(d) $\frac{5}{2x-1} - \frac{4}{4x-2} - \frac{3}{6x-3} = 1$

(e) $\frac{3}{2-x} + \frac{5}{4-2x} - \frac{1}{x-2} = 4$

15. In optics, the focal length, f cm, of a lens, can be calculated using the formula $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$, where u cm and v cm are the object distance and the image distance from the centre of the lens respectively.

(i) Make v the subject of the formula $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$.

(ii) Hence, find the image distance when the focal length of the lens is 20 cm and the object distance is 30 cm.

16. The time taken, T seconds, for a pendulum to complete one oscillation is given by the formula $T = 2\pi\sqrt{\frac{l}{g}}$, where l m is the length of the pendulum and g is taken to be 10 m s^{-2}.

(i) Make l the subject of the formula $T = 2\pi\sqrt{\frac{l}{g}}$.

(ii) Hence, find the length of the pendulum if it takes 12 seconds to complete 20 oscillations.

17. The amount of energy, E joules (J), stored in an object with a mass of m kg is given by the formula $E = mgh + \frac{1}{2}mv^2$, where h m is the height of the object above the ground, v m s^{-1} is the velocity of the object and g is taken to be 10 m s^{-2}.

(i) Given that $v \geq 0$, make v the subject of the formula $E = mgh + \frac{1}{2}mv^2$.

(ii) Hence, find the velocity of an object with a mass of 0.5 kg, if it is 2 m above the ground and has 100 J of energy.

ADVANCED LEVEL

18. Given that $\frac{\frac{1}{x} + \frac{1}{y}}{\frac{2}{x}} = \frac{4}{3}$, find the value of $\frac{y}{x}$.

19. Given that $y = 3x + \sqrt[3]{a + b^2}$, find
(a) the value of y when $a = 13$, $b = 15$ and $x = 3.8$,
(b) the value of a when $b = 13$, $x = 8.5$ and $y = 35$,
(c) the values of b when $a = 23$, $x = 15.6$ and $y = 56$.

20. The resistance of a wire, R ohms (Ω), is directly proportional to its length, l m, and inversely proportional to the square of its radius, r m.
(a) Express R in terms of l, r and a constant k.
(b) Make r the subject of the formula in (a).
(c) (i) Given that a wire with a length of 2 m and a radius of 0.5 cm has a resistance of 40 Ω, use your answer in (a) to find the value of k.
(ii) Hence, use your answer in (b) to find the radius of a wire with a length of 4 m and a resistance of 60 Ω.

1. The value of a fraction remains unchanged if both its numerator and denominator are multiplied or divided by the same non-zero number or expression,

i.e. $\frac{a}{b} = \frac{a \times c}{b \times c}$ and $\frac{a}{b} = \frac{a \div c}{b \div c}$,

where $b, c \neq 0$.

2. When we multiply $\frac{c}{d}$ by $\frac{a}{b}$, we have:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} = \frac{ac}{bd},$$

where $b, d \neq 0$.

When we divide $\frac{a}{b}$ by $\frac{c}{d}$, we have:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c} = \frac{ad}{bc},$$

where $b, c, d \neq 0$.

1. Simplify each of the following.

(a) $\frac{24x^2y^2}{6x^3y}$

(b) $\frac{x^2(c-d)(c+3d)}{5x(c-d)^2}$

(c) $\frac{5a+10b}{10ac+20bc}$

(d) $\frac{3f+4d}{(6f+8d)^2}$

(e) $\frac{p^2-4p+4}{p^2-2p}$

(f) $\frac{3r-qs+3s-qr}{9+q^2-6q}$

(g) $\frac{3w^2+8w+4}{3w^2-12}$

(h) $\frac{z^2+7z+6}{3z^2+9z+6}$

2. Simplify each of the following.

(a) $\frac{3a^2b}{4c} \times \frac{c^3}{ab}$

(b) $\frac{27}{d-3f} + \frac{6}{9f-3d}$

(c) $\frac{10k}{k-5} \times \frac{3k^2-15k}{5k^2}$

(d) $\frac{4x-1}{4x-4y} \div \frac{1}{9x-9y}$

(e) $\frac{mn-4m+2n^2-8n}{m+2n} \div \frac{4-n}{n}$

(f) $\frac{7(p-q)^2}{pq} \times \frac{2p^3q^2}{(p-q^2)^2}$

(g) $\frac{3x}{x^2-2x+1} \times \frac{2}{x^2} \div \frac{1}{x(x-1)}$

(h) $\frac{y^2-16}{2y^2-12y+16} \div \frac{2}{y-2}$

3. Express each of the following as a fraction in its simplest form.

(a) $\frac{5}{4a} - \frac{3}{8a}$

(b) $\frac{5b}{3b+2c} + \frac{b}{9b+6c}$

(c) $\frac{9}{2d-4f} + \frac{7}{8f-4d}$

(d) $\frac{h}{2h+5} + \frac{5}{6}$

(e) $\frac{3}{k-1} - \frac{2}{4k+5}$

(f) $\frac{m}{2m-5} + \frac{4}{8+3m}$

(g) $\frac{3n}{(2n-1)^2} - \frac{5}{2n-1}$

(h) $\frac{1}{p^2+3p-4} - \frac{1}{p+4}$

4. In each of the following cases, make the letter in the brackets the subject of the formula.

(a) $\frac{a}{2c} + \frac{b}{4} = 2$ [a]

(b) $A = P\left(1 + \frac{r}{100}\right)$ [r]

(c) $a = \frac{1-t}{1+t}$ [t]

(d) $k = h + \frac{2hk}{5}$ [h]

(e) $\sqrt{3a-2} = \sqrt{\frac{a}{b}}$ [b]

(f) $x + \sqrt[3]{y^2+z} = k$ [y]

5. Given that $a\sqrt{b^2-c} = 5k$, find
(a) the value of k when $a = 3$, $b = 6$ and $c = 20$,
(b) the value of c when $a = 4$, $b = 7$ and $k = 11$.

6. Given that $a^2 + a^2b = 320$, find
(a) the value of b when $a = 8$,
(b) the values of a when $b = 2\frac{1}{5}$.

7. Given that $V = \pi R^2h + \frac{2}{3}\pi r^3$, find
(a) the value of V when $\pi = 3.142$, $R = 12$, $h = 14$ and $r = 9$,
(b) the value of h when $\pi = 3.142$, $R = 8$, $V = 3800$ and $r = 6$,
(c) the values of R when $\pi = 3.142$, $V = 3500$, $h = 17$ and $r = 8.5$,
(d) the value of r when $\pi = 3.142$, $R = 11$, $V = 4600$ and $h = 6.9$.

8. Solve each of the following equations.

(a) $\frac{12}{5q} + 1 = \frac{7}{q}$

(b) $\frac{5}{y-3} - \frac{7}{3y-9} + \frac{10}{6-2y} = 3$

9. In an electrical circuit, when two resistors are connected in parallel, the effective resistance, R ohms (Ω), of the two resistors is given by the formula $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$, where R_1 and R_2 are the resistances of the individual resistors in ohms.

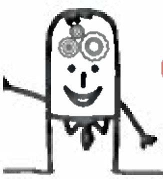
(i) Make R the subject of the formula $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$.

(ii) Hence, find the effective resistance of the two resistors if $R_1 = 2$ and $R_2 = 3$.

10. The equation of a circle is given by the formula $(x-a)^2+(y-b)^2=r^2$, where (a, b) are the coordinates of the centre of the circle and r is the radius.

(i) Make y the subject of the formula $(x-a)^2+(y-b)^2=r^2$.

(ii) Hence, find the y-coordinate(s) of the point(s) on the circle when $a = 2$, $b = 3$, $r = 5$ and $x = 5$.



Challenge Yourself

1. Simplify $\frac{(a+b)^n}{bc^2}+\frac{(a+b)^{n+3}}{abc}$.

Hint: You need to apply one of the laws of indices.

2. Given that $\frac{x^{n+2}y^{3-k}}{x^k y^{2n+5}}=\frac{x^{13}}{y^9}$, find the value of n and of k.

Hint: You need to apply one of the laws of indices.

3. If a, b and c are non-zero constants such that $\frac{a+b-c}{a}=\frac{a-b+c}{c}=\frac{-a+b+c}{b}$ and $p=\frac{(a-b)(b-c)(c-a)}{abc}$, show that $p=\frac{(a-b)^3}{c^3}$.

Relations and Functions

Everytime we put an orange into a blender, orange juice is produced. If we put starfruit into it, we get starfruit juice. Under no circumstance will we get orange juice by putting in some other types of fruits.

In Mathematics, we can define an operation on a set of numbers so that every time we apply the operations on a given number x, say we will always get a result y. Such an operation is known as a function.

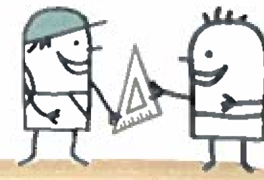
Chapter Seven

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- define a function,
- verify if a given relation is a function,
- solve problems on functions involving linear expressions.

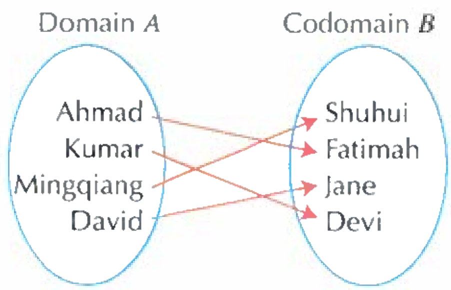
7.1 Relations



We have come across many relations between two sets in everyday life. Examples of such relations include "is the father of", "is the brother of", "is the wife of", "is the uncle of" and others.

Similarly, many relations exist in Mathematics such as "is less than", "is perpendicular to", "is higher than" and "is equal to". In fact, some mathematicians have described Mathematics as the study of relations.

Let us consider the following common examples of relations between two sets.



"is the husband of"

Fig. 7.1

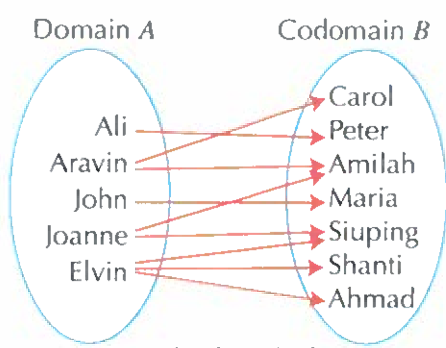
ATTENTION

The arrow is always from the domain to the codomain.

In Fig. 7.1, the members in Set A represent the husbands and the members in Set B represent the wives. We observe the relation that each man in Set A is the husband of a woman in Set B i.e. Ahmad is the husband of Fatimah, Kumar is the husband of Devi, Mingqiang is the husband of Shuhui and David is the husband of Jane.

We can also see that a relation has sense, that is, a direction in which it goes. This is conveniently indicated by the arrowheads. Thus, such diagrams are sometimes known as arrow diagrams

In any **relation**, we will have a **domain** and a **codomain**. In Fig. 7.1, Set A is the domain while Set B is the codomain of the relation. The member that is being matched to in the codomain is referred to as the **image** of the relation. Therefore, Devi is the image of Kumar and Jane is the image of David.



"is the friend of"

Fig. 7.2

In Fig. 7.2, we can see the relation that each person in Set A is the friend of another person in Set B.

From the arrow diagram, we observe that Aravin is the friend of Carol and Amilah and Joanne is the friend of Amilah and Siuping. The above relation shows that a member in a domain may be related to more than one member in the codomain. In other words, a member in a domain may have more than one image. For example, Carol and Amilah are the images of Aravin, while Amilah and Siuping are the images of Joanne.

7.2 Functions

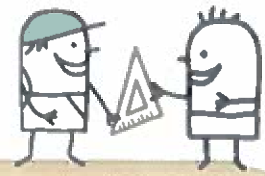


Fig. 7.3 shows the arrow diagram of a relation.

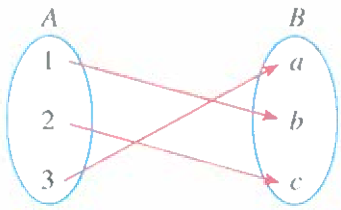


Fig. 7.3

We notice that only one arrow leaves each element in the domain. Thus *every* element in the domain of the relation has a *unique* (exactly one) image in the codomain. The relations, whose arrow diagrams are shown in Fig. 7.4 and Fig. 7.5, also satisfy the property that every element in the domain has a unique image in the codomain.

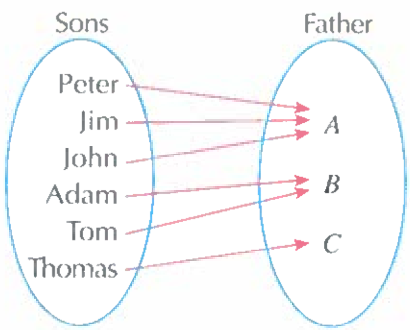


Fig. 7.4

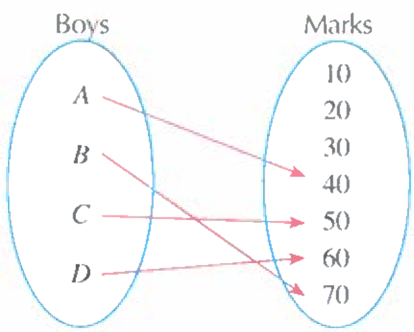


Fig. 7.5

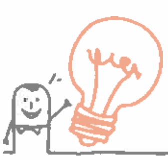
Relations in Fig. 7.3, Fig. 7.4 and Fig. 7.5 are examples of a special kind of relation that we call **functions**. In particular, Fig. 7.3 is a one-to-one correspondence which we call a **one-to-one** function.

INFORMATION

A function is also called a mapping.

> A **function** is a **relation** in which every element in the domain has a unique image in the codomain.

In this book, we will learn to solve problems on functions involving only linear expressions.



Thinking Time

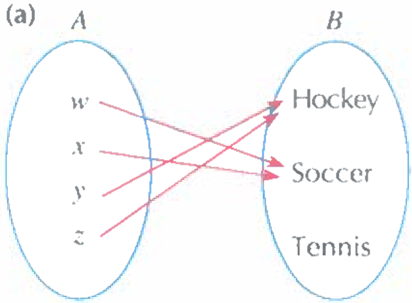
What are the differences among the relations shown in Fig. 7.3, Fig. 7.4 and Fig. 7.5?

Worked Example 1

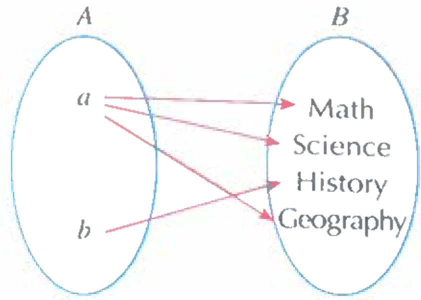
Verifying if a Relation is a Function

State, with reason, whether each of the following arrow diagrams defines a function.

(a)



(b)



Solution:

(a) The relation is a function since every element in the domain A has a unique image in the codomain B.

(b) The relation is not a function since the element a in the domain A has three images, Math, Science and Geography, in the codomain B.

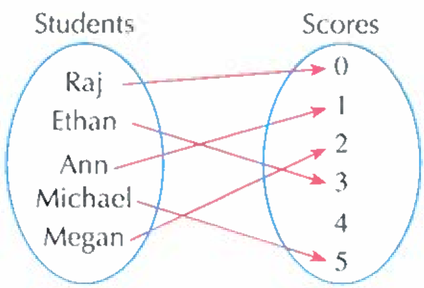
PRACTISE NOW 1

SIMILAR QUESTIONS

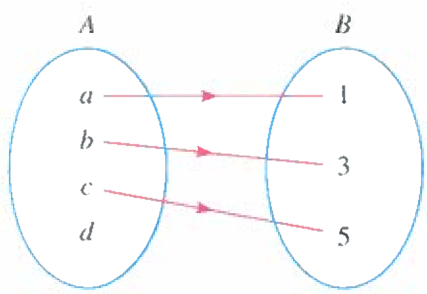
Exercise 7A Question 1

State, with reason, whether each of the following arrow diagrams defines a function.

(a)



(b)



Notation of a Function

Consider the function f whose arrow diagram is displayed in Fig. 7.6. The domain of the function is the set $X = \{1, 2, 3, 4\}$ and its codomain is the set $Y = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

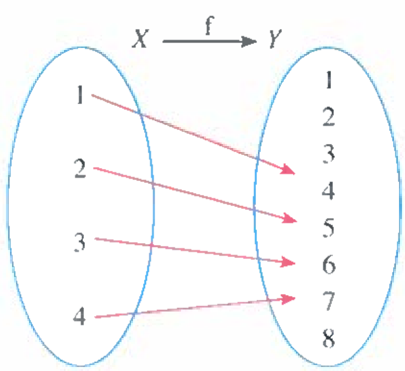


Fig. 7.6

We often use a lower case letter such as f to name a function. The notation f : $X \rightarrow Y$ is used to indicate that the function f has domain X and codomain Y. We read this as "a function f from X to Y". We can also write $X \xrightarrow{f} Y$ as illustrated in Fig. 7.7.

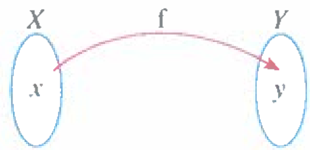


Fig. 7.7

For a function f : $X \rightarrow Y$, each element x in the domain X has a unique image y in the codomain Y. We often say y is a function of x and write it as $y = f(x)$. The function may be written as f : $x \mapsto f(x)$, linking an element of the domain to its image $f(x)$ in the codomain. Note that the vertical stroke on the arrow distinguishes it from f : $X \rightarrow Y$. $f(x)$ is also called the value of the function f at x.

ATTENTION

$f(x)$ is read as "f of x".

Consider the example in Fig. 7.3. Suppose f represents the function, then

$$f(1) = b,\ f(2) = c \text{ and } f(3) = a.$$

In Fig. 7.4, if g represents the function, then

$$g(\text{Peter}) = g(\text{Jim}) = g(\text{John}) = A,\ g(\text{Adam}) = g(\text{Tom}) = B \text{ and } g(\text{Thomas}) = C.$$

In Fig. 7.5, if h represents the function, then

$$h(A) = 40,\ h(B) = 70,\ h(C) = 50 \text{ and } h(D) = 60.$$

For the function f in Fig. 7.6, we have

$$f(\mathbf{1}) = 4 = \mathbf{1} + 3,\ f(\mathbf{2}) = 5 = \mathbf{2} + 3,\ f(\mathbf{3}) = 6 = \mathbf{3} + 3 \text{ and } f(\mathbf{4}) = 7 = \mathbf{4} + 3$$

Thus, in general,

$$f(x) = x + 3$$

i.e. to each x in the domain, the function f assigns the image $f(x)$ by adding 3 to x. The function f can be completely described as follows:

$$f : x \mapsto x + 3,\ x = 1, 2, 3 \text{ or } 4 \text{ or } x \in \{1, 2, 3, 4\}$$

Range of a Function

Let f : $X \rightarrow Y$ be a function. The set of values of $f(x)$ is called the **range** of f. The range of the function f in Fig. 7.6 is {4, 5, 6, 7}. The range may or may not consist of all of the elements of the codomain. The range {4, 5, 6, 7} consists of only some of the elements of $Y = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Very often, we are interested in the range and not the codomain of a function. Hence it is often inadequate to define a function f by stating its domain and the rule which determines the unique image $f(x)$ of each x in the domain.

Worked Example 2

(Determining Dependent and Independent Variables)

Given the functions $f : x \mapsto 3x + 2$ and $g : x \mapsto 5x - 4$, find the value of each of the following.

(i) $f(2)$ (ii) $f(-5)$ (iii) $f\left(\frac{1}{3}\right)$

(iv) $g(3)$ (v) $2g(7)$ (vi) $g\left(\frac{3}{5}\right)$

(vii) $f(1) + g(2)$ (viii) x for which $f(x) = g(x)$ (ix) x for which $f(x) = 17$

Solution:

$f : x \mapsto 3x + 2$, i.e. $f(x) = 3x + 2$

$g : x \mapsto 5x - 4$, i.e. $g(x) = 5x - 4$

(i) $f(2) = 3(2) + 2 = 8$

(ii) $f(-5) = 3(-5) + 2 = -13$

(iii) $f\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right) + 2 = 3$

(iv) $g(3) = 5(3) - 4 = 11$

(v) $2g(7) = 2[5(7) - 4] = 2(31) = 62$

(vi) $g\left(\frac{3}{5}\right) = 5\left(\frac{3}{5}\right) - 4 = -1$

(vii) $f(1) + g(2) = [3(1) + 2] + [5(2) - 4] = 5 + 6 = 11$

(viii) When $f(x) = g(x)$, we have $3x + 2 = 5x - 4$

$$2x = 6$$
$$x = 3$$

(ix) When $f(x) = 17$, we have $3x + 2 = 17$

$$3x = 15$$
$$x = 5$$

PRACTISE NOW 2

Given the functions $f : x \mapsto 10x + 4$ and $g : x \mapsto 4x - 6$, find the value of each of the following.

(i) $f(4)$ (ii) $f(-7)$

(iii) $f\left(-\frac{2}{3}\right)$ (iv) $g(2)$

(v) $2g(6)$ (vi) $g\left(\frac{7}{8}\right)$

(vii) $f\left(\frac{1}{2}\right) + g(1)$ (viii) x for which $f(x) = g(x)$

(ix) x for which $f(x) = 34$

SIMILAR QUESTIONS

Exercise 7A Questions 2–7, 8(i)–(iii)

Worked Example 3

(Expressing Values of Functions in Algebraic Expressions)

If $f(x) = 7x - 4$ and $F(x) = 6x + 5$, express

(i) $f(a)$, **(ii)** $F(a + 2)$, **(iii)** $f(3a) + F(2a + 1)$,

in terms of a.

Solution:

(i) $f(a) = 7a - 4$

(ii) $F(a + 2) = 6(a + 2) + 5$

$= 6a + 12 + 5$

$= 6a + 17$

(iii) $f(3a) + F(2a + 1) = 7(3a) - 4 + 6(2a + 1) + 5$

$= 21a - 4 + 12a + 6 + 5$

$= 33a + 7$

PRACTISE NOW 3

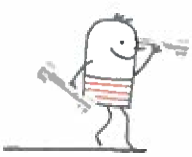
If $f(x) = 2x - 5$ and $F(x) = 7x + 12$, express

(i) $f(b)$, **(ii)** $F(b - 1)$, **(iii)** $f(2b) + F(2b - 5)$,

in terms of b.

SIMILAR QUESTIONS

Exercise 7A Questions 8(iv)–(vi)

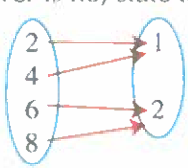


Exercise 7A

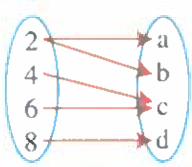
BASIC LEVEL

1. Each of the following relations has the set of integers {2, 4, 6, 8} as its domain. State whether each of the following arrow diagrams defines a function. If the answer is no, state the reason.

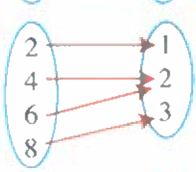
(a)



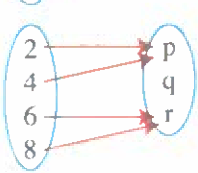
(b)



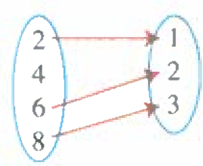
(c)



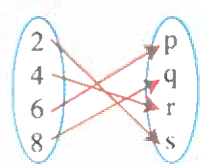
(d)



(e)



(f)



2. A function f is defined by $f : x \mapsto 6x - 4$ for all real values of x. What are the images of 2, −4, $\frac{1}{3}$ and $-\frac{1}{2}$ under f?

3. Given the function $f : x \mapsto 5 - 2x$, evaluate each of the following.

(i) $f(1)$ **(ii)** $f(-2)$

(iii) $f(0)$ **(iv)** $f(3) + f(-3)$

4. Given the function $g(x) = 7x + 4$, find the value of each of the following.

(i) $g(2)$ (ii) $g(-3)$

(iii) $g\left(\frac{4}{7}\right)$ (iv) $g(0) + g(-1)$

(v) $g\left(\frac{1}{7}\right) - g\left(-\frac{1}{7}\right)$

INTERMEDIATE LEVEL

5. Given the functions $f(x) = \frac{x}{2} + 3$ and $g(x) = \frac{3}{4}x - 2$,

(a) find the values of

(i) $f(2) + g(2)$ (ii) $f(-1) - g(-1)$

(iii) $2f(4) - 3g(6)$ (iv) $5f(-2) - 7g(-4)$

(b) What are the values of x for which $f(x) = g(x)$ and $f(x) = 17$?

6. Given the functions $f : x \mapsto 5x - 9$ and $g : x \mapsto 2 - 6x$, find the value of x for which

(i) $f(x) = 16$ (ii) $g(x) = 14$

(iii) $g(x) = x$ (iv) $f(x) = 2x$

(v) $f(x) = g(x)$ (vi) $2f(x) = 3g(x)$

ADVANCED LEVEL

7. Given the function $f(x) = 4x + 9$, evaluate $f(1)$, $f(2)$ and $f(3)$. Is it true that

(i) $f(1) + f(2) = f(1 + 2)$?

(ii) $f(3) - f(2) = f(3 - 2)$?

(iii) $f(1) \times f(2) = f(1 \times 2)$?

(iv) $f(2) \div f(1) = f(2 \div 1)$?

8. Given the functions $f(x) = \frac{3}{4}x + \frac{1}{2}$ and $g(x) = 1\frac{1}{4} - \frac{2}{3}x$, evaluate $f(2)$, $f\left(-\frac{1}{2}\right)$, $g(3)$ and $g(-6)$.

(i) Is it true that $f(2) + f(3) = f(2 + 3)$?

(ii) Is it true that $g(4) - g(2) = g(4 - 2)$?

(iii) Find the value of x for which $f(x) = g(x)$.

(iv) Find the value of $f(a)$, $f(2a)$ and of $g(3a)$.

(v) Find the value of a for which $f(a + 1) + g(a) = 5$.

(vi) Find the value of a for which $f(2a) = g(6a)$.

1. A **relation** connects elements in set A (**domain**) to elements in set B (**codomain**) according to the definition of the relation.
2. A **function** is a relation in which every element in the domain has a unique **image** in the codomain.
3. The **range** of a function f is the set of values of $f(x)$ (images under f) for the given domain.

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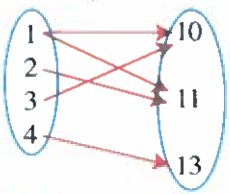
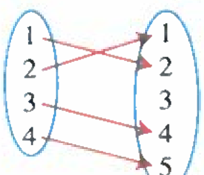
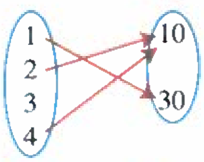
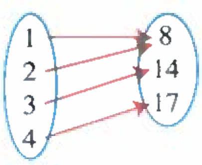
Review Exercise 7

1. Each of the following relations has the set of integers {1, 2, 3, 4} as its domain. State whether each of the following arrow diagrams defines a function. If the answer is no, state the reason.

   (a)


   (b)


   (c)


   (d)


2. A function f is defined by $f : x \mapsto \frac{1}{2}x + 2$ for all real values of x. What are the images of -20, 6, $\frac{1}{8}$ and $-\frac{2}{3}$ under f?

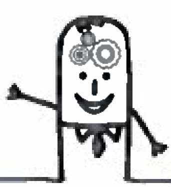
3. Given the function $f : x \mapsto 12 - 5x$, evaluate each of the following.
   (i) f(2) (ii) f(−3)
   (iii) f(0) (iv) f(3) + f(−5)

4. Given the function $g(x) = 5x - 9$, evaluate each of the following.
   (i) g(2) (ii) g(−5)
   (iii) $g\left(\frac{3}{5}\right)$ (iv) g(0) + g(−3)
   (v) $g\left(\frac{4}{5}\right) - g\left(-\frac{1}{5}\right)$

5. Given the functions $f(x) = \frac{x}{4} + 5$ and $g(x) = \frac{2}{5}x - 1$,
   (a) evaluate
      (i) f(8) + g(5) (ii) f(−1) − g(−10)
      (iii) 2f(3) − 3g(5) (iv) 4f(−3) − 7g(0)
   (b) What are the values of x for which $f(x) = g(x)$ and $f(x) = 6$?

6. Given the functions $f : x \mapsto 4x - 3$ and $g : x \mapsto 8 - 3x$, find the value of x for which
   (i) $f(x) = 13$ (ii) $g(x) = 20$
   (iii) $g(x) = x$ (iv) $f(x) = -2x$
   (v) $f(x) = g(x)$

7. If $f(x) = 10x - 3$ and $F(x) = \frac{3}{4}x + 2$, express
   (i) $f(a)$, (ii) $F(8a + 1)$,
   (iii) $f\left(\frac{1}{2}a\right) + F(2a - 4)$,
   in terms of a.



Challenge Yourself

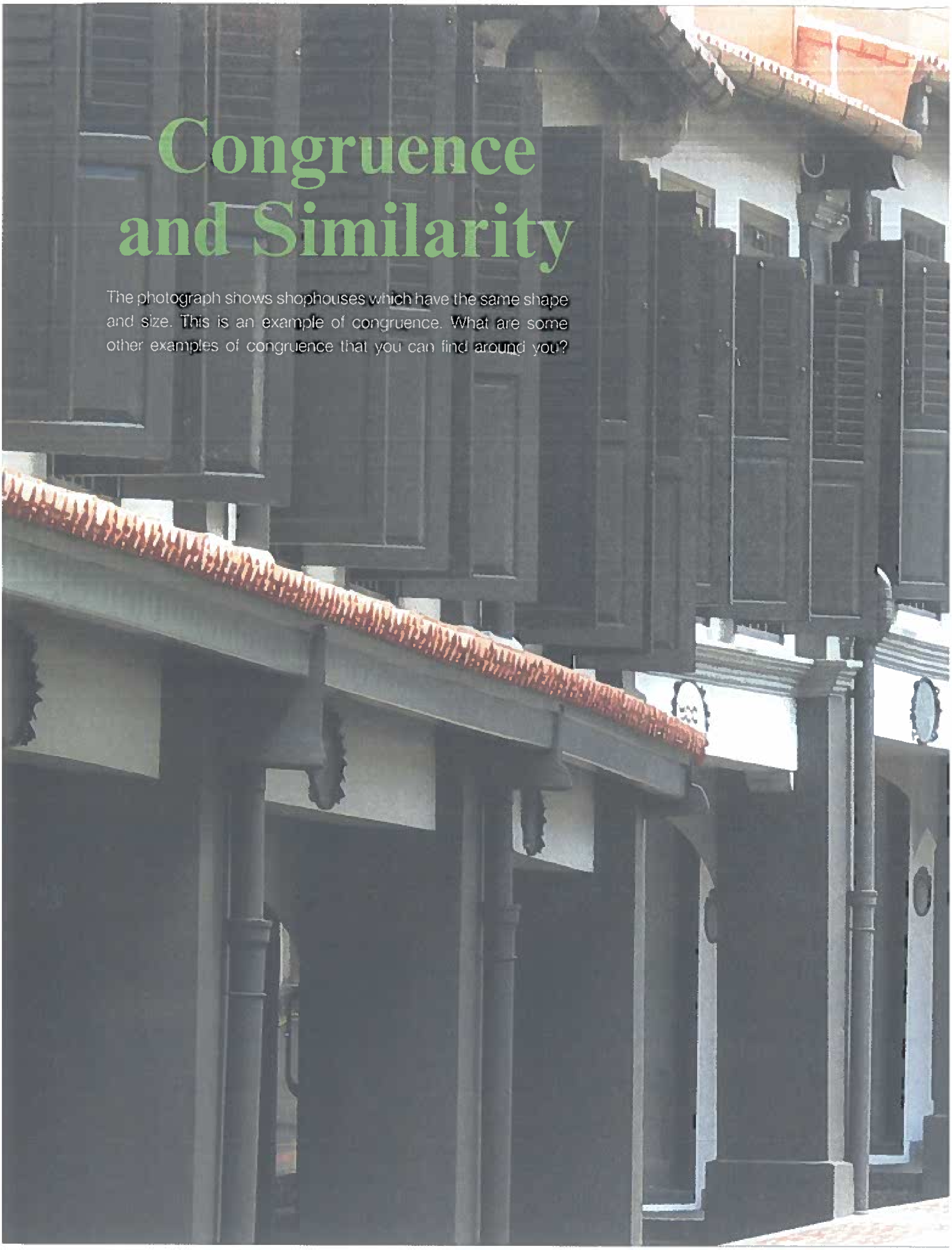
A function f is defined such that $f(1) = \frac{5}{2}$ and $f(a + b) = \left(\frac{a+b+1}{a+1}\right)f(a) + \left(\frac{a+b+1}{b+1}\right)f(b) + a + 2$, where a and b are natural numbers.

(i) Show that $\frac{f(a+1)}{a+2} - \frac{f(a)}{a+1} = \frac{9}{4}$.

(ii) Hence find the value of f(2).

Congruence and Similarity

The photograph shows shophouses which have the same shape and size. This is an example of congruence. What are some other examples of congruence that you can find around you?



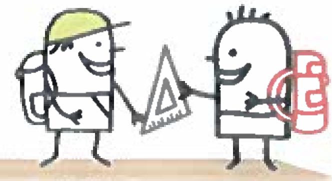
Chapter Eight

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- examine whether two figures are congruent or similar,
- state the properties of similar triangles and polygons,
- make simple scale drawings with appropriate scales,
- interpret scales on maps,
- solve simple problems involving congruence and similarity.

8.1 Congruent Figures



Investigation

Properties of Congruent Figures

Fig. 8.1 shows five pairs of scissors.

(a) **(b)** **(c)** **(d)** **(e)**

Fig. 8.1

1. What can we say about the shape, size, orientation and position of the pairs of scissors?
2. If we cut out the pairs of scissors and stack them up, what will we observe?
3. The pair of scissors in **(a)** can be moved to look like the pair of scissors in **(b)** by a translation from $A_1 \rightarrow A_2$ and a rotation of 90° about A_2.
   How can we move the pair of scissors in **(a)** to look like the pairs of scissors in **(c)**, **(d)** and **(e)**?

From the investigation, we observe that:

- Two figures are **congruent** if they have exactly the **same shape** and **size**.
- They can be mapped onto each other under **translation**, **rotation** and **reflection**.

You can also investigate the effect of translation, rotation and reflection on a triangle/quadrilateral using the geometry software template 'Congruence' at http://www.shinglee.com.sg/StudentResources/

Thinking Time

Fig. 8.2 shows two pairs of scissors of different colours.

(a) (b)

Fig. 8.2

Are they congruent? Explain your answer.

Congruence is a property of *geometrical figures*. The two pairs of scissors in Fig. 8.2 are *congruent* because they have exactly the same shape and size.

Fig. 8.3 shows some patterns that are formed by congruent figures. These are known as tessellations, which can be found in many real-life objects.

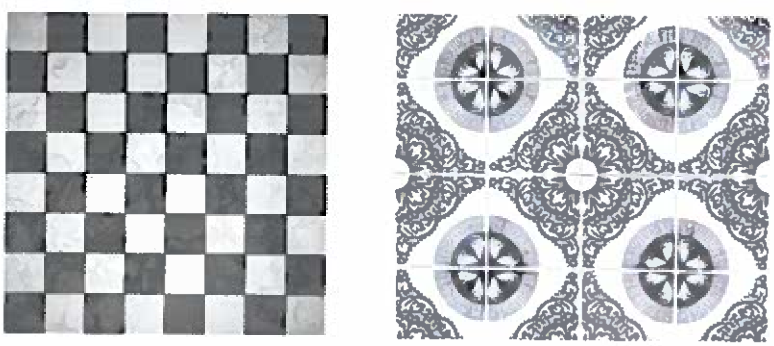
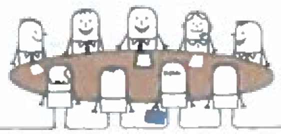


Fig. 8.3

Another real-life application of congruence is photocopying as the photocopied document is of the same shape and size as the original document. The concept of congruence also plays an important role in the manufacturing sector. The congruence of pen refills allows us to refill our pens when they run dry.



Class Discussion

Congruence in the Real World

1. Look around your classroom or school. Find at least 3 different sets of congruent objects.
2. Tessellations, like those shown in Fig. 8.3, can be found on floor tiles. What are some other objects that exhibit tessellations?
3. Discuss with your classmates other real-life applications of congruence.

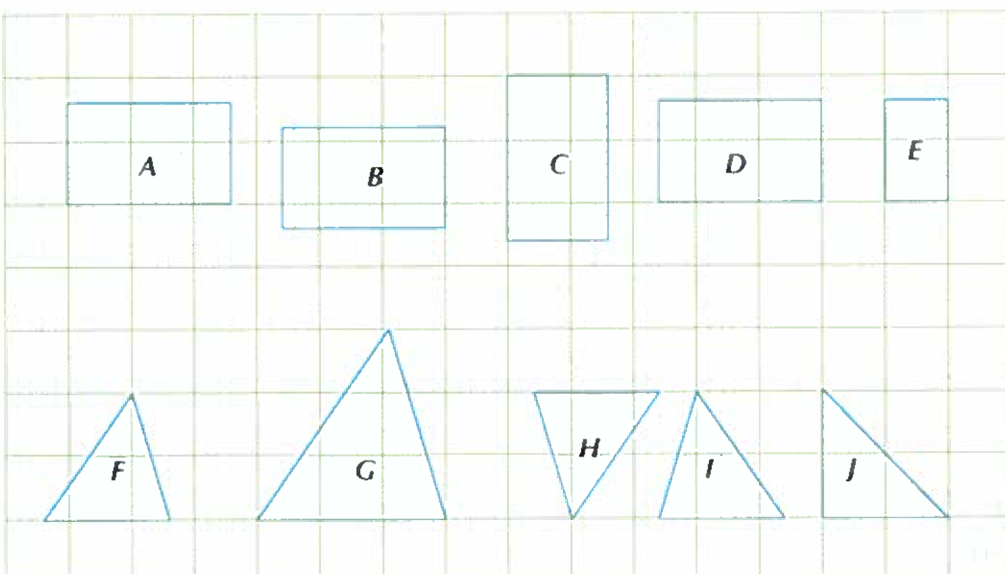
Internet Resources

Search on the Internet for 'Tessellation Tool' to make your own tessellations.

Worked Example 1

(Identifying Congruent Shapes)

Which shapes are congruent?



Solution:

A, ***B***, ***C*** and ***D*** are congruent rectangles.

F, ***H*** and ***I*** are congruent triangles.

PRACTISE NOW 1

SIMILAR QUESTIONS

Exercise 8A Question 1

Which shapes are congruent?

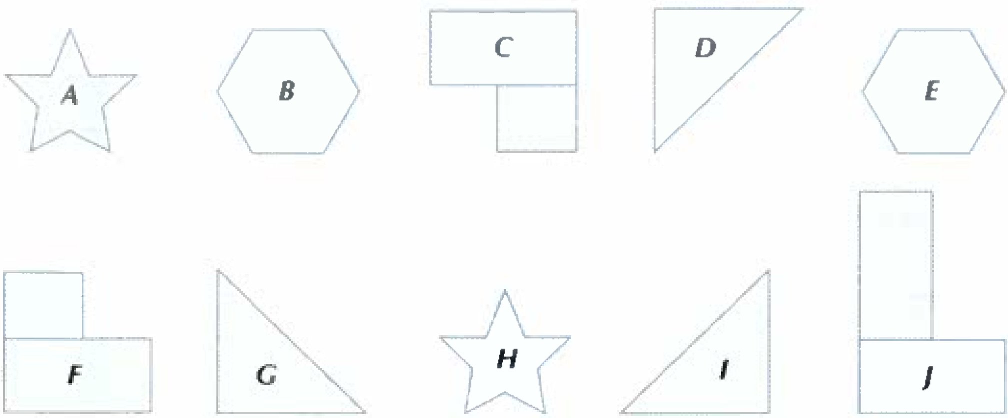


Fig. 8.4 shows two congruent quadrilaterals $ABCD$ and $A'B'C'D'$. The vertex A corresponds to the vertex A' because they have the same angle. Similarly, the vertices that correspond to B, C and D are B', C' and D' respectively.

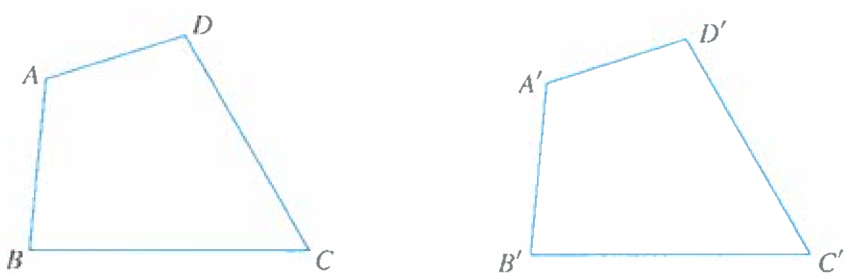
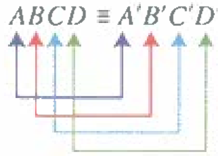


Fig. 8.4

The symbol '≡' means 'is congruent to'. Thus for the two quadrilaterals in Fig. 8.4, we have $ABCD \equiv A'B'C'D'$.

Notice that the order in which the vertices of $A'B'C'D'$ are written must correspond to the order in which the vertices of $ABCD$ are written.



We can also write $BCDA \equiv B'C'D'A'$ because the corresponding vertices match.
Can we write $CDAB \equiv C'D'A'B'$ or $DABC \equiv D'A'B'C'$?

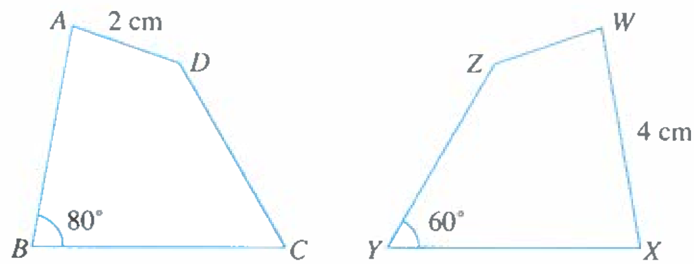
In Fig. 8.4, the side AD corresponds to the side $A'D'$. Similarly, the sides that correspond to AB, BC and CD are $A'B'$, $B'C'$ and $C'D'$ respectively.

Hence,

the corresponding angles and the corresponding sides of congruent figures are equal.

Worked Example 2

(Problem involving Congruent Figures)



Given that $ABCD \equiv WXYZ$, copy and complete each of the following.

(i) $\angle ABC = \angle WXY =$ ________°

(ii) ________ $= \angle XYZ =$ ________°

(iii) $AD =$ ________ $=$ ________ cm

(iv) ________ $= WX =$ ________ cm

Solution:

Since $ABCD \equiv WXYZ$, then the corresponding vertices match:

$A \leftrightarrow W$
$B \leftrightarrow X$
$C \leftrightarrow Y$
$D \leftrightarrow Z$

(i) $\angle ABC = \angle WXY = 80°$

(ii) $\angle XYZ = \angle BCD = 60°$ $(X \leftrightarrow B,\ Y \leftrightarrow C,\ Z \leftrightarrow D)$

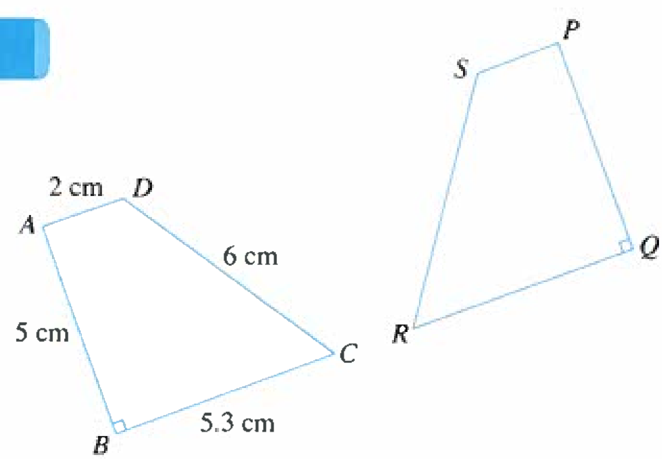
(iii) $AD = WZ = 2$ cm $(A \leftrightarrow W,\ D \leftrightarrow Z)$

(iv) $WX = AB = 4$ cm $(W \leftrightarrow A,\ X \leftrightarrow B)$

PRACTISE NOW 2

SIMILAR QUESTIONS

Exercise 8A Questions 2–3



Given that $ABCD \equiv PQRS$, copy and complete each of the following.

(i) $PQ = AB =$ ________ cm

(ii) $SR =$ ________ $= 6$ cm

(iii) $PS =$ ________ $=$ ________ cm

(iv) $QR =$ ________ $=$ ________ cm

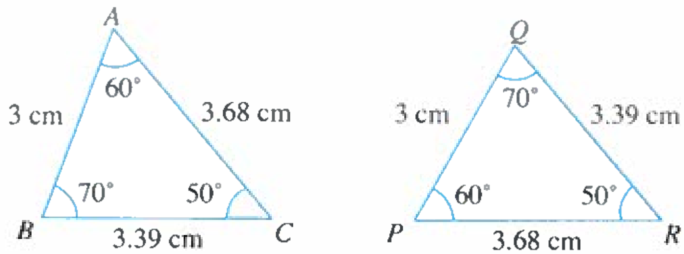
(v) $\angle PQR =$ ________ $=$ ________°

Worked Example 3

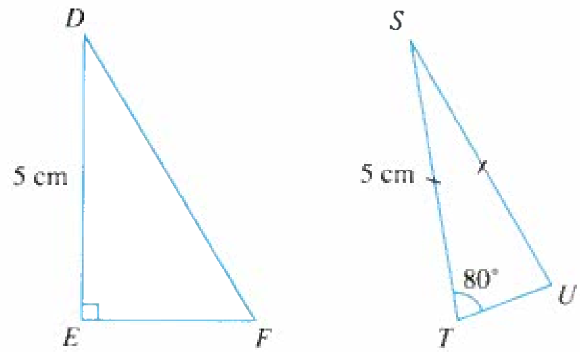
Identifying Congruent Triangles and Writing Statement of Congruence

Are the following pairs of triangles congruent? If so, explain your answer and write down the statement of congruence. If not, explain your answer.

(a)



(b)



Solution:

(a) **Step 1**: Identify the corresponding vertices by comparing the size of the angles.

$A \leftrightarrow P$ (since $\angle A = \angle P = 60°$)

$B \leftrightarrow Q$ (since $\angle B = \angle Q = 70°$)

$C \leftrightarrow R$ (since $\angle C = \angle R = 50°$)

Step 2: Write proper statements using the corresponding vertices identified in Step 1.

$\angle BAC = \angle QPR = 60°$ (notice that the corresponding vertices match)

$\angle ABC = \angle PQR = 70°$

$\angle ACB = \angle PRQ = 50°$

$AB = PQ = 3$ cm (notice that the corresponding vertices match)

$BC = QR = 3.39$ cm

$AC = PR = 3.68$ cm

$\therefore$ The two triangles have the same shape and size and so $\Delta ABC \equiv \Delta PQR$.

(b) In ΔSTU,

$\angle T = \angle U = 80°$ (base $\angle$s of isos. ΔSTU)

$\angle S = 180° - 80° - 80°$ ($\angle$ sum of ΔSTU)

$= 20°$

$\therefore$ ΔSTU does not have any right angle that corresponds to that in ΔDEF.

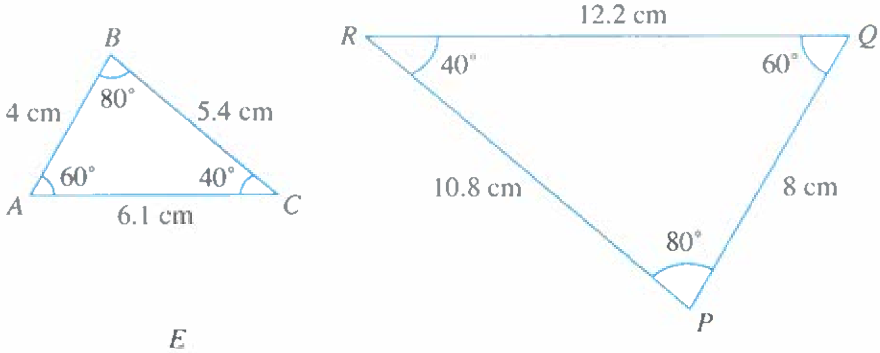
$\therefore$ ΔSTU does not have the same shape as ΔDEF and so it is not congruent to ΔDEF.

SIMILAR QUESTIONS

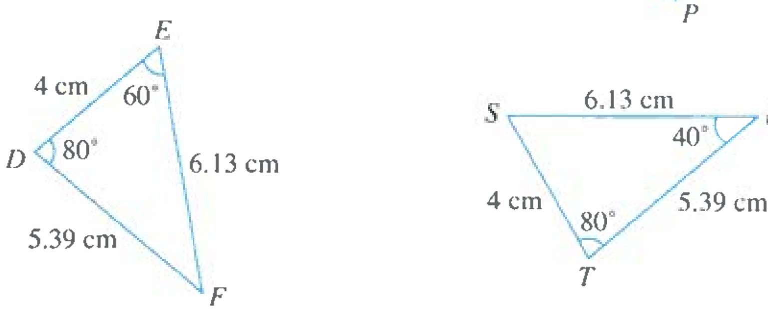
Exercise 8A Questions 4(a)–(c)

Are the following pairs of triangles congruent? If so, explain your answer and write down the statement of congruence. If not, explain your answer.

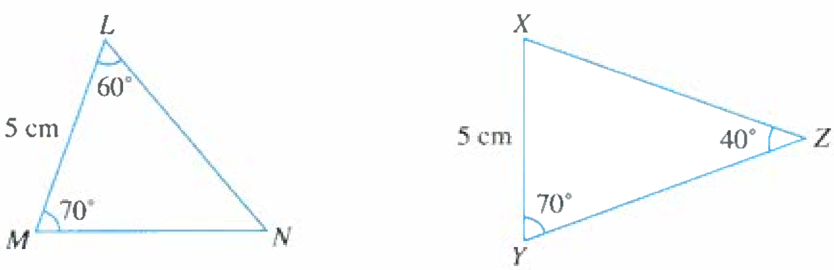
(a)



(b)



(c)



Worked Example 4

(Problem involving Congruent Triangles)

In the figure, $\Delta ABC \equiv \Delta CED$.

(a) Given that $\angle BAC = 20°$, $\angle CDE = 60°$, $AB = 8.8$ cm and $CD = 10$ cm, calculate

(i) $\angle ECD$,

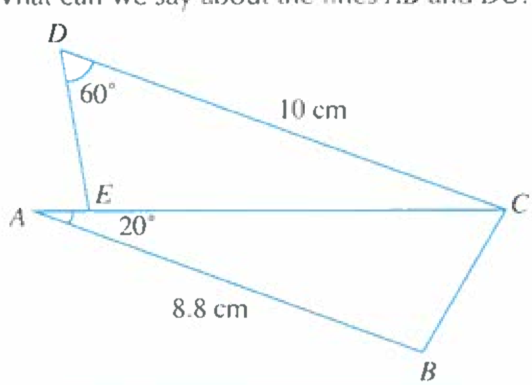
(ii) $\angle ECB$,

(iii) $\angle ABC$,

(iv) the length of AC,

(v) the length of AE.

(b) What can we say about the lines AB and DC?



Solution:

Since $\Delta ABC \equiv \Delta CED$, then the corresponding vertices match:

$$A \leftrightarrow C$$
$$B \leftrightarrow E$$
$$C \leftrightarrow D$$

(a)(i) $\angle ECD = \angle BAC$ ($E \leftrightarrow B$, $C \leftrightarrow A$, $D \leftrightarrow C$)
$= 20°$

(ii) $\angle ECB = \angle ACB$ (from the figure, $\angle ECB$ belongs to ΔABC)
$= \angle CDE$ ($A \leftrightarrow C$, $C \leftrightarrow D$, $B \leftrightarrow E$)
$= 60°$

(iii) $\angle ABC = 180° - 20° - 60°$ ($\angle$ sum of ΔABC)
$= 100°$

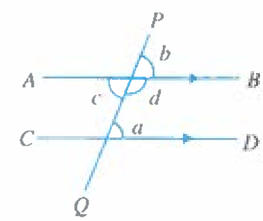
(iv) Length of AC = length of CD ($A \leftrightarrow C$, $C \leftrightarrow D$)
$= 10$ cm

(v) Length of EC = length of BA ($E \leftrightarrow B$, $C \leftrightarrow A$)
$= 8.8$ cm

$\therefore$ Length of AE = length of AC – length of EC
$= 10 - 8.8$
$= 1.2$ cm

(b) Since $\angle BAC = \angle ECD$ (= 20°), then $AB \parallel DC$ (converse of alt. $\angle$s).

RECALL



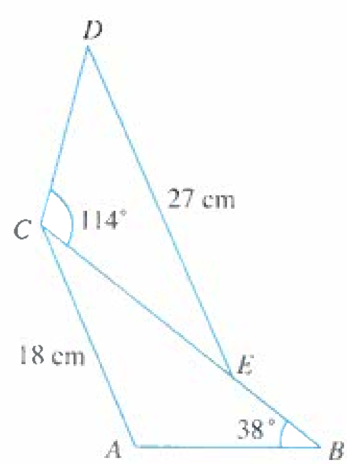
When two lines AB and CD are cut by a transversal PQ, and
- if $\angle a = \angle b$, then $AB \parallel CD$ (converse of corr. $\angle$s);
- if $\angle a = \angle c$, then $AB \parallel CD$ (converse of alt. $\angle$s);
- if $\angle a + \angle d = 180°$, then $AB \parallel CD$ (converse of int. $\angle$s).

PRACTISE NOW 4

In the figure, $\Delta ABC \equiv \Delta CDE$.

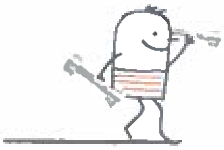
(a) Given that $\angle ABC = 38°$, $\angle DCE = 114°$, $AC = 18$ cm and $DE = 27$ cm, find
- **(i)** $\angle CDE$,
- **(ii)** $\angle CED$,
- **(iii)** $\angle ACB$,
- **(iv)** the length of BC,
- **(v)** the length of BE.

(b) What can we say about the lines AC and ED?



SIMILAR QUESTIONS

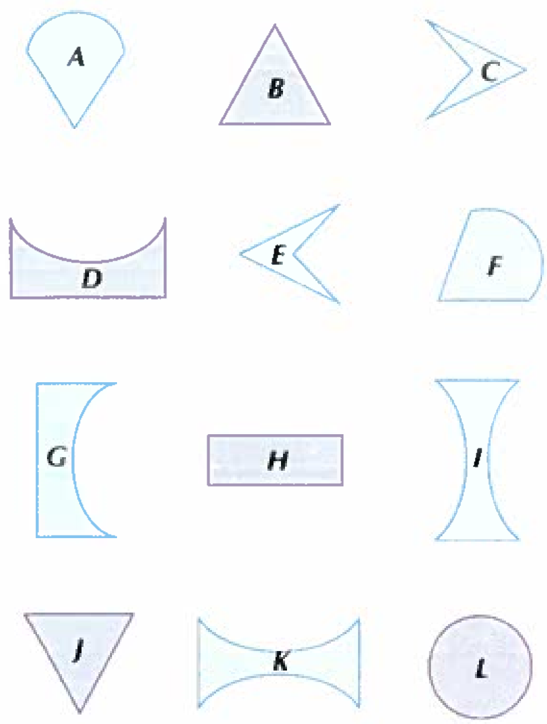
Exercise 8A Questions 5–7



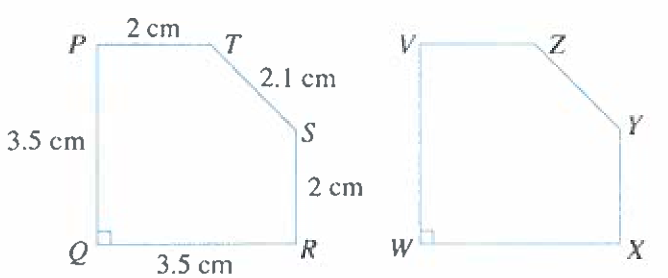
Exercise 8A

BASIC LEVEL

1. Which pairs of shapes are congruent?



2.



Given that *PQRST* ≡ *VWXYZ*, copy and complete each of the following.

(i) $PQ = VW =$ ________ cm

(ii) $PT =$ ________ $= 2$ cm

(iii) $QR =$ ________ $=$ ________ cm

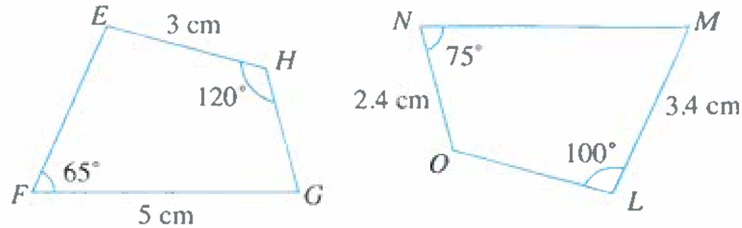
(iv) $TS =$ ________ $=$ ________ cm

(v) $SR =$ ________ $=$ ________ cm

(vi) $\angle PQR =$ ________ $=$ ________ °

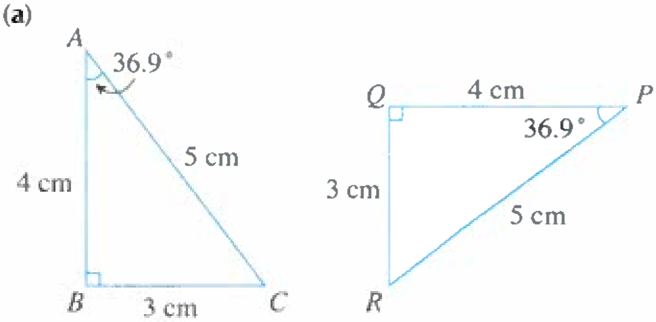
INTERMEDIATE LEVEL

3. Given that $EFGH \equiv LMNO$, write down all the missing measurements.

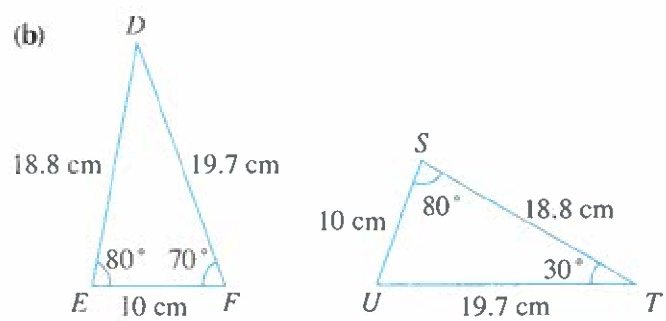


4. Are the following pairs of triangles congruent? If so, explain your answer and write down the statement of congruence. If not, explain your answer.

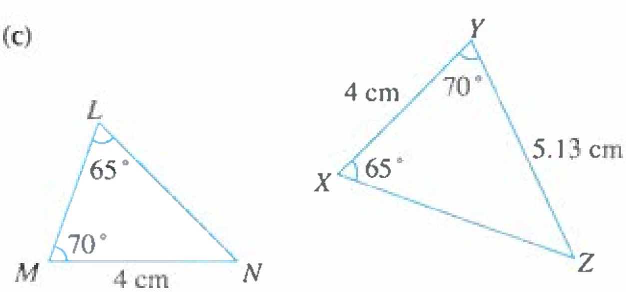
(a)



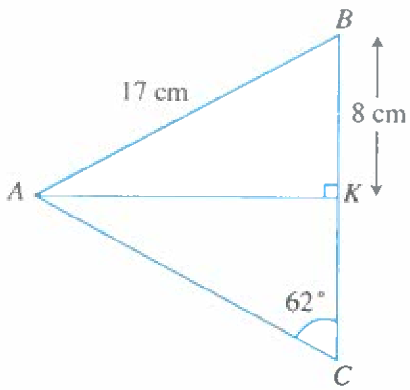
(b)



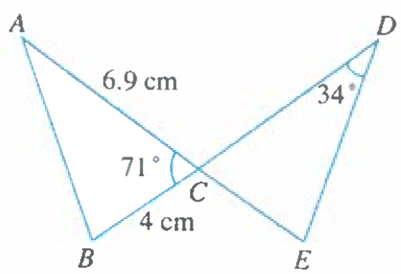
(c)



5. In the figure, $\Delta ABK \equiv \Delta ACK$. Given that $\angle AKB = 90°$, $\angle ACK = 62°$, $AB = 17$ cm and $BK = 8$ cm, find
(i) $\angle BAC$,
(ii) the length of BC.

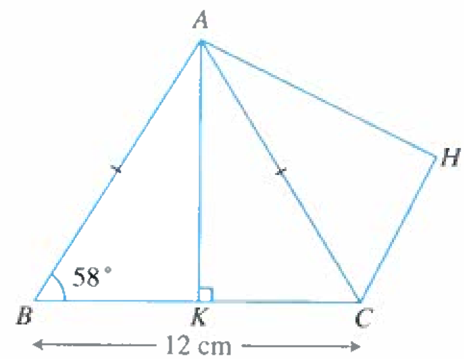


6. In the figure, $\Delta ABC \equiv \Delta DEC$. Given that $\angle ACB = 71°$, $\angle CDE = 34°$, $AC = 6.9$ cm and $BC = 4$ cm, find
(i) $\angle ABC$,
(ii) the length of BD.



ADVANCED LEVEL

7. In the figure, ΔABC is an isosceles triangle where $AB = AC$, $BC = 12$ cm and $\angle ABK = 58°$. Given that $\Delta ABK \equiv \Delta ACH$ and $\angle AKC = 90°$, find
(i) the length of CH,
(ii) $\angle BAH$.



8.2 Similar Figures

Fig. 8.5(**a**) shows three cups which look alike but are of different sizes.

Fig. 8.5(a)

Fig. 8.5(**b**) shows the projection of a photograph on a screen using a visualiser.

Fig. 8.5(b)

> Two figures are **similar** if they have exactly the same shape but *not* necessarily the same size.

If two similar figures also have exactly the same size, then they are congruent. In other words, congruence is a *special case* of similarity.

Class Discussion

Similarity in the Real World

1. Look around your classroom, science laboratory or school. Find at least 3 different sets of similar objects.
2. Discuss with your classmates other real-life examples of similarity.

Investigation

Properties of Similar Polygons

The hexagon $ABCDEF$ is increased in size without changing its shape to become the hexagon $A'B'C'D'E'F'$.

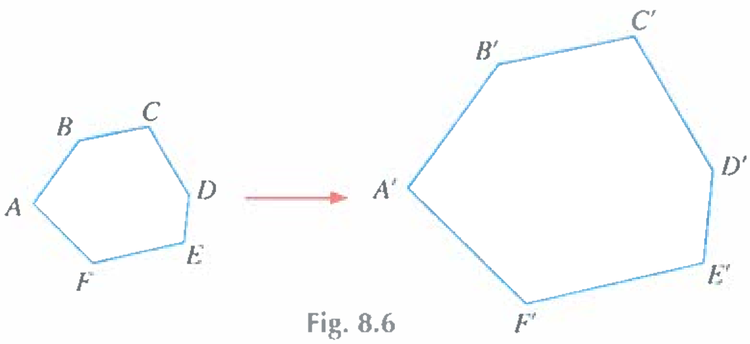


Fig. 8.6

1. Measure each of the following pairs of angles.

(a) $\angle A, \angle A'$ **(b)** $\angle B, \angle B'$ **(c)** $\angle C, \angle C'$

(d) $\angle D, \angle D'$ **(e)** $\angle E, \angle E'$ **(f)** $\angle F, \angle F'$

What do you notice about the size of each of the above pairs of angles?

2. **(a)** Measure the lengths of each of the following pairs of sides.

(i) $AB, A'B'$ **(ii)** $BC, B'C'$ **(iii)** $CD, C'D'$

(iv) $DE, D'E'$ **(v)** $EF, E'F'$ **(vi)** $FA, F'A'$

(b) Hence, find the value of each of the following ratios.

(i) $\frac{A'B'}{AB}$ **(ii)** $\frac{B'C'}{BC}$ **(iii)** $\frac{C'D'}{CD}$

(iv) $\frac{D'E'}{DE}$ **(v)** $\frac{E'F'}{EF}$ **(vi)** $\frac{F'A'}{FA}$

What do you notice about the values of the ratios?

From the investigation, we observe that if two hexagons are similar, then

- all the corresponding angles are equal,
  i.e. $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, $\angle D = \angle D'$, $\angle E = \angle E'$, $\angle F = \angle F'$, and
- the length of each side of a hexagon is increased by the same factor,
  i.e. $\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{D'E'}{DE} = \frac{E'F'}{EF} = \frac{F'A'}{FA} = k$, where k is a constant.

In general, we have:

> If two polygons are *similar*, then
>
> - all the corresponding angles are equal, *and*
> - all the ratios of the corresponding sides are equal.

Thinking Time

1. Fig. 8.7 shows two rectangles.

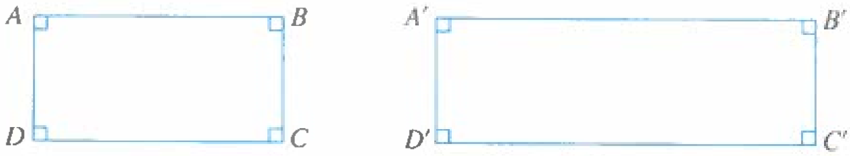


Fig. 8.7

(i) Are all the corresponding angles equal?
(ii) Are all the ratios of the corresponding sides equal?
(iii) Are the two rectangles similar?

2. Fig. 8.8 shows a square and a rhombus.

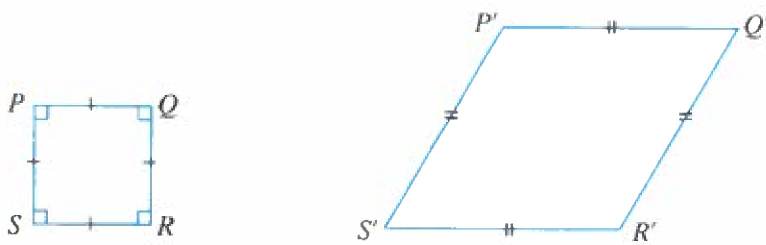


Fig. 8.8

(i) Measure the length of the square and the length of the rhombus. Are all the ratios of the corresponding sides equal?
(ii) Are all the corresponding angles equal?
(iii) Are the two quadrilaterals similar?

3. Fig. 8.9 shows two triangles.

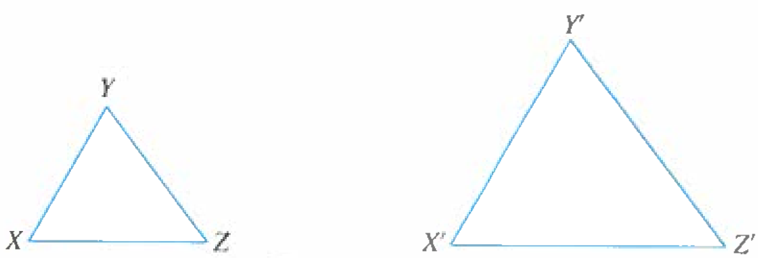


Fig. 8.9

(i) Measure all the angles. Are all the corresponding angles equal?
(ii) Measure the lengths of all the sides. Are all the ratios of the corresponding sides equal?
(iii) Are the two triangles similar?
(iv) Are we able to find two triangles which have equal corresponding angles but are not similar?
(v) Are we able to find two triangles where the ratios of the corresponding sides are equal but are not similar?

From the Thinking Time, we can conclude that:

1. If two triangles have equal corresponding angles *or* the ratios of the corresponding sides are equal, then they are similar.
2. If two polygons with four or more sides have equal corresponding angles *or* the ratios of the corresponding sides are equal, they *may not* be similar.

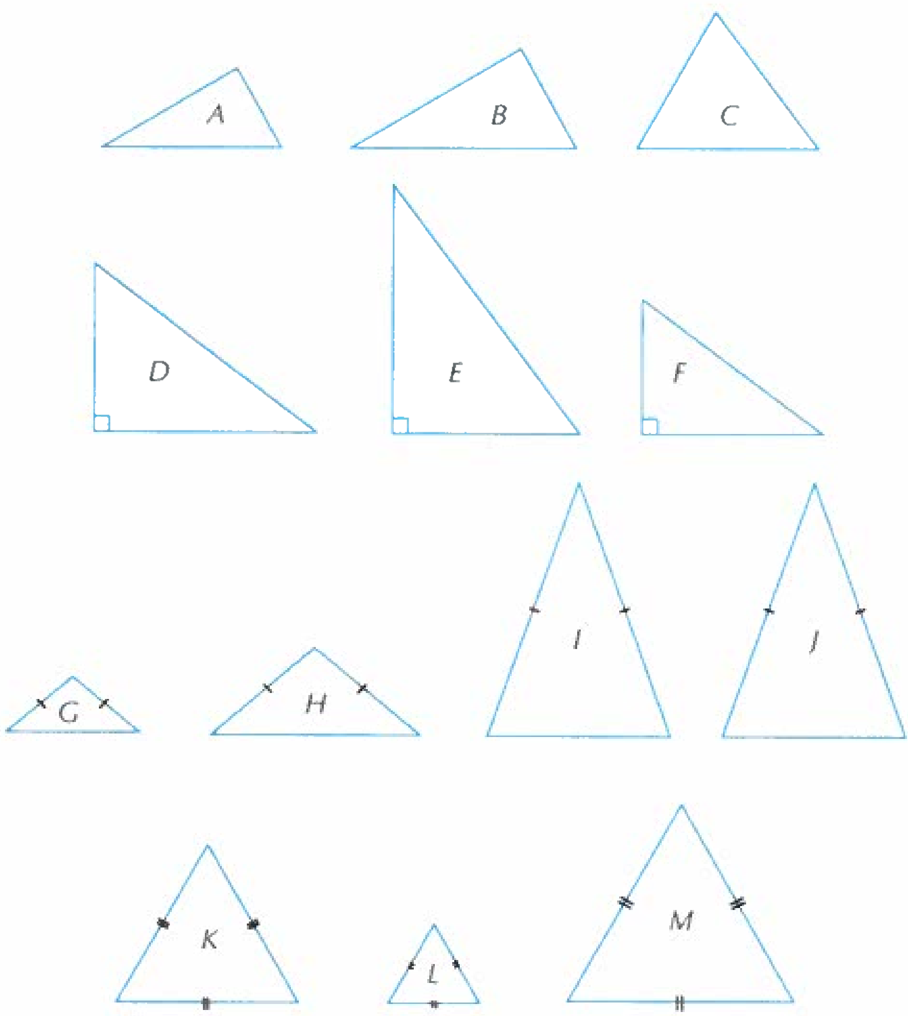
   For polygons with four or more sides, the following conditions must hold true before we can conclude that they are similar.

   - all the corresponding angles are equal, *and*
   - all the ratios of the corresponding sides are equal.

Class Discussion

Identifying Similar Triangles

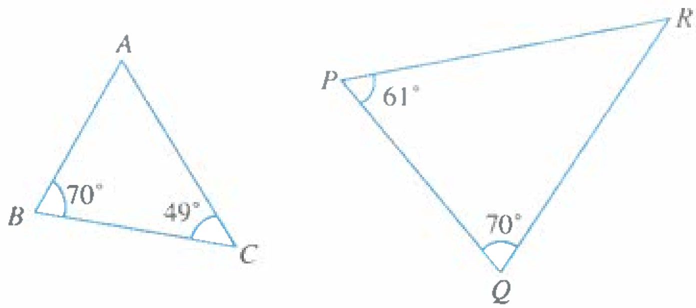
1. Photocopy the following triangles and cut them out.
2. Which triangles are similar? Explain your answer.
3. (a) Are all right-angled triangles similar? Explain your answer.
   (b) Are all isosceles triangles similar? Explain your answer.
   (c) Are all equilateral triangles similar? Explain your answer.



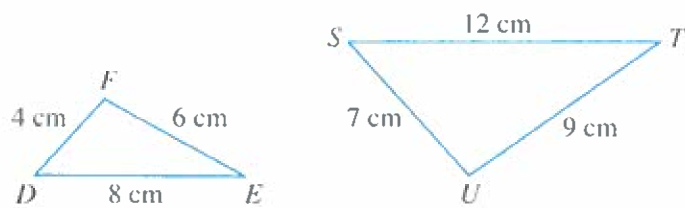
Worked Example 5

(Identifying Similar Triangles)

(a) Is ΔABC similar to ΔPQR? Explain your answer.



(b) Is ΔDEF similar to ΔSTU? Explain your answer.



Solution:

(a) $\angle A = 180° - 70° - 49°$ ($\angle$ sum of ΔABC)

$= 61°$

$\angle R = 180° - 61° - 70°$ ($\angle$ sum of ΔPQR)

$= 49°$

$\angle A = \angle P = 61°$

$\angle B = \angle Q = 70°$

$\angle C = \angle R = 49°$

Since all the corresponding angles are equal, ΔABC is similar to ΔPQR.

(b) $\frac{ST}{DE} = \frac{12}{8} = 1.5$

$\frac{TU}{EF} = \frac{9}{6} = 1.5$

$\frac{SU}{DF} = \frac{7}{4} = 1.75$

Since not all the ratios of the corresponding sides are equal, then ΔDEF is not similar to ΔSTU.

ATTENTION

There is no standard notation for similarity. Do not use the symbol '≡' for similarity because some countries use this symbol for congruence. Thus we write 'ΔABC is similar to ΔPQR'.

Problem Solving Tip

If ΔDEF is similar to ΔSTU, then the longest side of ΔDEF will correspond to the longest side of ΔSTU, the second longest side of ΔDEF will correspond to the second longest side of ΔSTU, and the shortest side of ΔDEF will correspond to the shortest side of ΔSTU.

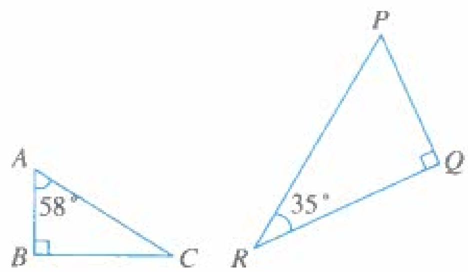
Thus we need to compare the longest side of ΔDEF with the longest side of ΔSTU, the second longest side of ΔDEF with the second longest side of ΔSTU, and the shortest side of ΔDEF with the shortest side of ΔSTU.

PRACTISE NOW 5

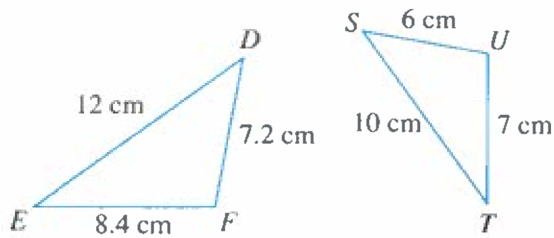
SIMILAR QUESTIONS

Exercise 8B Questions 2(a)–(b)

(a) Is ΔABC similar to ΔPQR?
Explain your answer.



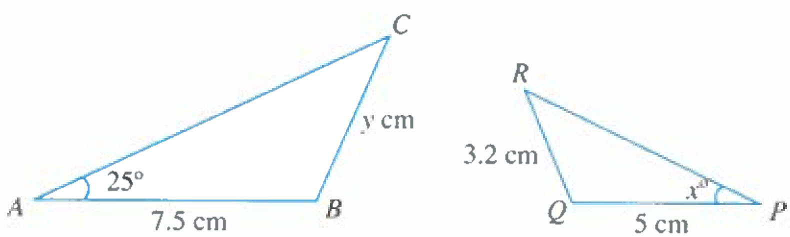
(b) Is ΔDEF similar to ΔSTU?
Explain your answer.



Worked Example 6

(Problem involving Similar Figures)

Given that ΔABC is similar to ΔPQR, calculate the values of the unknowns in the triangles.



Solution:

Since ΔABC is similar to ΔPQR, then the corresponding vertices match:

$$A \leftrightarrow P$$
$$B \leftrightarrow Q$$
$$C \leftrightarrow R$$

Since ΔABC is similar to ΔPQR, then all the corresponding angles are equal.

$\therefore x° = \angle QPR$

$= \angle BAC \ (Q \leftrightarrow B, P \leftrightarrow A, R \leftrightarrow C)$

$= 25°$

$\therefore x = 25$

Since ΔABC is similar to ΔPQR, then all the ratios of the corresponding sides are equal.

$\therefore \frac{BC}{QR} = \frac{AB}{PQ} \ (B \leftrightarrow Q, C \leftrightarrow R, A \leftrightarrow P)$

i.e. $\frac{y}{3.2} = \frac{7.5}{5}$

$\therefore y = \frac{7.5}{5} \times 3.2$

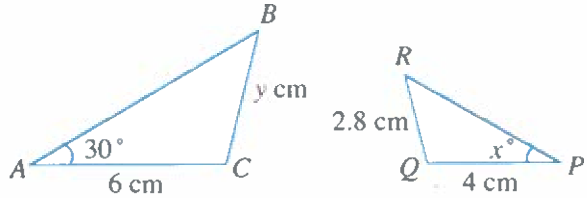
$= 4.8$

PRACTISE NOW 6

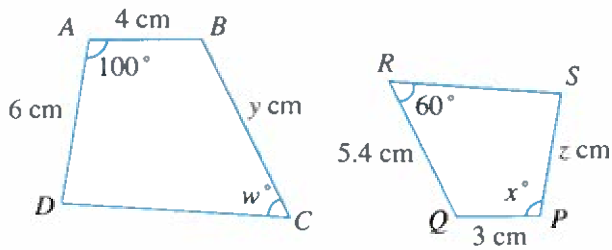
SIMILAR QUESTIONS

Exercise 8B Questions 1(a)–(d), 3(a)–(b), 4–5

1. Given that ΔABC is similar to ΔPRQ, find the values of the unknowns in the triangles.



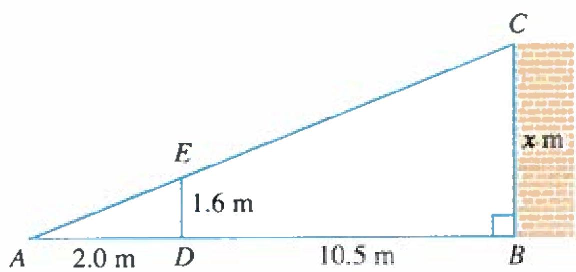
2. Given that the quadrilateral $ABCD$ is similar to the quadrilateral $PQRS$, find the values of the unknowns in the quadrilaterals.



Worked Example 7

(Similar Triangles in Real-World Context)

Nora stands at point D, 2.0 m in front of a spotlight at point A. She is 1.6 m tall and is facing the wall of a building which is 10.5 m away from her. How tall is her shadow on the wall of the building?



Solution:

Let the height of her shadow be x m.

We observe that the ΔABC and ΔADE are right-angled triangles with one common angle A. Hence the two triangles are similar.

Since ΔABC and ΔADE are similar, then all the ratios of the corresponding sides are equal.

$$\frac{AD}{AB} = \frac{ED}{CB}$$

$$\frac{2}{2+10.5} = \frac{1.6}{x}$$

$$2x = 1.6(2 + 10.5)$$

$$2x = 1.6(12.5)$$

$$\therefore x = 10 \text{ m}$$

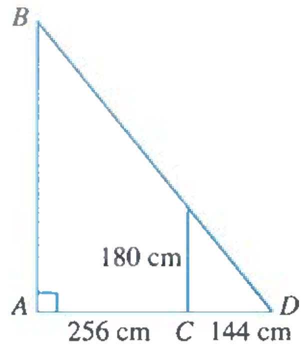
The height of her shadow on the wall is 10 m.

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 8B Questions 6, 9

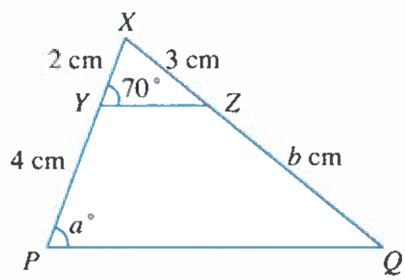
Ethan stands 256 cm at point C from a lamp post AB. He is 180 cm tall and his shadow from the lamp is 144 cm long. Find the height AB, in metres, of the lamp post.



Worked Example 8

(Problem involving Similar Triangles)

Given that ΔXYZ is similar to ΔXPQ, calculate the values of the unknowns in the figure.



Solution:

Since ΔXYZ is similar to ΔXPQ, then the corresponding vertices match:

$$X \leftrightarrow X$$
$$Y \leftrightarrow P$$
$$Z \leftrightarrow Q$$

Since ΔXYZ is similar to ΔXPQ, then all the corresponding angles are equal.

$\therefore a° = \angle XPQ$

$= \angle XYZ$ ($X \leftrightarrow X$, $P \leftrightarrow Y$, $Q \leftrightarrow Z$)

$= 70°$

$\therefore a = 70$

Since ΔXYZ is similar to ΔXPQ, then all the ratios of the corresponding sides are equal.

$\therefore \dfrac{XQ}{XZ} = \dfrac{XP}{XY}$ ($X \leftrightarrow X$, $Q \leftrightarrow Z$, $P \leftrightarrow Y$)

i.e. $\dfrac{XQ}{3} = \dfrac{2+4}{2}$

$\therefore XQ = \dfrac{6}{2} \times 3$

$= 9$ cm

$\therefore b = 9 - 3$

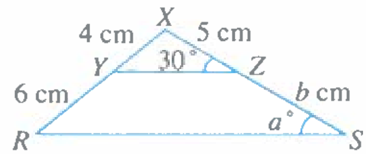
$= 6$

PRACTISE NOW 8

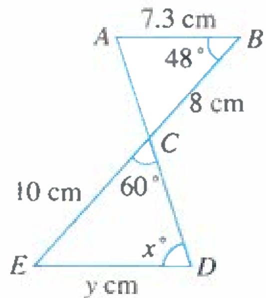
SIMILAR QUESTIONS

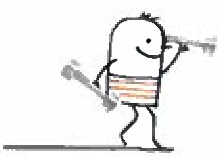
Exercise 8B Questions 7–8

1. Given that ΔXYZ is similar to ΔXRS, find the values of the unknowns in the figure.



2. Given that ΔABC is similar to ΔDEC, find the values of the unknowns in the figure.



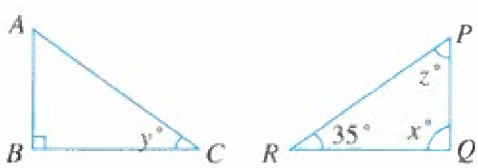


Exercise 8B

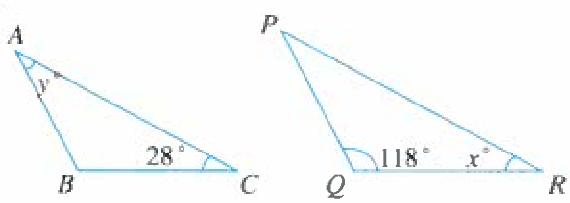
BASIC LEVEL

1. Given that ΔABC is similar to ΔPQR, find the values of the unknowns in each of the following pairs of triangles.

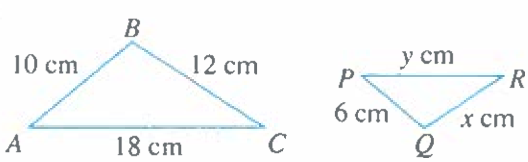
(a)



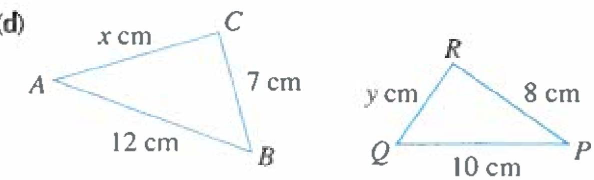
(b)



(c)

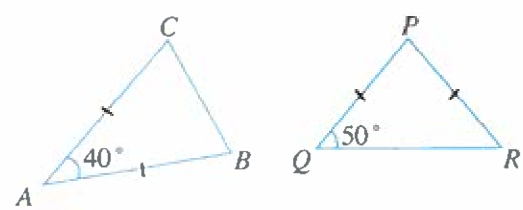


(d)

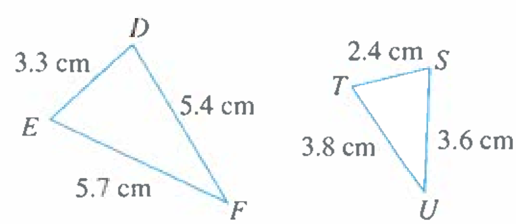


INTERMEDIATE LEVEL

2. (a) Is ΔABC similar to ΔPQR? Explain your answer.

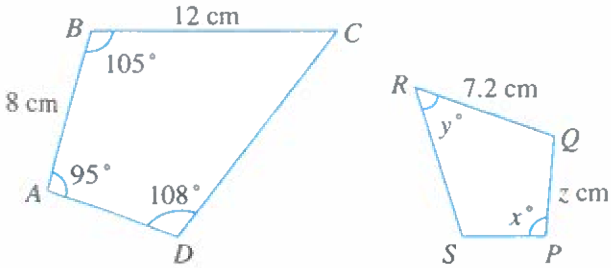


(b) Is ΔDEF similar to ΔSTU? Explain your answer.

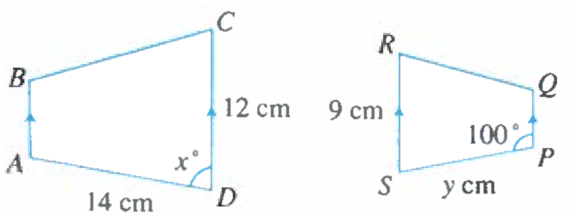


3. Given that the quadrilateral $ABCD$ is similar to the quadrilateral $PQRS$, find the values of the unknowns in each of the following pairs of quadrilaterals.

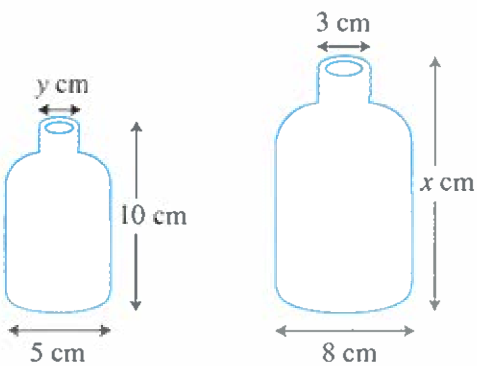
(a)



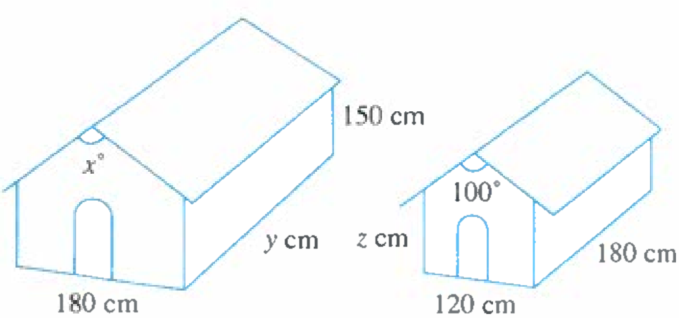
(b)



4. Two similar water bottles are as shown. Find the values of the unknowns.



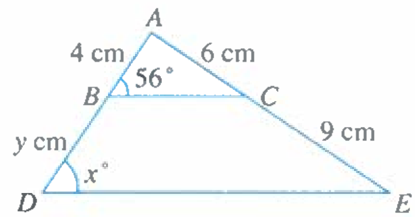
5. Two similar toy houses are as shown. Find the values of the unknowns.



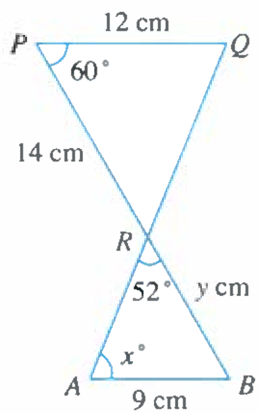
6. A pole of height 3 m is placed in front of a standing lamp 10 m away. The length of the shadow of the pole that is cast as a result of the light from the lamp is 6 m. Find the height of the lamp.

ADVANCED LEVEL

7. Given that ΔABC is similar to ΔADE, find the values of the unknowns in the figure.



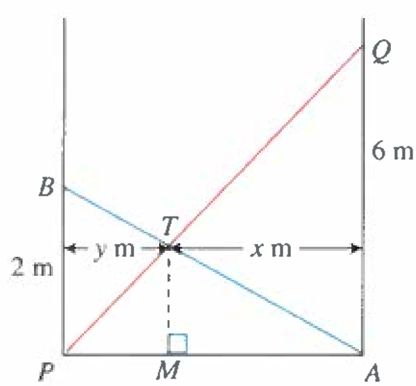
8. Given that ΔPQR is similar to ΔBAR, find the values of the unknowns in the figure.



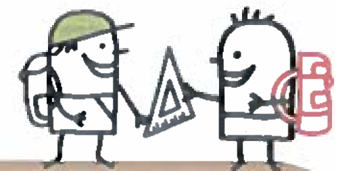
9. Two ladders AB and PQ are resting against opposite walls of an alley. The ladders AB and PQ are 2 m and 6 m above the ground respectively and T is the point where the 2 ladders meet.

(i) Given that ΔTBP is similar to ΔTAQ, find an expression, in terms of y, for the length of PA.

(ii) Given that ΔPTM is similar to ΔPQA, find the length of TM.



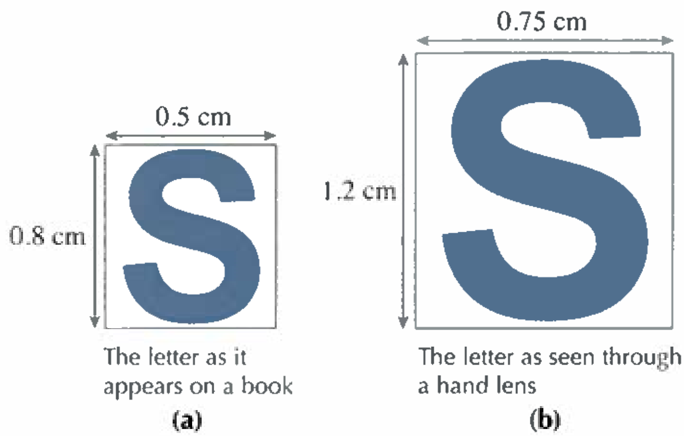
8.3 Similarity, Enlargement and Scale Drawings



Similarity and Enlargement

In the previous section, we have learnt that two figures are similar if they have exactly the same shape but not necessarily the same size. Their dimensions are in proportion.

Look at Fig. 8.10**(a)**. A letter 'S' may appear small on a book. If a person uses a hand lens and enlarges the letter for a clearer view (see Fig. 8.10**(b)**), the letter would appear larger.



Scale factor

$= \frac{0.75}{0.5}$

$= 1.5$

Fig. 8.10
(Diagram not drawn to scale)

The larger letter is an enlargement of the original letter 'S'. We say that the two letters are similar to each other. The corresponding lengths of the letters are in proportion. The ratio of length of the enlarged letter to the corresponding length of the original letter is known as the ***scale factor***.

Another example is the enlargement of a photograph on a screen using a visualiser (see Fig. 8.5**(b)** on page 203). The photograph and its projected image are similar.

In Fig. 8.11, ΔABC is similar to $\Delta A'B'C'$. We say that $\Delta A'B'C'$ is an enlargement of ΔABC with a scale factor of $k = \frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC}$.

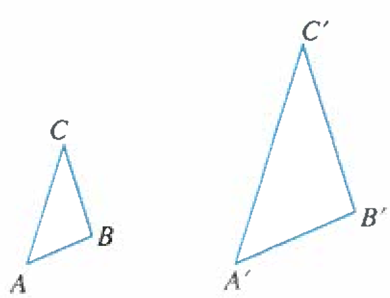
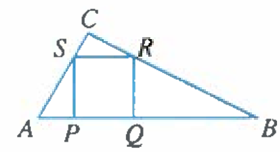


Fig. 8.11

Draw a triangle ABC where $AB = 8$ cm, $BC = 6$ cm and $AC = 4$ cm. By using the concept of enlargement, construct a square $PQRS$ inside the triangle such that PQ is on AB, R is on BC and S is on AC.

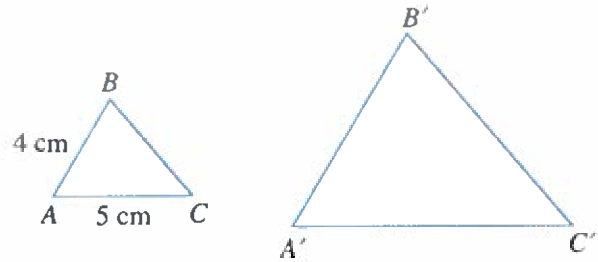


An enlargement with a scale factor between 0 and 1 will produce a smaller image, but it is still called an enlargement in mathematics.

Worked Example 9

(Problem involving Enlargement of Figures)

In the figure, $\Delta A'B'C'$ is an enlargement of ΔABC with a scale factor of 2. Given that $AB = 4$ cm and $AC = 5$ cm, find the length of $A'B'$ and of $A'C'$.



Solution:

ΔABC is similar to $\Delta A'B'C'$ under enlargement.

$\therefore \dfrac{A'B'}{AB} = \dfrac{A'C'}{AC} = 2$ (scale factor)

i.e. $\dfrac{A'B'}{4} = 2$ and $\dfrac{A'C'}{5} = 2$

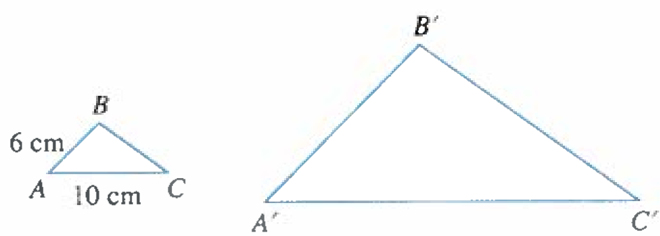
$\therefore A'B' = 8$ cm and $A'C' = 10$ cm

PRACTISE NOW 9

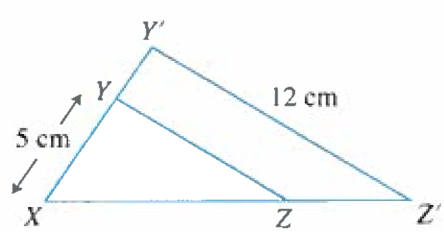
SIMILAR QUESTIONS

Exercise 8C Questions 1–2, 8–9

1. In the figure, $\Delta A'B'C'$ is an enlargement of ΔABC with a scale factor of 3. Given that $AB = 6$ cm and $AC = 10$ cm, find the length of $A'B'$ and of $A'C'$.



2. In the figure, $\Delta XY'Z'$ is an enlargement of ΔXYZ with a scale factor of 1.5. Given that $XY = 5$ cm and $Y'Z' = 12$ cm, find the length of XY' and of YZ.



3. A photograph shows Mr Goh, who is 180 cm tall, standing in front of his terrace house. In the photograph, the height of Mr Goh is 9 cm and that of his house is 22.5 cm. Find the actual height of the house, giving your answer in metres.

Similarity and Scale Drawings

In our daily activities, we sometimes need to enlarge or reduce pictures or drawings of actual objects. For example, if we wish to draw a plan of a badminton court in order to explain the rules of the game, we need to make a much smaller drawing on paper or on a whiteboard. If we wish to show a diagram of the apparatus used for the distillation of water, we can enlarge the diagram on a screen using a visualiser.

Fig. 8.12 shows the floor plan of a house. It is similar to the actual floor of the house.

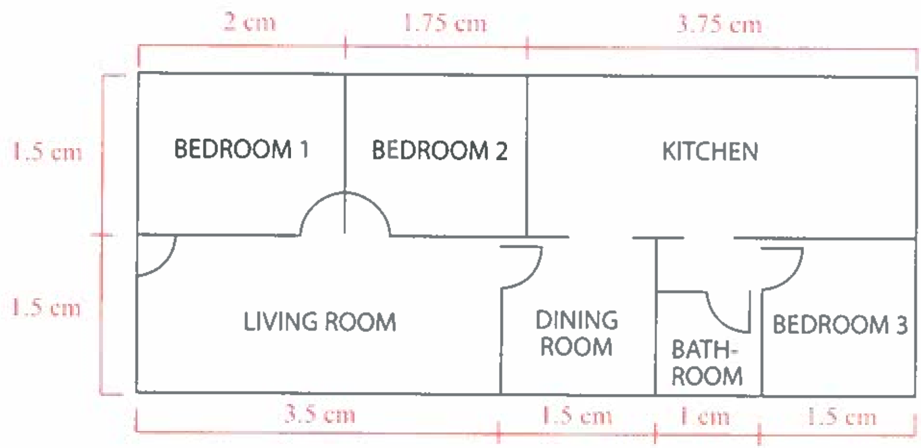


Fig. 8.12

The dimensions of the plan are proportional to the actual dimensions of the house. Fig. 8.12 has been drawn to a scale of 1 cm to 2 m, i.e. 1 cm on the plan represents 2 m on actual ground. From Fig. 8.12, we have:

(i) Length of the living room $= (3.5 \times 2)$ m

$= 7$ m

Width of the living room $= (1.5 \times 2)$ m

$= 3$ m

(ii) Area of Bedroom 1 on the plan = 2 cm × 1.5 cm

Actual area of Bedroom 1 $= (2 \times 2)$ m $\times (1.5 \times 2)$ m

$= 12\text{ m}^2$

(iii) Area of the living room $= (3.5 \times 2)$ m $\times (1.5 \times 2)$ m

$= 21\text{ m}^2$

(iv) Total area of the house $= (7.5 \times 2)$ m $\times (3 \times 2)$ m

$= 90\text{ m}^2$

SIMILAR QUESTIONS

Exercise 8C Question 10

Scale Drawings

Let us see one example of how scale drawings can help us to find the information of the actual object it represents and another example on how we can use graph paper to draw a scale diagram.

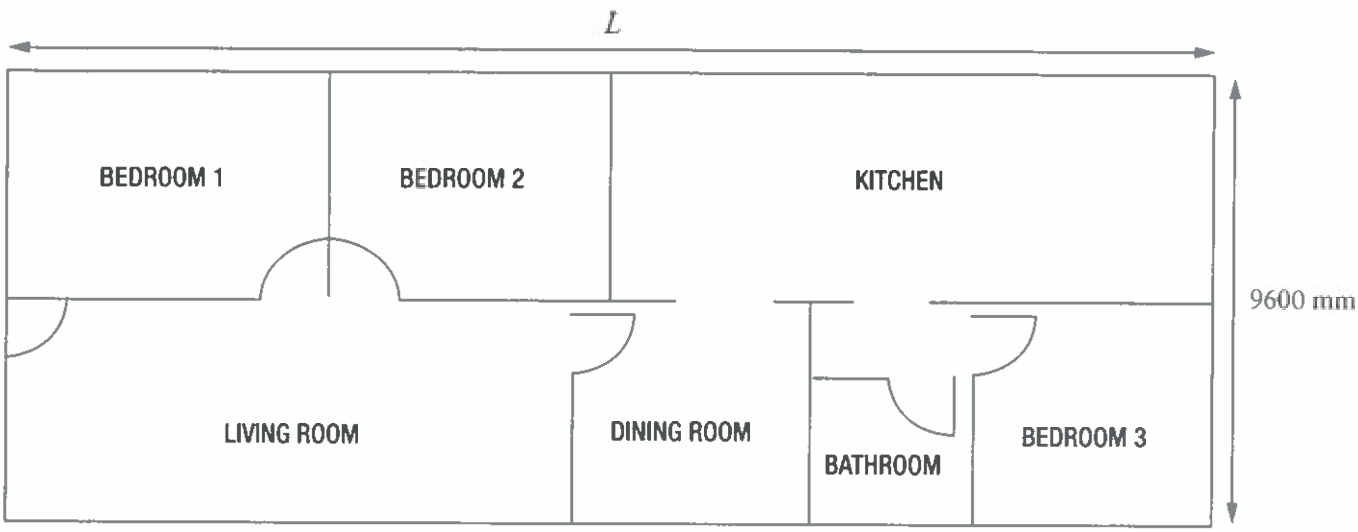
Worked Example 10

(Finding the Scale of a Drawing by Measurements)

The diagram shows a scale drawing of an apartment.

(i) By measuring the scale drawing, find its scale.

(ii) Find the actual length L, in metres, of the apartment.



Solution:

(i) By measuring the vertical width, which represents 9600 mm, we get 60 mm.
So, the scale is 60 : 9600, which is 1 : 160.

(ii)

Plan		Actual
1 mm	represents	160 mm
160 mm	represents	(160 × 160) mm
		= 25 600 mm
		= 25.6 m

∴ The actual length L of the apartment is 25.6 m.

PRACTISE NOW 10

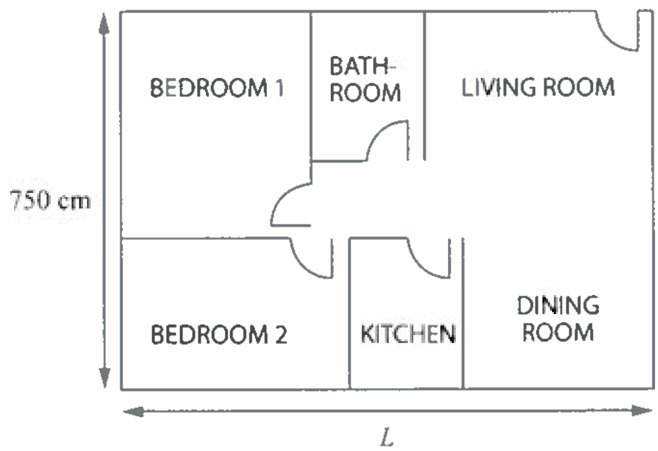
The diagram shows a scale drawing of another apartment.

(i) By measuring the scale drawing, find its scale.

(ii) Find the actual length L, in metres, of the apartment.

SIMILAR QUESTIONS

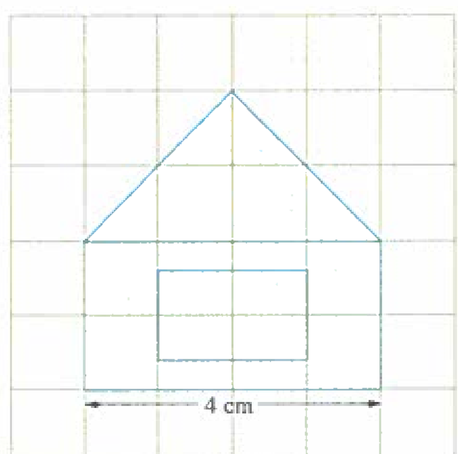
Exercise 8C Question 3



Worked Example 11

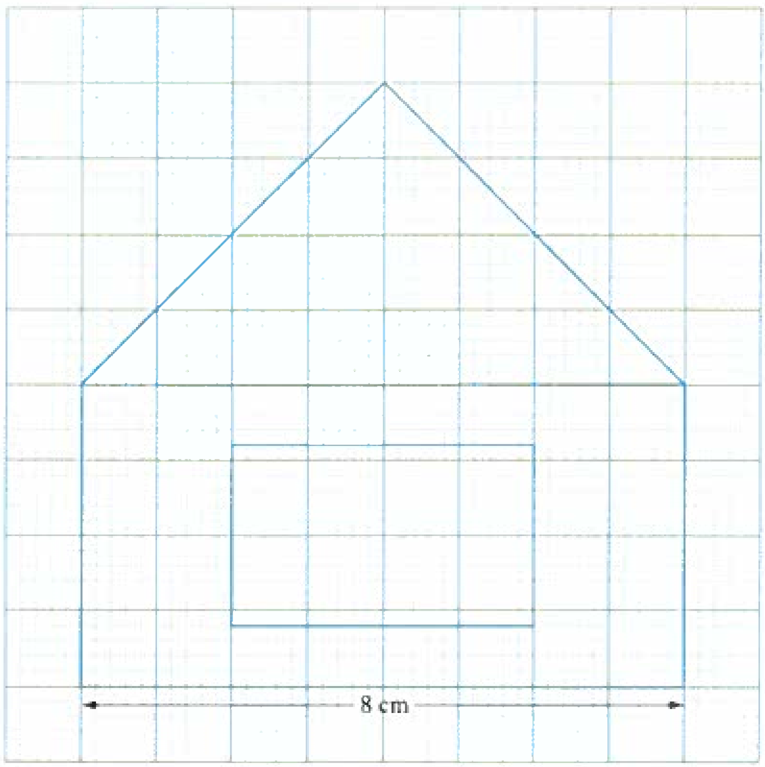
(Drawing a Scale Diagram)

Using graph paper, draw a scale diagram of the figure shown with a scale of 1 : 2.



Solution:

Since the scale is 1 : 2, we use the grid and enlarge each part of the figure by multiplying the length by 2. For example, the length of the larger rectangle changes from 4 cm to 8 cm. The scale drawing is shown below.

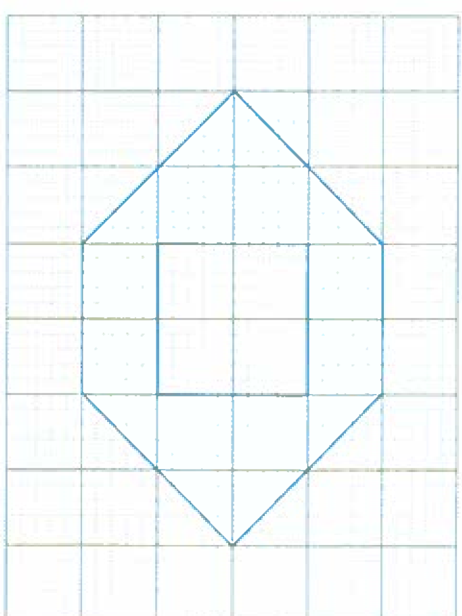


PRACTISE NOW 11

SIMILAR QUESTIONS

Exercise 8C Questions 4(a)–(b)

Using graph paper, draw a scale diagram of the figure shown with a scale of 1 : 2.5.



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In Book 1, we have learnt geometrical constructions. We shall now use scale drawings to construct quadrilaterals and solve related problems.

Worked Example 12

(Using Scale Drawings to Construct Quadrilaterals)

The scale drawing in the space below shows the positions of the towns A, B and C. A is due North of B and C is due East of B.

(i) Given that $\angle BAD = 80^\circ$ and $\angle BCD = 65^\circ$, find and label the position of the town D.

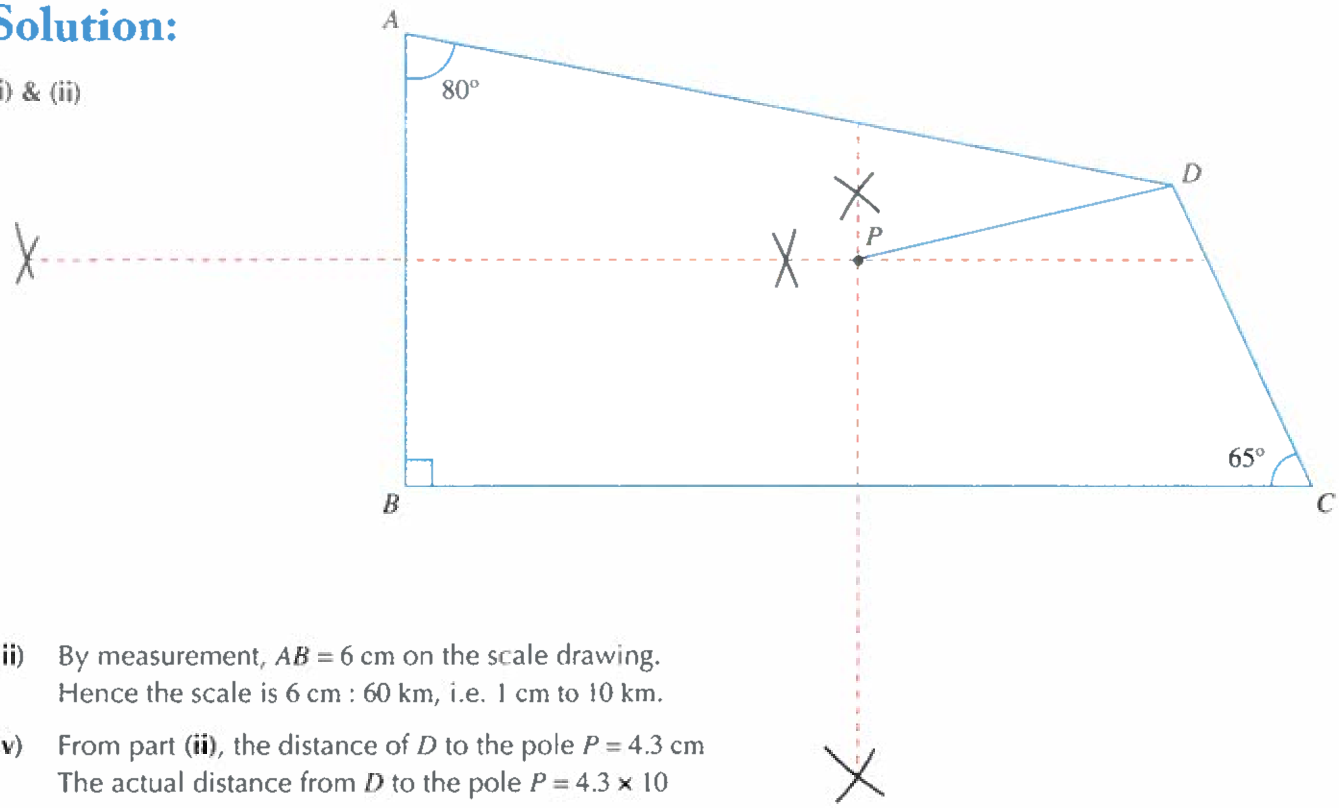
(ii) A pole P is to be erected equidistant from towns A, B and C.
By constructing perpendicular bisectors, find and label the position of the pole P.

(iii) The actual distance of $AB = 60$ km. By measuring the scale drawing, find its scale.

(iv) Hence, find the actual distance from D to the pole P.

Solution:

(i) & (ii)



(iii) By measurement, $AB = 6$ cm on the scale drawing.
Hence the scale is 6 cm : 60 km, i.e. 1 cm to 10 km.

(iv) From part **(ii)**, the distance of D to the pole $P = 4.3$ cm
The actual distance from D to the pole $P = 4.3 \times 10$
$= 43$ km.

PRACTISE NOW 12

SIMILAR QUESTIONS

Exercise 8C Question 5

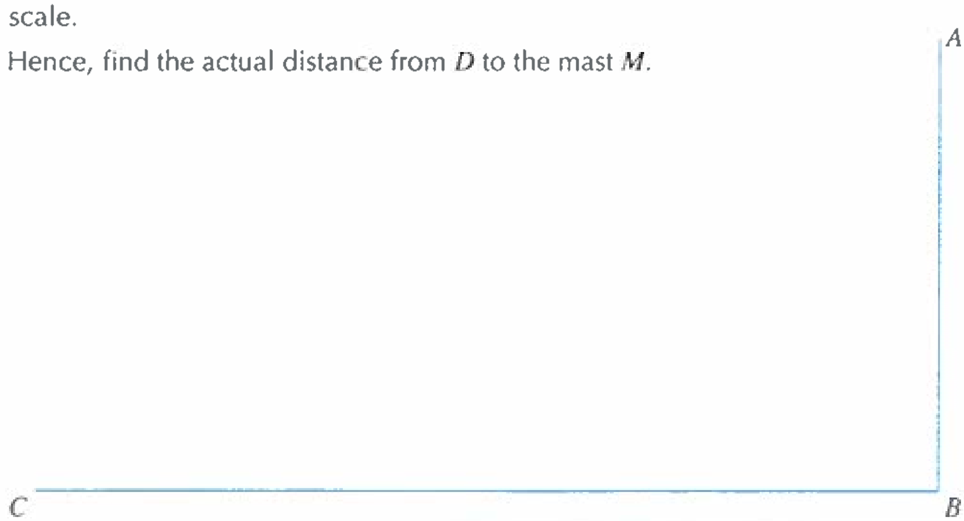
The scale drawing in the space below shows the positions of the towns A, B and C. A is due North of B and C is due West of B.

(i) Given that $\angle BCD = 80°$ and $\angle BAD = 110°$, find and label the position of the town D.

(ii) A mast M is to be erected equidistant from towns A, B and C.
By constructing perpendicular bisectors, find and label the position of the mast M.

(iii) The actual distance of AB is 48 km. By measuring the scale drawing, find its scale.

(iv) Hence, find the actual distance from D to the mast M.



Worked Example 13

(Problem involving Scale Drawing)

The scale of a building plan is 1 cm to 50 cm. Calculate

(i) the actual length of a bedroom if it is represented by a length of 9.2 cm on the plan,

(ii) the length on the plan that represents an actual length of 28 m.

Solution:

(i)

Plan		Actual
1 cm	represents	50 cm (scale)
9.2 cm	represents	(9.2×50) cm
		$= 460$ cm
		$= 4.6$ m

$\therefore$ The actual length of the bedroom is 4.6 m.

(ii)

Actual		Plan
50 cm	is represented by	1 cm (scale)
1 m (100 cm)	is represented by	$\frac{1}{50} \times 100$ cm $= 2$ cm
28 m	is represented by	(28×2) cm
		$= 56$ cm

$\therefore$ The length on the plan is 56 cm.

ATTENTION

We should always write what we want to find on the right-hand side. In **(ii)**, since we want to find the length on the plan, we write 'Plan' on the right-hand side.

PRACTISE NOW 13

SIMILAR QUESTIONS

Exercise 8C Questions 11–12, 21

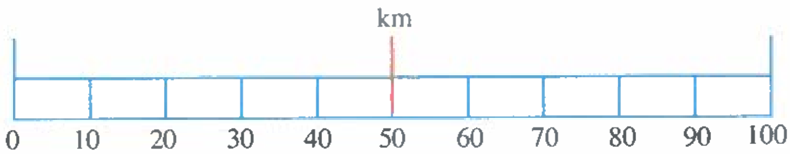
1. The scale of the floor plan of a house is 1 cm to 2.5 m. Find
   (i) the actual length of the dining room if it is represented by a length of 1.25 cm on the plan,
   (ii) the length on the plan that represents the width of a bedroom if its actual width is 3.4 m.

2. A model of a cruise liner is made to a scale of 1 cm to 4 m. The length of the model cruise liner is 67 cm. Find
   (i) the actual length of the cruise liner,
   (ii) the length of the model cruise liner if it is made to a scale of 1 cm to 10 m.

3. A rectangular school hall is 50 m long and 30 m wide.
   (i) Using a scale of 1 cm to 5 m, make a scale drawing of the school hall.
   (ii) From your scale drawing, find the actual distance between the opposite corners of the school hall.

Map Scales

Maps are scale drawings of actual land. The **linear scale** of a map is usually given at a corner of the map. There are several ways to represent the scale of a map. For example, on a map of Singapore, the scale as shown in Fig. 8.13 may be given.



Scale 1 : 1 000 000 **Fig. 8.13**

There are two ways to read the scale. If we use a ruler to measure the length from 0 to 10 km, we will find that it is 1 cm. Thus 1 cm represents 10 km. This is the same as the scale 1 : 1 000 000. When a scale is given in this form, it means that we have to use the same units on both sides, i.e. 1 m : 1 000 000 m, 1 km : 1 000 000 km or 1 cm : 1 000 000 cm = 10 km, so 1 cm represents 10 km.

The scale of 1 : 1 000 000 can also be represented as a **representative fraction** (R.F.) of $\frac{1}{1000\,000}$. For example, if the R.F. is $\frac{1}{200}$, the scale is 1 : 200. When we use R.F., the numerator must always be 1.

Worked Example 14

(Problem involving Map Scale)

A map has a scale of 1 cm to 3 km.

(i) If a road has a length of 3 cm on the map, calculate its actual length.

(ii) If the distance between two stadiums is 7.5 km, find the corresponding distance on the map.

(iii) Express the scale of the map in the form $\frac{1}{n}$, where n is an integer.

Solution:

(i)

Map		Actual
1 cm	represents	3 km (scale)
3 cm	represents	(3 × 3) km
		= 9 km

∴ The actual length of the road is 9 km.

(ii)

Actual		Map
3 km	is represented by	1 cm (scale)
1 km	is represented by	$\frac{1}{3}$ cm
7.5 km	is represented by	$\left(7.5 \times \frac{1}{3}\right)$ cm
		= 2.5 cm

∴ The distance between the two stadiums on the map is 2.5 cm.

(ii) 3 km = 300 000 cm

i.e. the scale of the map is $\frac{1}{300\,000}$.

ATTENTION

We should always write what we want to find on the right-hand side. In **(ii)**, since we want to find the length on the map, we write 'Map' on the right-hand side.

PRACTISE NOW 14

SIMILAR QUESTIONS

Exercise 8C Questions 5–6, 13–14, 19

1. A map has a scale of 1 cm to 5 km.
   (i) If a road has a length of 6.5 cm on the map, find its actual length.
   (ii) If the distance between two towns is 25 km, calculate the corresponding distance on the map.
   (iii) Express the scale of the map in the form $\frac{1}{n}$, where n is an integer.

2. A map is drawn to a scale of 1 : 50 000.
   (i) Find the actual length that is represented by 2 cm on the map, giving your answer in kilometres.
   (ii) Calculate the length on the map that represents an actual length of 14.5 km.

ATTENTION

For Question 2, '1 : 50 000' is the same as '1 cm : 50 000 cm'.

Using the scale of a map, we can also find the actual area of a site from its area on the map. For example, if the scale of a map is 1 cm to 2 km, then 1 cm^2 on the map represents an actual area of $(2 \text{ km})^2 = 4 \text{ km}^2$ (see Fig. 8.14). Therefore, the **area scale** of the map is 1 cm^2 to 4 km^2.

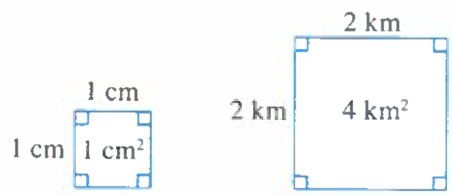


Fig. 8.14

Worked Example 15

(Problem involving Map Drawing)

A scale of 1 cm to 0.5 km is used for a map.

(i) If a plot of land has an area of 8 cm^2 on the map, calculate its actual area.

(ii) If the actual area of a pond is 50 000 m^2, find its area on the map.

Solution:

(i)

Map		Actual
1 cm	represents	$0.5 \text{ km} = \frac{1}{2} \text{ km}$ (scale)
1 cm^2	represents	$\left(\frac{1}{2} \text{ km}\right)^2 = \frac{1}{4} \text{ km}^2$
8 cm^2	represents	$\left(8 \times \frac{1}{4}\right) \text{km}^2$
		$= 2 \text{ km}^2$

∴ The actual area of the plot of land is 2 km^2.

(ii)

	Actual		Map
	0.5 km	is represented by	1 cm
i.e.	500 m	is represented by	1 cm
	1 m	is represented by	$\frac{1}{500}$ cm
	1 m^2	is represented by	$\left(\frac{1}{500}\text{ cm}\right)^2 = \frac{1}{250\,000}\text{ cm}^2$
	50 000 m^2	is represented by	$\left(50\,000 \times \frac{1}{250\,000}\right)\text{ cm}^2$
			$= 0.2\text{ cm}^2$

∴ The area of the pond on the map is 0.2 cm^2.

Problem Solving Tip

In **(ii)**, since the actual area of the pond is given in m^2, it will be easier to convert the linear scale from km to m before finding the area scale.

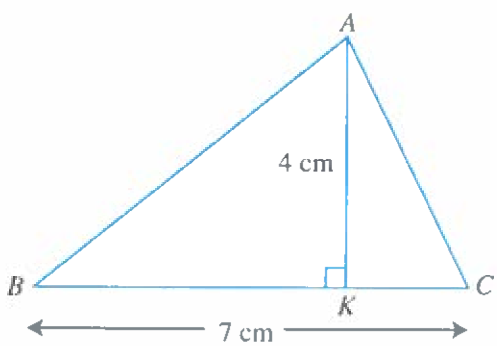
PRACTISE NOW 15

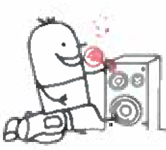
SIMILAR QUESTIONS

Exercise 8C Questions 7, 15–17, 20

1. A scale of 1 cm to 2 km is used for a map.
   (i) If a plot of land has an area of 3 cm^2 on the map, find its actual area.
   (ii) If the actual area of a lake is 18 000 000 m^2, calculate its area on the map.

2. A triangular plot of land ABC is drawn to a scale of 1 cm to 3 km. Given that $BC = 7$ cm and $AK = 4$ cm, find its actual area.





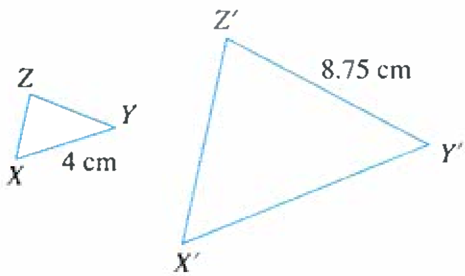
Performance Task

Work in groups to make a scale drawing of an existing or a dream classroom or bedroom. Present your drawing to the class, explaining the choice of scale used.

Exercise 8C

BASIC LEVEL

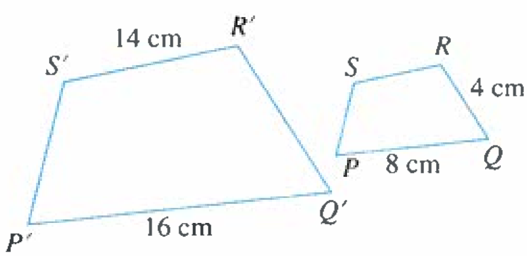
1. In the figure, $\Delta X'Y'Z'$ is an enlargement of ΔXYZ with a scale factor of 2.5. Given that $XY = 4$ cm and $Y'Z' = 8.75$ cm, find the lengths of $X'Y'$ and YZ.



2. In the figure, $P'Q'R'S'$ is an enlargement of $PQRS$ with a scale factor of k.

(i) Given that $PQ = 8$ cm and $P'Q' = 16$ cm, find the value of k.

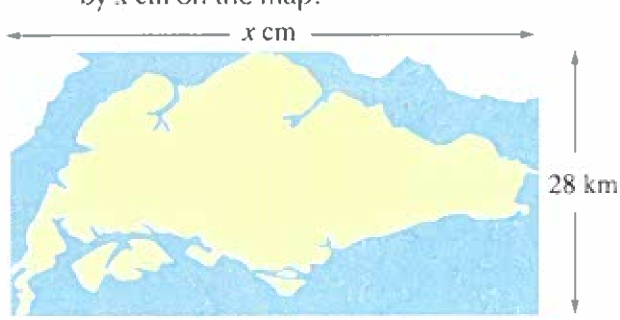
(ii) Given that $QR = 4$ cm and $S'R' = 14$ cm, calculate the length of $Q'R'$ and of SR.



3. The figure shows a map of Singapore. The actual length of Singapore from the North to the South is 28 km.

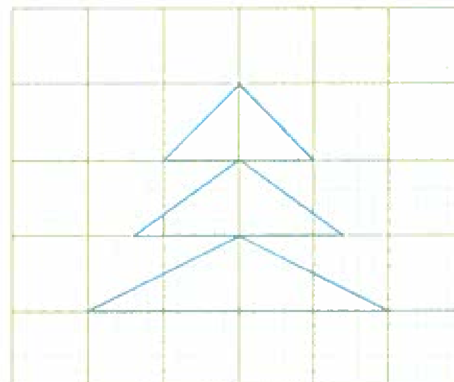
(i) By taking measurements, find the scale of the map.

(ii) What is the actual distance between the East and West of Singapore which is represented by x cm on the map?

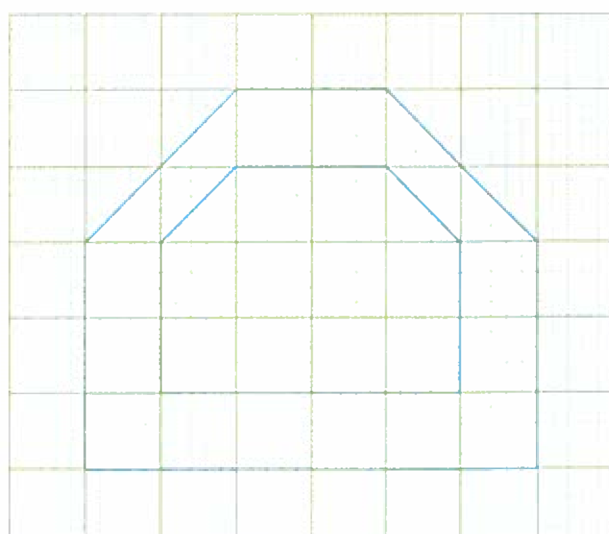


4. Using graph paper, draw a scale diagram for each of the following figures shown with their scales given.

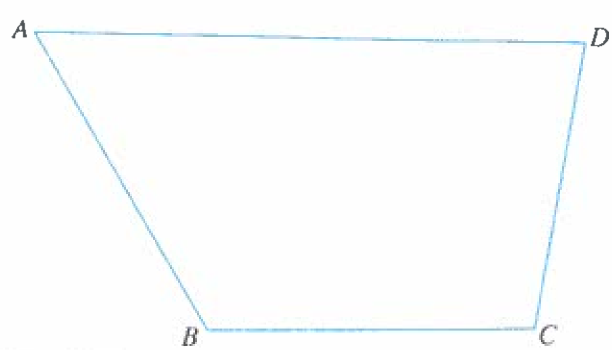
(a) 1 : 2



(b) 1 : 0.5



5. The diagram shows a scale drawing of a field $ABCD$. It is drawn to a scale of 1 cm to 25 m.



(a) On the diagram, construct
   (i) the angle bisector of $\angle C$,
   (ii) the perpendicular bisector of CD.

(b) A treasure is hidden in the field.
   (i) It is equidistant from the points C and D and is equidistant from the lines BC and CD. Mark and label the point X, the position where the treasure is hidden.
   (ii) Find the actual shortest distance of the treasure from the corner A. Give your answer in metres.

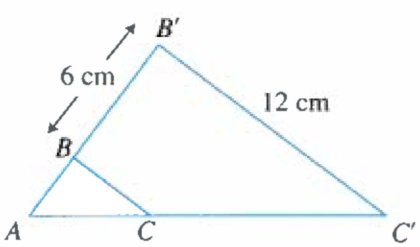
6. A map of Singapore has a scale of 1 cm to 250 m.
(i) Given that the Tuas Second Link has a length of 7.68 cm on the map, find its actual length.
(ii) Given that the actual width of the Tuas Second Link is 25 m, calculate the corresponding width on the map.
(iii) Express the scale of the map in the form $\frac{1}{n}$, where n is an integer.

7. A map is drawn to a scale of 1 : 20 000.
(i) Find the actual length that is represented by $5\frac{1}{2}$ cm on the map, giving your answer in kilometres.
(ii) Calculate the length on the map that represents an actual length of 100 m.

8. A scale of 1 cm to 8 km is used for a map.
(i) If a forest has an area of 5 cm^2 on the map, find its actual area.
(ii) If the actual area of a park is 128 km^2, calculate its area on the map.

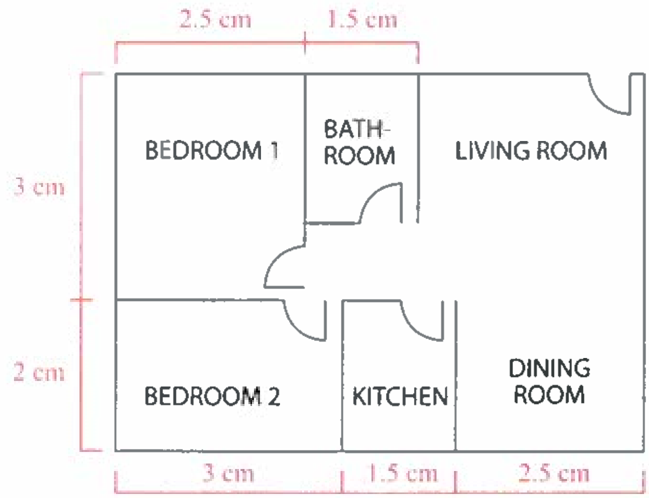
INTERMEDIATE LEVEL

9. In the figure, $\Delta AB'C'$ is an enlargement of ΔABC with a scale factor of 3. Given that $B'C' = 12$ cm and $BB' = 6$ cm, find the length of BC and of AB'.



10. In a television commercial, a toddler of height 75 cm is standing next to a tin of milk of height 24 cm. If the height of the toddler is 25 cm on a television screen, find the height of the tin of milk on the screen.

11. The figure shows the floor plan of an apartment which has been drawn to a scale of 1 cm to 1.5 m. Find
(i) the actual dimensions of Bedroom 1,
(ii) the actual area of the kitchen,
(iii) the actual total area of the apartment.

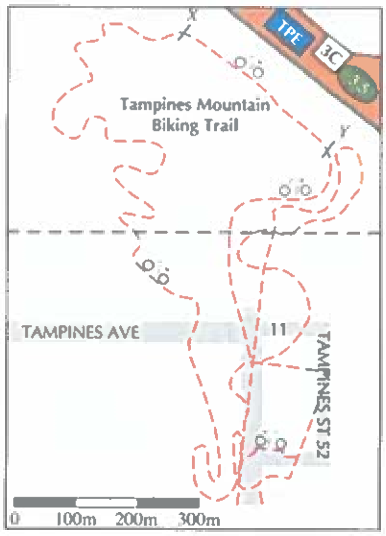


12. On the floor plan of an apartment, the length of a bedroom is 12 cm. The actual length of the bedroom is 3 m.
(i) What is the scale used?
(ii) Find the width of the living room on the floor plan if its actual width is 425 cm.

13. A model of the Marina Bay Sands Tower 1 is made to a scale of 1 cm to 15 m. The height of the model tower is 42.4 cm. Find
(i) the actual height of the tower,
(ii) the height of the model tower if it is made to a scale of 1 cm to 12 m.

14. A map of Singapore has a scale of 4 cm to 5 km. The distance between Paragon Shopping Centre and Plaza Singapura on the map is 1.12 cm. Find
(i) the actual distance between the two shopping centres,
(ii) the distance between the two shopping centres on another map of Singapore that is drawn to a scale of 1 : 175 000.

15. The figure shows a map of the Tampines Mountain Biking Trail. The scale is given in the form of a bar at the bottom of the map, showing 0 m to 300 m.

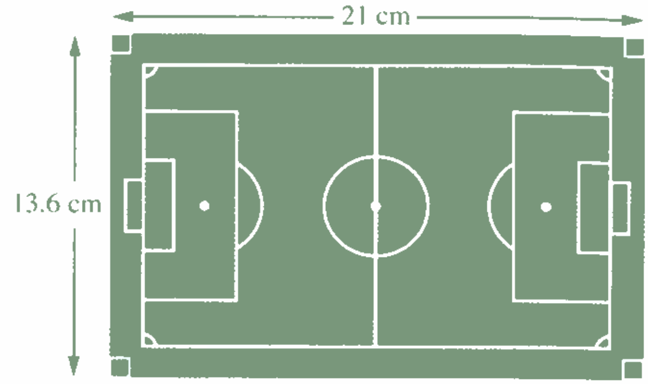


(i) Express the scale of the map in the form 1 : n, where n is an integer.
(ii) Estimate the actual distance XY of the biking trail.
(iii) Khairul cycles from X to Y, but he discovers that the actual distance XY is about 350 m. Suggest a reason why the actual distance XY is different from your estimate in **(ii)**.

16. A map has a scale of 1 cm to 500 m.
(i) Express the scale of the map in the form 1 : n, where n is an integer.
(ii) If the distance between two districts is 28 km, find the corresponding distance on the map.
(iii) If a jungle has an area of 12 cm² on the map, calculate its actual area in square kilometres.

17. A map is drawn to a scale of 1 : 240 000.
(i) If a seawater lake has an area of 3.8 cm² on the map, find its actual area in square kilometres.
(ii) If the actual area of a plot of land is 2 908 800 m², calculate its area on the map.

18. The Old Trafford soccer field in Manchester, England, is drawn to a scale of 1 cm to 5 m. Given that the length and the breadth of the soccer field on the scale drawing are 21 cm and 13.6 cm respectively, find its actual area in square metres.



ADVANCED LEVEL

19. Use the map of South Malaysia to answer the following questions. You may use a ruler to measure the approximate distances between any two places before finding the actual distances from the given scale.

(i) Express the scale of the map in the form $\frac{1}{n}$, where n is an integer.
(ii) Find the actual distance between Singapore and Kuantan.
(iii) How much would it cost to hire a taxi to travel from Melaka to Kuala Lumpur if the taxi fare is 60 cents per kilometre?

(iv) Calculate the time taken for a car to travel from Batu Pahat to Port Dickson if its average speed is 60 km/h, giving your answer in hours and minutes.

(v) A train takes 4 hours to travel from Johor Bahru to Segamat. Find its average speed, giving your answer in km/h.

20. On a map drawn to a scale of 1 cm to 500 m, a rectangular plot of land has a breadth of 8 cm and a length that is 175% of its breadth. Find its actual area in hectares. (1 hectare = 10 000 m^2)

21. A triangular field ABC is such that AB = 90 m, BC = 70 m and AC = 85 m.

(i) Using a scale of 1 cm to 10 m, make a scale drawing of the field.

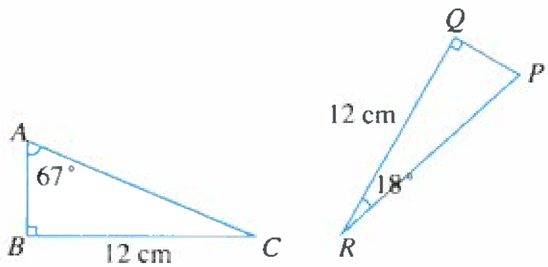
(ii) From your scale drawing, find the actual distance from B to the point K, where K is equidistant from A, B and C.
Hint: Points on the perpendicular bisector of AB are equidistant from A and B.

1. Two figures are **congruent** if they have exactly the same shape and size.
   They can be mapped onto each other under **translation**, **rotation** and **reflection**.
2. A figure and its image under a translation, a rotation or a reflection are *congruent* i.e. all corresponding angles and corresponding sides are equal.
3. Two figures are **similar** if they have exactly the same shape but *not* necessarily the same size.
   If two similar figures also have exactly the same size, then they are congruent. In other words, congruence is a *special case* of similarity.
4. If two polygons are *similar*,
   - all the corresponding angles are equal, *and*
   - all the ratios of the corresponding sides are equal.
5. A figure and its image under an *enlargement* are *similar*.
   - An enlargement with a scale factor *greater than 1* produces an *enlarged* image.
   - An enlargement with a scale factor *between 0 and 1* produces a *diminished* image.
   - An enlargement with a scale factor of *1* produces a *congruent* image.
6. If the **linear scale** of a map is 1 : x, it means that 1 cm on the map represents x cm on the actual piece of land, and the **area scale** of the map is 1 : x^2.

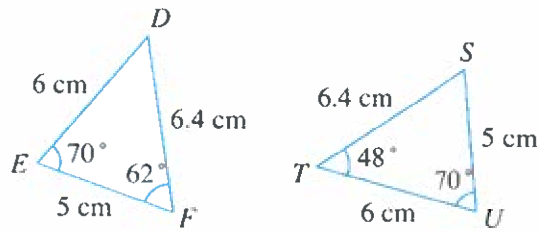
Review Exercise 8

1. Are the following pairs of triangles congruent? If so, explain your answer and write down the statement of congruence. If not, explain your answer.

(a)

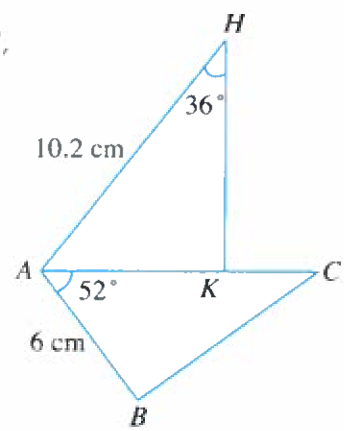


(b)

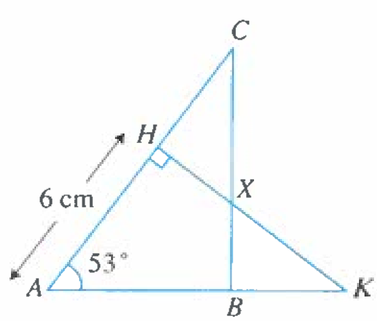


2. It is given that the quadrilateral $ABCD$ is congruent to the quadrilateral $PQRS$, $\angle A = 100°$, $\angle B = 70°$, $\angle C = 95°$ and $PQ = 6$ cm.
   (i) Write down the length of AB.
   (ii) Find $\angle S$.

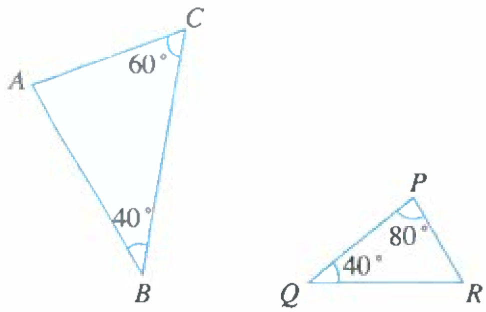
3. In the figure, $\Delta ABC \equiv \Delta AKH$. Given that $\angle BAC = 52°$, $\angle AHK = 36°$, $AB = 6$ cm and $AH = 10.2$ cm, find
   (i) $\angle AKH$,
   (ii) the length of KC.



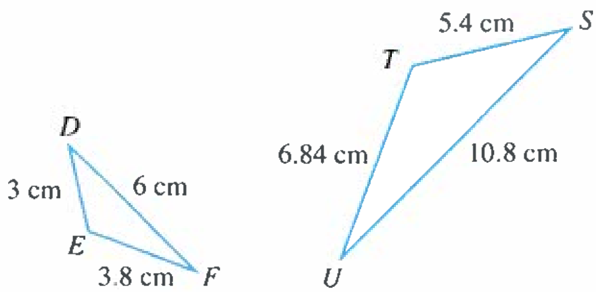
4. In the figure, $\Delta ABC \equiv \Delta AHK$. It is given that $\angle BAC = 53°$, $\angle AHK = 90°$, $AH = 6$ cm and the area of ΔABC is 24 cm^2.
   (i) Write down the size of $\angle ABC$.
   (ii) Write down the length of AB. Hence, find the length of BC.
   (iii) Calculate $\angle BXK$.



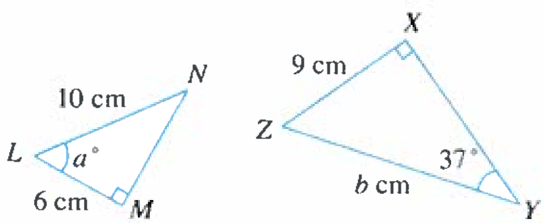
5. **(a)** Is ΔABC similar to ΔPQR? Explain your answer.



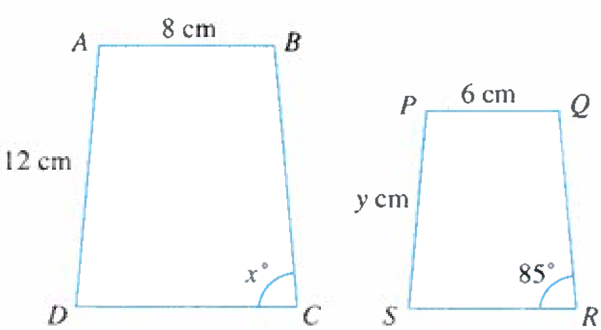
(b) Is ΔDEF similar to ΔSTU? Explain your answer.



6. Given that ΔLMN is similar to ΔZXY, find the values of the unknowns in the triangles.



7. Given that the quadrilateral $ABCD$ is similar to the quadrilateral $PQRS$, find the values of the unknowns in the quadrilaterals.



8. It is given that ΔABC is similar to ΔPQR, $\angle A = 60°$, $AB = 6$ cm, $AC = 8$ cm and $PR = 10$ cm.
   (i) Write down the size of $\angle P$.
   (ii) Find the length of PQ.

9. A model of an aircraft is made to a scale of 1 : 80.
   (i) If the wingspan of the model aircraft is 25 cm, find the wingspan of the actual aircraft, giving your answer in metres.
   (ii) If the length of the actual aircraft is 40 m, calculate the length of the model aircraft.

10. A model of a block of flats of The Pinnacle@Duxton housing project is made to a scale of 2 cm to 7.5 m. The height of the model block of flats is 41.6 cm. Find
   (i) the actual height of the block of flats,
   (ii) the height of the model block of flats if it is made to a scale of 5 cm to 12 m.

11. A map has a scale of 4 cm to 1 km.
   (i) Express the scale of the map in the form $\frac{1}{n}$, where n is an integer.
   (ii) If a river has a length of 3 cm on the map, find its actual length, giving your answer in kilometres.
   (iii) If the distance between two towns is 8 km, calculate the corresponding distance on the map.

12. A map of Singapore is drawn to a scale of 1 : 180 000.
   (i) Given that the length of the Circle Mass Rapid Transit Line is 35.7 km, find the corresponding length on the map, leaving your answer correct to 2 decimal places.
   (ii) Given that the distance between Sentosa and Changi Ferry Terminal is 13.5 cm on the map, calculate the actual distance between Sentosa and Changi Ferry Terminal, giving your answer in kilometres.
   (iii) The actual area of the Sentosa Island is 5 km^2. Find its area on the map, leaving your answer correct to 2 decimal places.

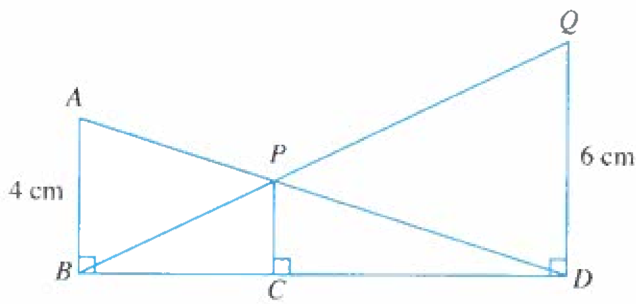
13. A map has a scale of 2 cm to 3 km.
   (i) Express the scale of the map in the form 1 : n, where n is an integer.
   (ii) If the distance between two towns is 7 cm on the map, find the actual distance between the two towns, giving your answer in kilometres.
   (iii) If the actual area of a lake is 81 km^2, calculate its area on the map.

14. A map of a region of Singapore is drawn to a scale of 1 : 25 000.
   (i) If two Town Councils are 3.5 km apart, find the corresponding distance on the map.
   (ii) A reservoir has an area of 16 cm^2 on the map. Calculate its actual area in square kilometres.

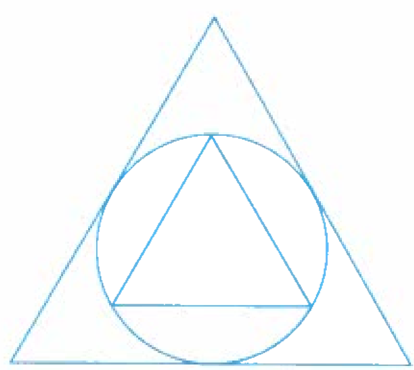
15. The floor plan of the basement of a shopping centre is drawn to a scale of 1 : 400.
   (i) If a corridor has a length of 24.5 cm on the plan, find its actual length, giving your answer in metres.
   (ii) A fast food restaurant occupies a floor area of 400 m^2. Calculate its area on the plan.
   (iii) If a supermarket has an area of 0.25 m^2 on the plan, find its actual area in hectares.
   (1 hectare = 10 000 m^2)

Challenge Yourself

1. In the figure, APD and BPQ are straight lines. AB, PC and QD are perpendicular to BCD. It is given that $AB = 4$ cm and $QD = 6$ cm.
   (i) Name three pairs of similar triangles.
   (ii) Find the ratio of the length of BC to the length of CD.

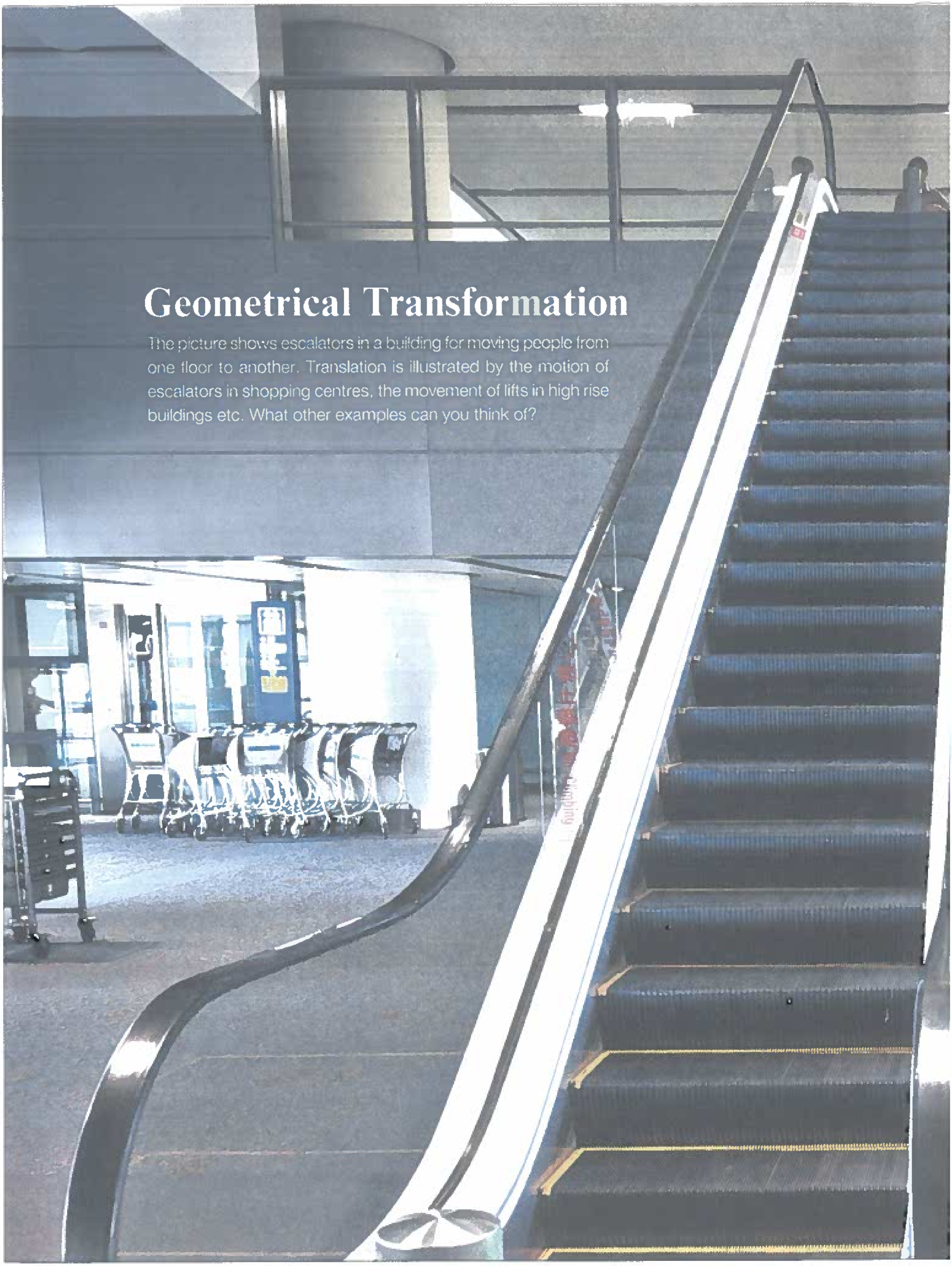


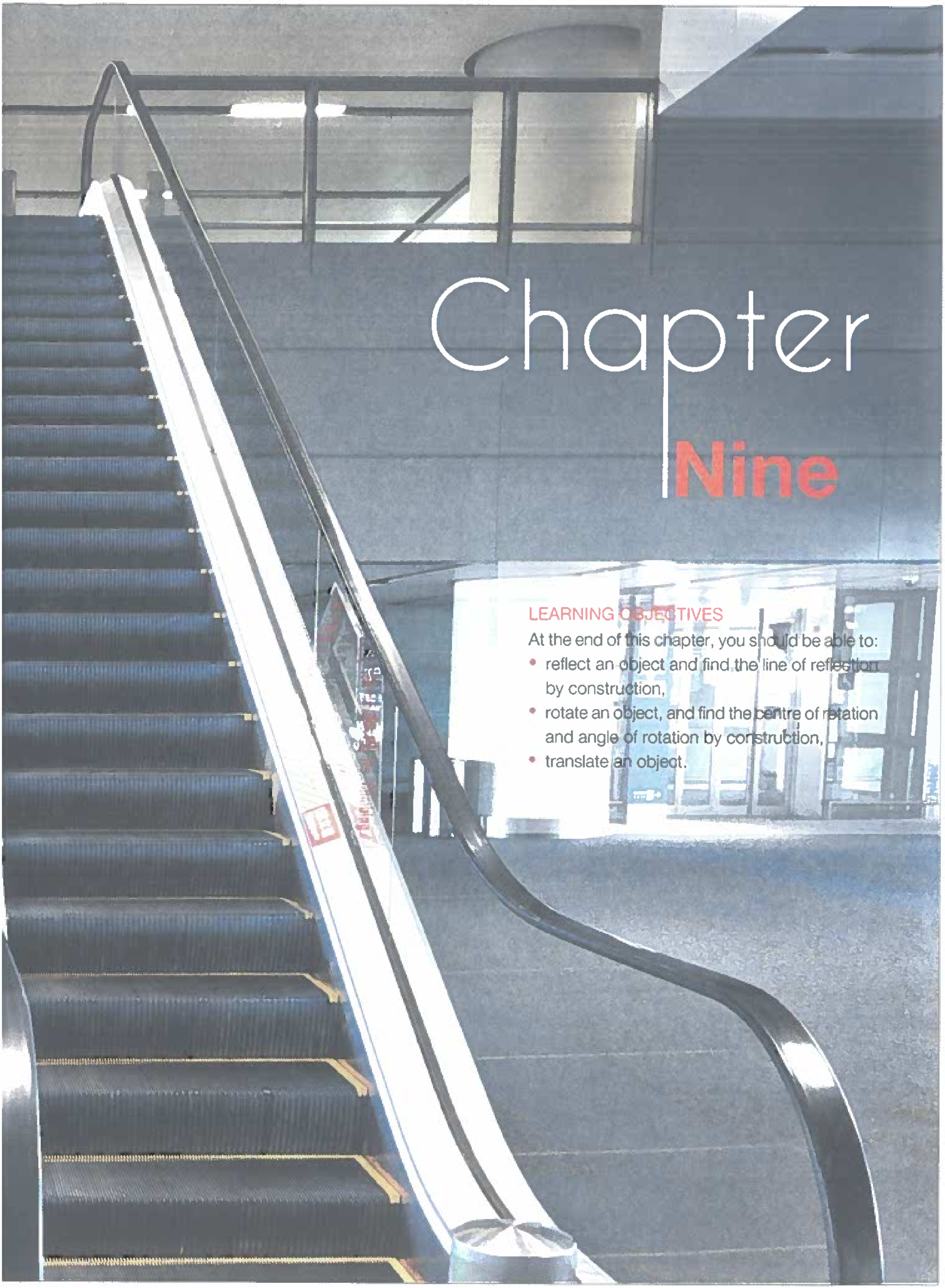
2. The figure shows an equilateral triangle inscribed inside a circle, which is inscribed inside a bigger equilateral triangle. Find the ratio of the area of the bigger triangle to that of the smaller triangle.



Geometrical Transformation

The picture shows escalators in a building for moving people from one floor to another. Translation is illustrated by the motion of escalators in shopping centres, the movement of lifts in high rise buildings etc. What other examples can you think of?



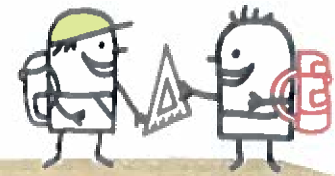


Chapter Nine

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- reflect an object and find the line of reflection by construction,
- rotate an object, and find the centre of rotation and angle of rotation by construction,
- translate an object.



9.1 Reflection

Fig. 9.1 shows the triangle ABC undergoing a reflection in the line $x = 3$ to produce the image $A'B'C'$.

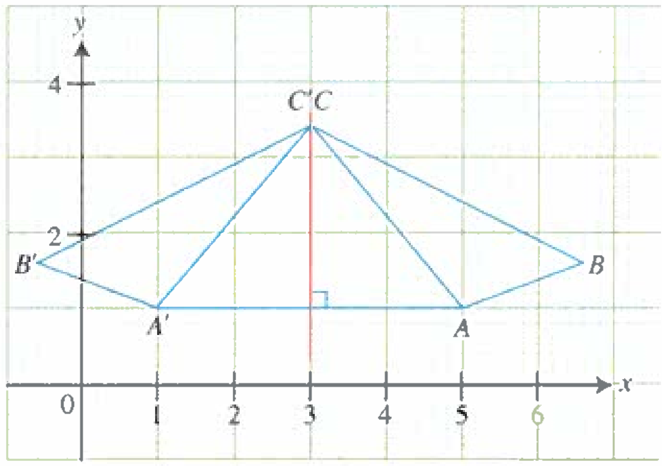


Fig. 9.1

A reflection is defined by its axis of reflection or the line of reflection or simply, the mirror line. In Fig. 9.1, this mirror line is $x = 3$.

(1) In Fig. 9.1, ΔABC (read in anticlockwise direction), when undergone reflection, becomes $\Delta A'B'C'$ (read in clockwise direction), i.e. the sense is reversed. We say that reflection does not preserve *orientation*.

(2) Under reflection, the shape and size of an image is exactly the same as its original figure. We call this rigid transformation an **isometric transformation**. In other words, ΔABC is *congruent* to $\Delta A'B'C'$.

(3) The line of reflection is the perpendicular bisector of the line joining any point and its image (e.g. AA' and BB' in Fig. 9.1).

(4) Any point on the mirror line undergoes no change. We say that these points are **invariant**. In the case of Fig. 9.1, C is the only invariant point, a point that does not undergo any change in a transformation.

From **(3)**, we know that the line of reflection passes through the midpoints formed by the line segments AA' and BB'. We can obtain the coordinates of the midpoints from observation, using geometrical construction to locate the perpendicular bisector or by using a formula.

> Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ where PQ forms a straight line, the coordinates of the midpoint of P and Q are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Equation of the Line of Reflection

Earlier in Chapter 2, we have learnt how to find the gradient and y-intercept of a straight line and subsequently, obtain the equation of a straight line. In this section, we shall take a look at how we can obtain the equation of the line of reflection.

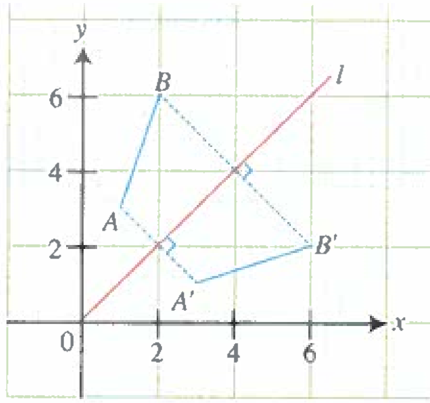


Fig. 9.2

Given that the line of reflection, l bisects AA' and BB', l will pass through the midpoints of AA' and BB'.

Fig. 9.2 shows the line segment AB and its image $A'B'$, where A, B, A' and B' are the points (1, 3), (2, 6), (3, 1) and (6, 2) respectively.

We can obtain the line of reflection using observation, geometrical construction or the formula for midpoint.

Using Geometrical Construction:

Join A to A' or B to B' and construct the perpendicular bisector l of AA' or BB'

Using the Formula for Midpoint:

Midpoint of AA', i.e. $\left(\frac{1+3}{2}, \frac{3+1}{2}\right) = (2, 2)$

Midpoint of BB', i.e. $\left(\frac{2+6}{2}, \frac{6+2}{2}\right) = (4, 4)$

We can obtain the line of reflection by joining the two coordinates i.e. (2, 2) and (4, 4) using a ruler.

Recall that the equation of a straight line is in the form $y = mx + c$, where the constant m is the gradient of the line and the constant c is the y-intercept.

From Fig. 9.2,

Vertical change (or rise) $= 4 - 2$

$= 2$

Horizontal change (or run) $= 4 - 2$

$= 2$

Gradient of l, $m = \frac{\text{rise}}{\text{run}}$

$= \frac{2}{2}$

$= 1$

y-intercept, $c = 0$

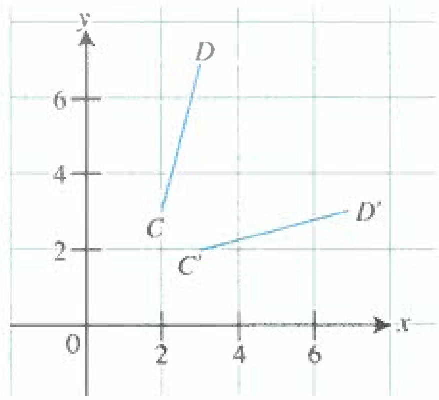
Hence the equation of the line of reflection is $y = x$.

PRACTISE NOW

SIMILAR QUESTIONS

Exercise 9A Questions 6–7, 14

The figure below shows the line segment CD and its image $C'D'$, where C, D, C' and D' are the points (2, 3), (3, 7), (3, 2) and (7, 3) respectively. Find the equation of the line of reflection.



Thinking Time

Fig. 9.3 shows a point A under reflection in two lines l and m. We represent the reflection in line l by M_l and that in line m by M_m. Hence $M_mM_l(A)$ represents a reflection of point A in line l followed by line m whereas $M_lM_m(A)$ represents a reflection of point A in line m followed by line l.

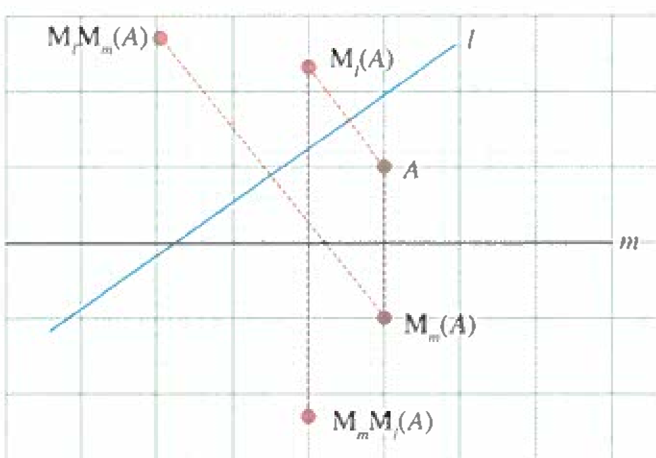


Fig. 9.3

What do you notice about the two points represented by $M_mM_l(A)$ and $M_lM_m(A)$?

From Fig. 9.3, we observe that the images of $M_mM_l(A)$ and $M_lM_m(A)$ are not the same. Therefore, we can conclude that the combination of reflections is not commutative.

INFORMATION

In most instances, we use symbols to represent transformations in order to simplify statements. For example, if we represent the enlargement with centre at origin and scale factor 2 as E and a reflection about the x-axis as M, then ME represents the transformation of an enlargement followed by a reflection and EM represents a reflection followed by an enlargement, MM (normally written as M^2) represents a reflection followed by another reflection, while EE (E^2) is an enlargement followed by another enlargement.

Worked Example 1

(Reflection of Points and Line Segments)

The coordinates of A and B are $(-2, 2)$ and $(1, 4)$ respectively. The line joining A and B is reflected in the x-axis to $A'B'$.

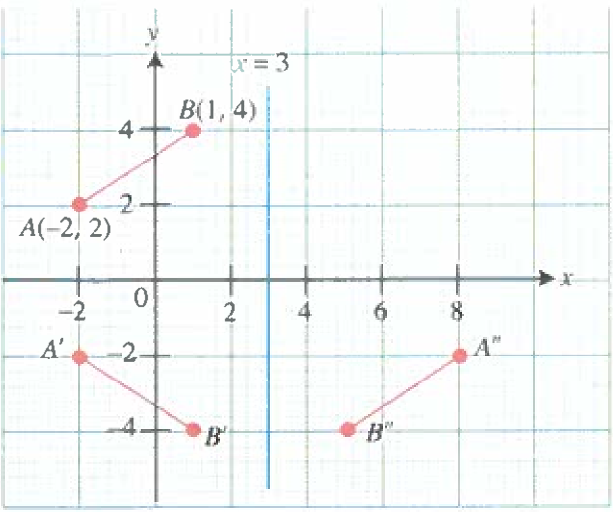
(i) Find the coordinates of A' and B'.

$A'B'$ is then reflected in the line $x = 3$ to give $A''B''$.

(ii) Find the coordinates of A'' and B''.

Show your working on a sheet of graph paper.

Solution:



From the graph, the coordinates of A' are $(-2, -2)$ and those of B' are $(1, -4)$. A'' is the point $(8, -2)$ and B'' is the point $(5, -4)$.

PRACTISE NOW 1

The coordinates of A and B are $(-3, 1)$ and $(-1, 5)$ respectively. The line joining A and B is reflected in the x-axis to $A'B'$.

(i) Find the coordinates of A' and B'.

$A'B'$ is then reflected in the line $x = 2$ to give $A''B''$.

(ii) Find the coordinates of A'' and B''.

Show your working on a sheet of graph paper.

SIMILAR QUESTIONS

Exercise 9A Questions 1–5, 8–10, 12–13

Worked Example 2

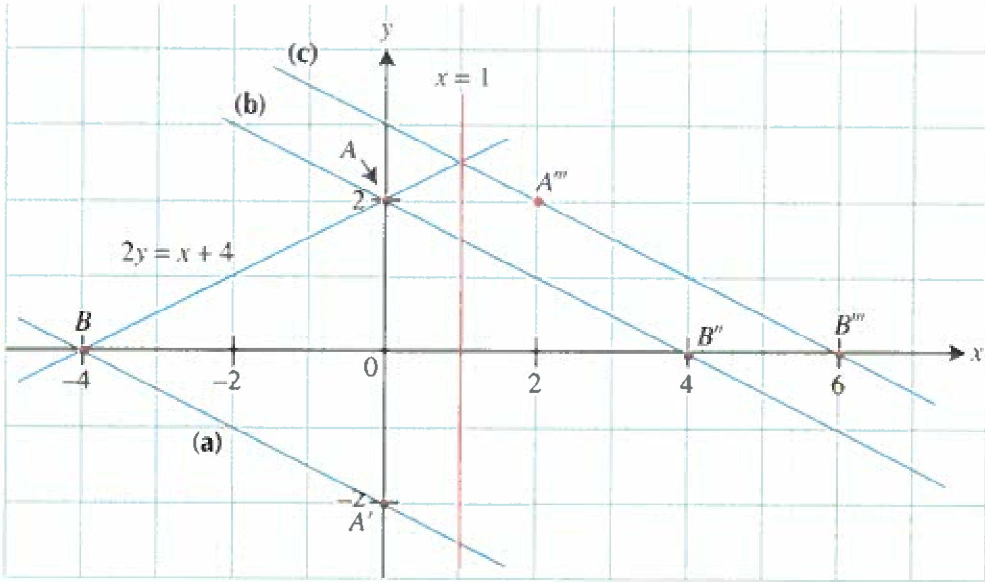
(Reflection of Linear Functions)

Find the equation of the image of the line $2y = x + 4$ under a reflection in the

(a) x-axis, **(b)** y-axis, **(c)** line $x = 1$,

by showing your working on a graph paper.

Solution:



Let AB represent the line $2y = x + 4$.

(a) Under reflection in the x-axis, the point A is mapped onto $A'(0, -2)$ and the point B is invariant.

Gradient of $A'B = \frac{\text{rise}}{\text{run}} = -\frac{1}{2}$

y-intercept $= -2$

Hence, the equation of $A'B$ is $y = -\frac{1}{2}x - 2$.

i.e. $2y = -x - 4$ or $2y + x = -4$.

(b) Under reflection in the y-axis, the point A is invariant and the point B is mapped onto $B''(4, 0)$.

Gradient of $AB'' = \frac{\text{rise}}{\text{run}} = -\frac{1}{2}$

y-intercept $= 2$

Hence, the equation of AB'' is $y = -\frac{1}{2}x + 2$, i.e. $2y + x = 4$.

(c) The images of A and B under reflection in the line $x = 1$ are $A'''(2, 2)$ and $B'''(6, 0)$ respectively.

Gradient of $A'''B''' = \frac{\text{rise}}{\text{run}} = -\frac{1}{2}$

y-intercept $= 3$

Hence, the equation of $A'''B'''$ is $y = -\frac{1}{2}x + 3$, i.e. $2y + x = 6$.

PRACTISE NOW 2

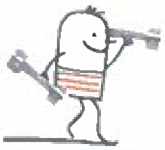
Find the equation of the image of the line $3y = 4x + 12$ under a reflection in the

(a) x-axis, **(b)** y-axis, **(c)** line $x = 3$,

by showing your working on a graph paper.

SIMILAR QUESTIONS

Exercise 9A Question 11



Exercise 9A

BASIC LEVEL

1. Write down the coordinates of the reflection of each of the following points in
(a) the x-axis, (b) the y-axis, and
(c) the line $y = x$.
(i) (3, 4) (ii) (−1, 3) (iii) (3, 3)
(iv) (−3, −4) (v) (3, −2) (vi) (p, q)

2. (a) State the coordinates of the final image when the point $A(-1, 3)$ is reflected in
(i) the x-axis and then in the line $y = 4$,
(ii) the line $y = 4$ and then in the x-axis.
(b) Is the answer in (i) the same as that in (ii)?

3. State the coordinates of the reflection of the point (3, 2) in the line $x = 2$.

4. The reflection of the origin in the line $y = x - 2$ is the point O'. On a sheet of graph paper,
(i) draw the line $y = x - 2$,
(ii) find the coordinates of O'.

5. The point $P(2, 1)$ is transformed by M_1, a reflection in the y-axis and then M_2, a reflection in the line $x = 4$. Give the coordinates of
(i) $M_1(P)$, (ii) $M_2(P)$,
(iii) $M_1M_2(P)$, (iv) $M_2M_1(P)$.

INTERMEDIATE LEVEL

6. Under a reflection, the point (3, 5) is mapped onto (5, 3).
(i) Find the equation of the line of reflection.
The point (5, 3) is then reflected and the coordinates of the final image is (−5, 3).
(ii) Find the equation of the line of the second reflection.

7. The point A and its image A' under a reflection are given below. Plot the points A and A' on a sheet of graph paper, construct the line of reflection and find its equation in each case.
(a) $A(1, 1)$, $A'(3, 1)$ (b) $A(1, -1)$, $A'(1, 9)$
(c) $A(2, 1)$, $A'(0, 3)$ (d) $A(0, 1)$, $A'(1, 2)$
(e) $A(0, -1)$, $A'(2, 1)$ (f) $A(-1, 1)$, $A'(3, -1)$

8. The point $A(3, 4)$ is reflected in the line $x = 2$ and then reflected in the line $y = 1$.
(i) Find the coordinates of the image of A under these two reflections.
(ii) State the coordinates of the point which remains invariant under these two reflections.

9. Find the coordinates of the image of the point $A(2, 3)$ under a reflection in the line $x = 6$ followed by a reflection in the line $y = x$ by showing your working on a graph paper.

10. The image of the origin under a reflection in the line $y = x + 2$ is point A. On a sheet of graph paper,
(i) draw the line $y = x + 2$,
(ii) find the coordinates of A.

11. Find the equation of the line onto which the line $y = 3x + 2$ is mapped under a reflection in the
(a) x-axis, (b) y-axis,
(c) line $x = 2$,
by showing your working on a graph paper.

ADVANCED LEVEL

12. (a) State the coordinates of the final image when
(i) the point $A(1, 4)$ is reflected in the line $y = x$ followed by another reflection in the line $x + y = 6$,
(ii) the point $A(1, 4)$ is reflected in the line $x + y = 6$ and then in the line $y = x$.
(b) Is the answer in (i) the same as that in (ii)?
(c) State the coordinates of the invariant point under these two reflections.

13. (a) State the coordinates of the final image when
(i) the point $A(1, 2)$ is reflected in the line $x + y = 6$ followed by another reflection in the line $x = 4$,
(ii) the point $A(1, 2)$ is reflected in the line $x = 4$ and then in the line $x + y = 6$.
(b) Is the reflection commutative in this case?
(c) State the coordinates of the invariant point under these two reflections.

9.2 Rotation

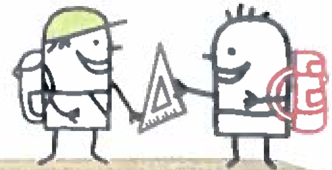


Fig. 9.4 shows the flag $ABCD$ rotated through 90° anticlockwise about the origin. The image of $ABCD$ is $A'B'C'D'$.

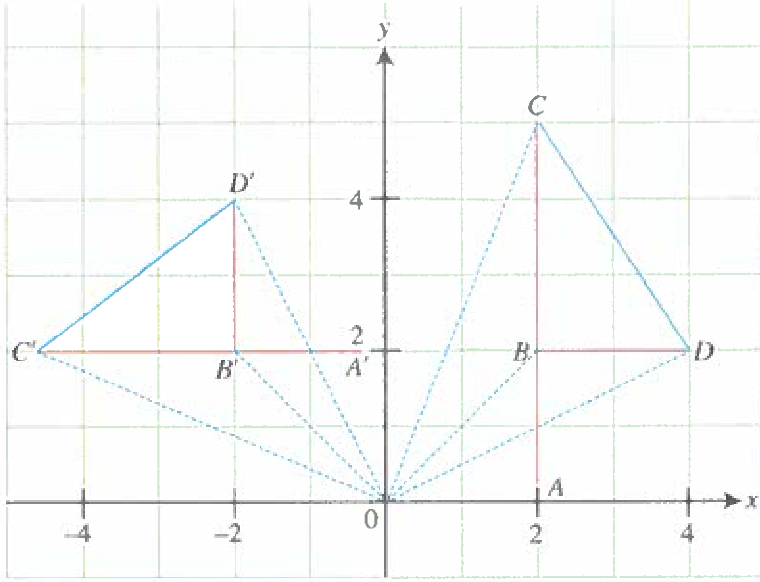


Fig. 9.4

The kite $PQRS$ in Fig. 9.5 is rotated through 180° about $P(0, 2)$. The image is $P'Q'R'S'$. Notice that P is invariant under the rotation.

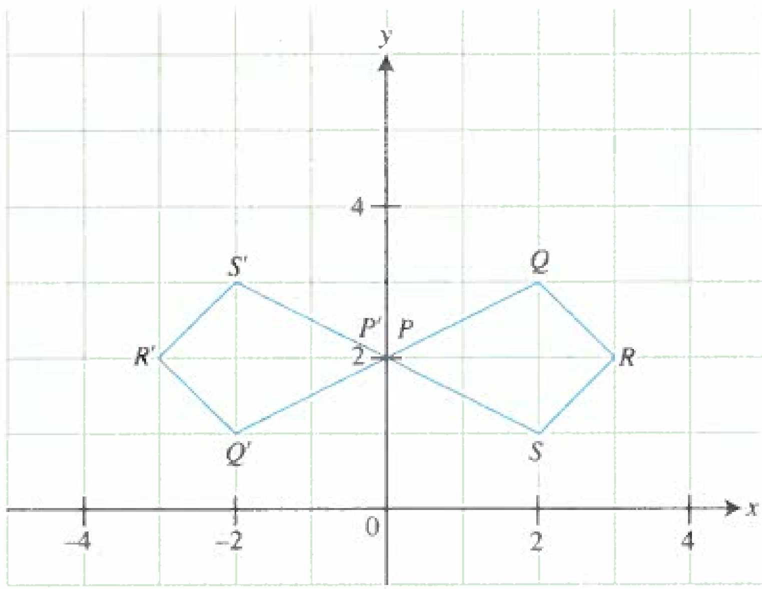


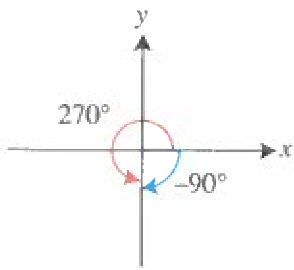
Fig. 9.5

Thinking Time

Can the transformation in Fig. 9.5 also be a reflection? Explain your answer.

From the Fig. 9.4 and Fig. 9.5, we observe the following:

1. Rotation can either be *clockwise* or *anticlockwise*. A 90° rotation represents an anticlockwise rotation and a –90° rotation represents a clockwise rotation. Hence, a 270° rotation is equivalent to a –90° rotation.



2. A 180° rotation is sometimes referred to as a half turn.
3. In a rotation, every point of the original figure is rotated through the *same angle* about the **centre of rotation**. If the centre of rotation lies on the figure, then it is the *invariant* point.
4. Rotation preserves *orientation*, *size* and *shape* and it is an **isometric transformation**.
5. We represent rotation with R. If R represents a rotation of 90°, then R^2 represents a rotation in the same direction with twice the angle of rotation i.e. rotation of 180° and R^{-1} represents a rotation in the opposite direction i.e. rotation of –90°.

Locating the Centre and Angle of Rotation

Fig. 9.6 shows the construction of the centre of rotation.

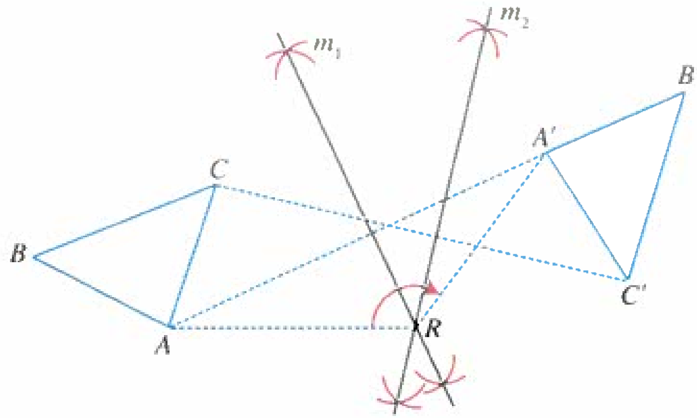


Fig. 9.6

Step 1: Join A to A' and construct the perpendicular bisector m_1.

Step 2: Join C to C' and construct the perpendicular bisector m_2. (You may join B to B' and do the same.)

Step 3: The point of intersection of m_1 and m_2, R, is known as the centre of rotation.

Step 4: To find the angle of rotation, join AR and $A'R$. $A\hat{R}A'$ is the angle of rotation in the clockwise direction.

Thinking Time

Explain why the centre of rotation lies on the intersection of the perpendicular bisector of AA' and of CC'.

Worked Example 3

(Problem involving Rotation)

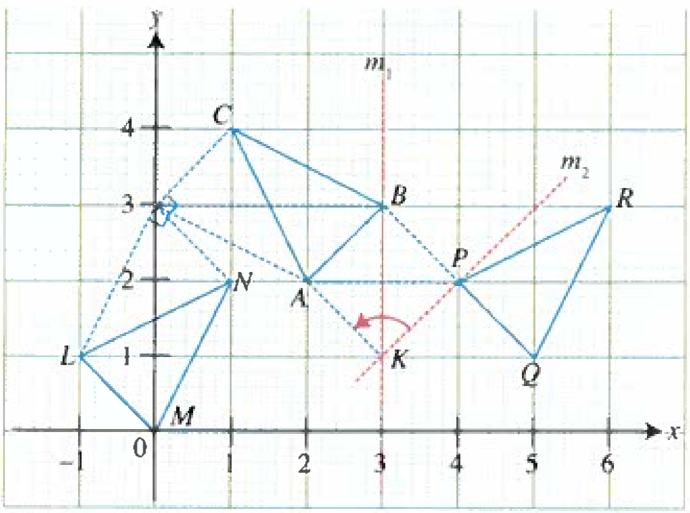
Using a scale of 1 cm to represent 1 unit on both axes, draw ΔABC and ΔPQR with vertices $A(2, 2)$, $B(3, 3)$, $C(1, 4)$, $P(4, 2)$, $Q(5, 1)$ and $R(6, 3)$.

(a) ΔABC is mapped onto ΔLMN by a 90° clockwise rotation with centre of rotation at (0, 3). Draw ΔLMN and label the vertices clearly.

(b) ΔABC is the image of ΔPQR under a rotation. Find the centre of rotation by construction and state the angle of rotation.

Solution:

(a) The vertices of ΔLMN are $L(-1, 1)$, $M(0, 0)$ and $N(1, 2)$.



(b) Line m_1 is the perpendicular bisector of AP while line m_2 is the perpendicular bisector of BQ. The point of intersection of these two perpendicular bisectors $K(3, 1)$ gives the centre of rotation. Join AK and PK. The angle of rotation is 90° anticlockwise.

PRACTISE NOW 3

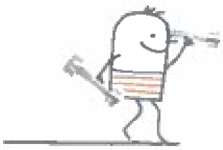
SIMILAR QUESTIONS

Exercise 9B Questions 1–9

Using a scale of 1 cm to represent 1 unit on both axes, draw ΔABC and ΔPQR with vertices $A(1, 6)$, $B(4, 5)$, $C(1, 4)$, $P(7, 6)$, $Q(6, 3)$ and $R(5, 6)$.

(a) ΔABC is mapped onto ΔLMN by a 90° anticlockwise rotation with centre of rotation at (0, 5). Draw ΔLMN and label the vertices clearly.

(b) ΔPQR is the image of ΔABC under a rotation. Find the centre of rotation by construction and state the angle of rotation.



Exercise 9B

BASIC LEVEL

1. The vertices of ΔABC are $A(3, 1)$, $B(4, 1)$ and $C(4, 5)$. ΔABC is mapped onto ΔPQR by a 90° clockwise rotation about the point (2, 1).
   On a sheet of graph paper, draw and label ΔPQR on the same diagram.

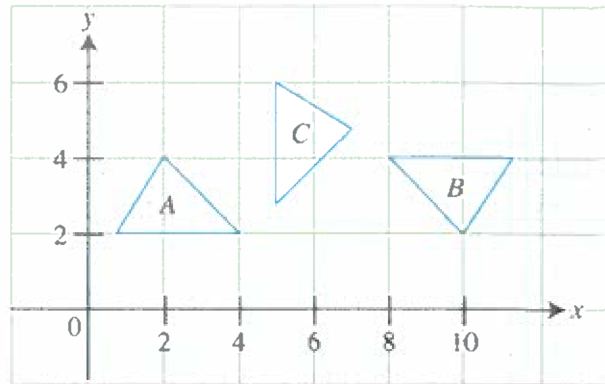
2. Given that P is the point (2, 4), Q is the point (4, –1) and R is the point (–1, 0), find
   (a) the image of P under a clockwise rotation of 90° about R,
   (b) the image of Q under an anticlockwise rotation of 90° about P,
   (c) the image of R under a 180° rotation about Q.

3. Find the coordinates of the image of the point (1, 4) under a clockwise rotation of
   (a) 90° about the centre (4, 2),
   (b) 180° about the centre (4, 2).

4. If R represents an anticlockwise rotation of 240° about the origin, describe R^2 and R^4.

INTERMEDIATE LEVEL

5. The coordinates of ΔABC are $A(4, 1)$, $B(6, 1)$ and $C(4, 6)$ while the coordinates of its image $\Delta A'B'C'$ under a rotation are $A'(0, -1)$, $B'(-2, -1)$ and $C'(0, -6)$. On a sheet of graph paper,
   (i) draw ΔABC and $\Delta A'B'C'$,
   (ii) find the centre of rotation by construction and state the angle of rotation.

6. 

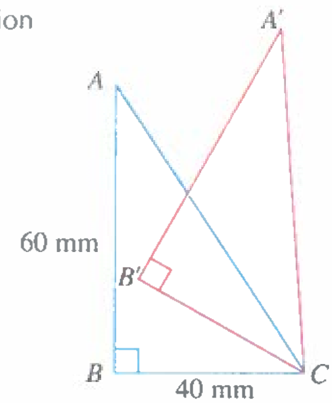
   (a) Triangle A can be mapped onto triangle B by a rotation. Find
   (i) the coordinates of the centre of rotation,
   (ii) the angle of rotation.
   (b) Triangle C can be mapped onto triangle A by a rotation. Find
   (i) the coordinates of the centre of rotation,
   (ii) the angle of rotation.
   (c) Triangle B is rotated through 90° clockwise about the point (4, 6). Find the coordinates of the vertices of the image of triangle B.

ADVANCED LEVEL

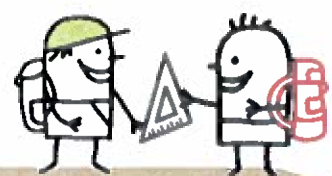
7. Find the equation of the image of the line $y = x + 2$ under a clockwise rotation of 90° about the origin.

8. Under a rotation, the line $P'Q'$ is the image of the line PQ. Given that their coordinates are $P(1, 1)$, $Q(1, 4)$, $P'(3, 1)$ and $Q'(k, 1)$, where $k > 0$, find
   (i) the value of k,
   (ii) the image of the point $\left(1, 2\frac{1}{2}\right)$,
   (iii) the coordinates of the point whose image is $\left(5\frac{1}{2}, 1\right)$.

9. The triangle $A'B'C$ is the image of the triangle ABC under a clockwise rotation of 25° about C. Calculate, giving your answer correct to the nearest 0.5°,
(a) $C\hat{A}A'$, (b) $A\hat{C}B'$.



9.3 Translation



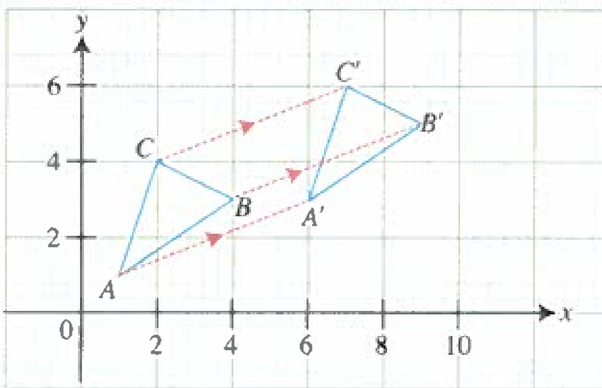


Fig. 9.7

Fig. 9.7 shows ΔABC being translated to $\Delta A'B'C'$. A **translation** is *isometric* and it preserves *orientation*. We represent the translation, T, of point A to A' by writing T(A).

Thinking Time

Can there be invariant points under a translation?

A translation can be represented by a *column vector* $\begin{pmatrix} a \\ b \end{pmatrix}$ where a and b is the number of units moved along the x- and y-axes respectively. In Fig. 9.7, the *vector equation* representing the translation is $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$ where $\begin{pmatrix} a \\ b \end{pmatrix}$ is the **translation vector** and $\begin{pmatrix} x' \\ y' \end{pmatrix}$ is the **image** of $\begin{pmatrix} x \\ y \end{pmatrix}$.

For column vectors,

$$\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p+r \\ q+s \end{pmatrix} \text{ and } \begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p-r \\ q-s \end{pmatrix}$$

In this section, we will apply only the basics of vectors, specifically column vectors. Vectors, as a whole, will be covered in Book 4.

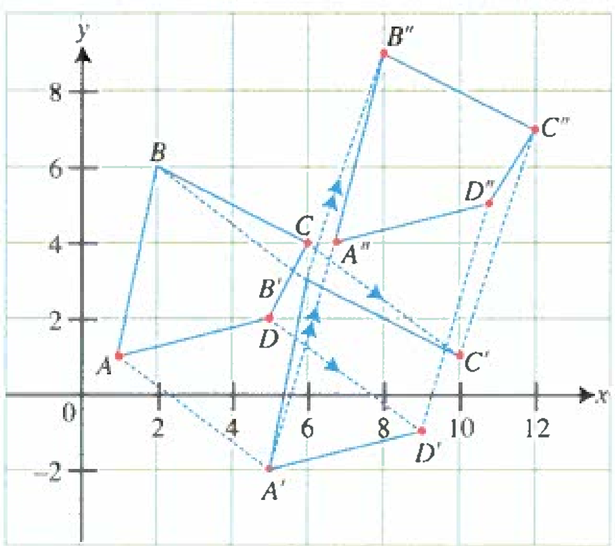
Worked Example 4

(Problem Involving Translation)

The vertices of a quadrilateral are $A(1, 1)$, $B(2, 6)$, $C(6, 4)$ and $D(5, 2)$. Find the coordinates of the vertices of the image of the quadrilateral $ABCD$ under a translation T_1 represented by $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$. Find the image of the new quadrilateral if it undergoes another translation T_2 represented by $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$. Show your working on a sheet of graph paper.

Solution:

The image of the quadrilateral $ABCD$ is obtained as shown below.



$$T_1(A) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$$T_1(B) = \begin{pmatrix} 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$$T_1(C) = \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 10 \\ 1 \end{pmatrix}$$

$$T_1(D) = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \end{pmatrix}$$

Therefore, the coordinates of the vertices of the image are $A'(5, -2)$, $B'(6, 3)$, $C'(10, 1)$ and $D'(9, -1)$. The image of the quadrilateral $A'B'C'D'$ is plotted on the figure above. The image of the new quadrilateral $A''B''C''D''$ under T_2 is obtained similarly.

$$T_2(A) = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$$T_2(B) = \begin{pmatrix} 6 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$$

$$T_2(C) = \begin{pmatrix} 10 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \end{pmatrix}$$

$$T_2(D) = \begin{pmatrix} 9 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \end{pmatrix}$$

The vertices of the image of the new quadrilateral, $A''(7, 4)$, $B''(8, 9)$, $C''(12, 7)$ and $D''(11, 5)$, are plotted on the figure above.

PRACTISE NOW 4

SIMILAR QUESTIONS

Exercise 9C Questions 1–2

The vertices of a quadrilateral are $A(1, 1)$, $B(2, 4)$, $C(4, 4)$ and $D(5, 1)$. Find the coordinates of the vertices of the image of the quadrilateral $ABCD$ under a translation T_1 represented by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Find the image of the new quadrilateral if it undergoes another translation T_2 represented by $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Show your working on a sheet of graph paper.

Thinking Time

(a) Would the result be the same if T_2 is performed first in Worked Example 4?

(b) Can you give a single vector that would produce the same result as the two successive transformations?

Journal Writing

Find out about Escher's tessellations and the transformations he used in his tessellations.

Worked Example 5

(Problem involving Translation)

T_1 is the translation $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and T_2 is a translation that will move the point (1, 2) to (–2, 5).

(i) Find the image of the point $A(7, 4)$ under T_1.

(ii) Find the translation vector represented by T_2.

(iii) What is the image of the point $B(8, -4)$ under T_2?

Solution:

(i) $A' = T_1(A)$

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

$\therefore$ Image of A under T_1 is (9, 7).

(ii) Let the translation vector of T_2 be $\begin{pmatrix} a \\ b \end{pmatrix}$.

$$\begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

$\therefore$ The translation vector of T_2 is $\begin{pmatrix} -3 \\ 3 \end{pmatrix}$.

(iii) $B' = T_2(B)$

$$= \begin{pmatrix} -3 \\ 3 \end{pmatrix} + \begin{pmatrix} 8 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

$\therefore$ The image of B under T_2 is (5, –1).

PRACTISE NOW 5

SIMILAR QUESTIONS

Exercise 9C Questions 3–4, 5(a)–(c)

T_1 is the translation $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ and T_2 is a translation that will move the point (3, 2) to (5, 7).

(i) Find the image of the point $A(2, 4)$ under T_1.

(ii) Find the translation vector represented by T_2.

(iii) What is the image of the point $B(6, -3)$ under T_2?

Exercise 9C

BASIC LEVEL

1. The vertices of a quadrilateral are $A(2, 1)$, $B(3, 3)$, $C(5, 3)$ and $D(5, 2)$. Find the coordinates of the vertices of the image of the quadrilateral $ABCD$ under a translation T_1 represented by $\begin{pmatrix} 5 \\ -4 \end{pmatrix}$. Find the image of the new quadrilateral if it undergoes another translation T_2 represented by $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.
   Show your working on a sheet of graph paper.

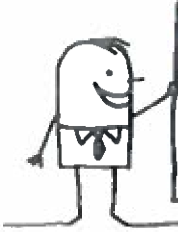
2. The vertices of ΔPQR are $P(1, 3)$, $Q(7, 5)$ and $R(2, 0)$. Find the coordinates of the vertices of the image of ΔPQR under a translation T represented by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

3. A translation T maps the point (6, 2) onto the point (2, 7) and the point (−1, −5) onto the point P.
   Find the column vector representing the translation T and the coordinates of the point P.

INTERMEDIATE LEVEL

4. T is the translation $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, A is the point (2, 4), B is (p, q) and C is (h, k).
   (i) Find the coordinates of the image of the point A under T.
   (ii) Given that $T(B) = A$, find the value of p and of q.
   (iii) Given that $T_2(A) = C$, find the value of h and of k.
   (iv) Find the coordinates of the point D such that $T_2(D) = A$.

5. Under a translation T_1, the image of the point (5, −1) is (2, 3). Under a translation T_2, the image of the point (−2, 5) is (4, −5). Find the image of the point (7, 6) under the following transformations.
   (a) T_1
   (b) T_2
   (c) T_1T_2
   (d) T_2T_1
   (e) T_1^2



Summary

An object can undergo any of the following three isometric transformations.

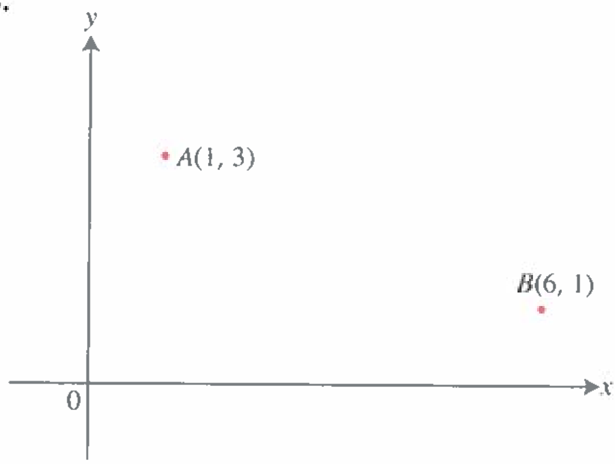
1. Under a **reflection**, a figure and its image are symmetrical about the mirror line. A reflection does not preserve orientation. Points on the mirror line are invariant.
2. A **rotation** is defined by its centre, angle and direction (clockwise or anticlockwise) of rotation. The centre of rotation is the only invariant point.
3. A **translation** moves all the points of an object on a plane the same distance and in the same direction. It preserves orientation and has no invariant points.

Review Exercise 9

1. Given that P is the point (3, 4), find the coordinates of the image of P under
   (a) an anticlockwise rotation of 90° about the point (2, 0),
   (b) a reflection in the line $x + 2 = 0$.

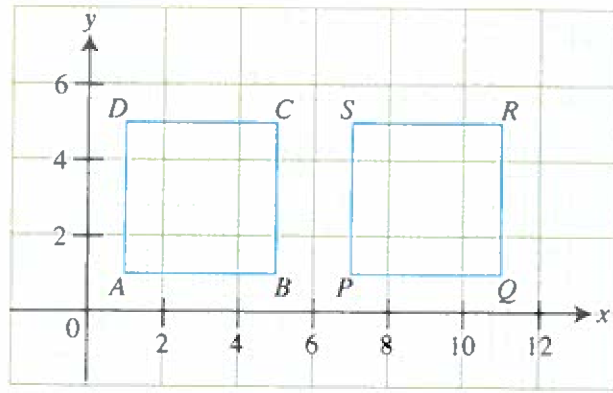
2. A translation maps the point (5, 7) onto the point (2, 9) and the point A onto the point (–3, –5). Find the coordinates of A.

3.



The figure above shows the point $A(1, 3)$ and $B(6, 1)$.
   (a) The line AB is rotated through 90° clockwise about B. Find the equation of the image of the line AB.
   (b) The line $y = 1$ is the line of symmetry of ΔABC. Find the coordinates of the point C.

4.



The figure above shows two squares $ABCD$ and $PQRS$. In each of the following cases, describe completely the single transformation that will map
   (a) $PQRS$ onto $ABCD$,
   (b) $ABCD$ onto $QPSR$,
   (c) $PQRS$ onto $CDAB$,
   (d) $ABCD$ onto $SPQR$,
   (e) $PQRS$ onto $DABC$.

5. M is a reflection in the line $y = -x$. Find the coordinates of the image of the point (5, 2) under
   (a) M, **(b)** M^2, **(c)** M^3.

6. Find the equation of the image of the line $y = x$ under an anticlockwise rotation of 90° about the point (2, 0).

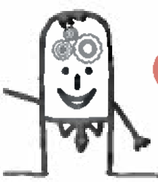
7. Find the equation of the image of the line $y = x + 4$ under
   (a) a reflection in the line $x = 2$,
   (b) a translation representation by the column vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$,
   (c) a 90° clockwise rotation about the origin O.

8. Find the image of the point $P(2, 1)$ under
   (a) a reflection in the line $x + y = 4$,
   (b) a 180° rotation about the point $(-5, 3)$,
   (c) a translation represented by $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$,
   (d) a reflection in the line $y = x + 2$.

9. Given that the line $y = 2x + 3$ is mapped onto the line $y = mx + c$ by a translation represented by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, find the values of m and c.

10. M is a reflection in the line $y = x$. Find the coordinates of the image of
   (a) (1, 3) under M,
   (b) (2, 5) under M^{-1},
   (c) (3, 7) under M^5.

11. R is an anticlockwise rotation of 90° about the origin.
   (a) Find the coordinates of the image of each of the following points under R.
      (i) (3, 5) (ii) (7, –4)
   (b) Find the coordinates of the image of each of the following point under R^{-1}.
      (i) (3, 4) (ii) (–2, –3)
   (c) Find the coordinates of the image of (2, 5) under
      (i) R^5, (ii) R^8.



Challenge Yourself

ΔABC is an equilateral triangle of sides 3 cm, the corners being lettered anti-clockwise in alphabetical order. ΔABC is mapped onto $\Delta A_1B_1C_1$ by an anticlockwise rotation of 120° about A. $\Delta A_1B_1C_1$ is then mapped onto $\Delta A_2B_2C_2$ by an anticlockwise rotation of 120° about B. Finally, $\Delta A_2B_2C_2$ is mapped onto $\Delta A_3B_3C_3$ by an anticlockwise rotation of 120° about C. Construct the complete figure accurately and state clearly a single geometrical transformation which maps ΔABC onto $\Delta A_3B_3C_3$.

B1 Revision Exercise

1. Solve each of the following equations.

   (a) $16a - a^2 - 64 = 0$ (b) $b^2 - \frac{16}{25} = 0$

   (c) $1 + 3c = 10c^2$ (d) $d^3 + d^2 - 132d = 0$

2. The length of a rectangle is 5 m longer than its breadth and its area is 66 m². Let the breadth of the rectangle be x m. Formulate an equation in terms of x. Hence, find the perimeter of the rectangle.

3. The variables x and y are connected by the equation $y = 3x - 2x^2$. Some values of x and the corresponding values of y are given in the table.

x	-2	-1.5	-1	0	1	2	2.5	3
y	p	-9	-5	0	1	-2	q	-9

   (a) Find the value of p and of q.

   (b) On a sheet of graph paper, using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 3x - 2x^2$ for $-2 \leqslant x \leqslant 3$.

   (c) Use your graph in (b) to find

   (i) the value of y when $x = -0.5$,

   (ii) the values of x when $y = 0.5$,

   (iii) the maximum value of y.

   (d) State the equation of the line of symmetry of the graph.

4. Express each of the following as a fraction in its simplest form.

   (a) $\frac{x-1}{4x-2} - \frac{5-2x}{3-6x}$ (b) $\frac{2x+3}{9x^2-1} - \frac{7}{3x-1}$

5. (i) Make x the subject of the formula $5a = \frac{3x-4}{2y-3x}$.

   (ii) Hence, find the value of x when $a = 1$ and $y = 5$.

6. If $f(x) = 7x - 2$ and $F(x) = \frac{5}{8}x + 6$, find an expression, in terms of a, for each of the following.

   (i) $f(a)$ (ii) $F(16a - 8)$

   (iii) $f\left(\frac{1}{7}a+1\right) + F(4a - 3)$

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7. It is given that ΔABC is congruent to ΔPQR, $\angle A = 70°$, $\angle B = 60°$ and $AB = 8$ cm.
   (i) Write down the length of PQ.
   (ii) Find $\angle R$.

8. A map has a scale of 1 cm to 8 km.
   (i) Express the scale of the map in the form $\frac{1}{n}$, where n is an integer.
   (ii) If two towns are 72 km apart, find the corresponding distance on the map.
   (iii) If the actual area of a forest is 496 km^2, calculate its area on the map.

9. State the coordinates of the reflection of the point (4, 5) in the line $x = 2$.

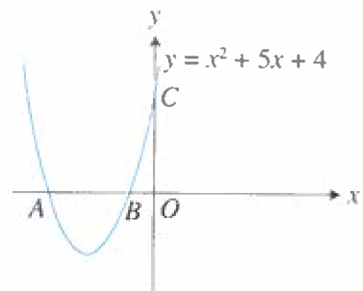
10. If R represents an clockwise rotation of 187° about the origin, describe R^2 and R^{-1}.

B2 Revision Exercise

1. Solve the equation $(2x - 1)(3 - 4x) = 2(1 - 2x)$.

2. (i) If $x = 3$ is a solution of the equation $x^2 + px + 15 = 0$, find the value of p.
   (ii) Hence, find the other solution of the equation.

3. In the figure, the curve $y = x^2 + 5x + 4$ cuts the x-axis at two points A and B, and the y-axis at the point C. Find the coordinates of A, B and C.



4. Simplify each of the following.

   (a) $\dfrac{4a^2b^2 - 8a^3b - 14ab^3}{4ab}$ (b) $\dfrac{6c^3d^4 + 12c^4d^2 - 9c^2d^3}{-3c^2d^2}$

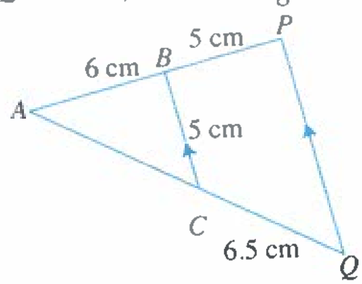
5. The centripetal force, F Newtons, acting on an object of mass m kg, moving at a tangential velocity v m s^{-1} along a path with radius r m, is given by the formula $F = \dfrac{mv^2}{r}$.
   (i) Given that $v \geqslant 0$, make v the subject of the formula $F = \dfrac{mv^2}{r}$.
   (ii) Hence, find the tangential velocity of a motorcycle of mass 225 kg moving along a circular track with radius 5 m, if the centripetal force acting on it is 4500 Newtons.

6. Given the function $f : x \mapsto 13 - 9x$, evaluate each of the following.

   (i) $f(2)$ (ii) $f\left(-\dfrac{4}{9}\right)$

   (iii) $2f(0) + f(1)$ (iv) $f(3) - f\left(\dfrac{2}{9}\right)$

7. The cross section of a blade of a paper cutter is in the shape of a triangle APQ, with parallel grooves CB and QP. Given that ΔABC is similar to ΔAPQ, $AB = 6$ cm, $BP = CB = 5$ cm and $CQ = 6.5$ cm, find the length of



(i) QP,
(ii) AC.

8. A model of the Port of Singapore Authority Building is made to a scale of 1 cm to 7.5 m. The height of the model building is 24.4 cm. Find
(i) the actual height of the building,
(ii) the height of the model building if it is made to a scale of 1 cm to 12 m.

9. Find the equation of the image of the line $y = 2x - 5$ under an anticlockwise rotation of 90° about the origin.

10. T_1 is the translation $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$ and T_2 is a translation that will move the point (2, 5) to (5, 9).
(i) Find the image of the point $A(2, 4)$ under T_1.
(ii) Find the translation vector represented by T_2.
(iii) What is the image of the point $B(4, -8)$ under T_2?

Pythagoras' Theorem

The ancient Egyptians made use of knotted ropes to form right-angled triangles for the construction of various structures such as the pyramids. These people, known as 'rope-stretchers', used a rope with 12 evenly-spaced knots tied in a circle to form right-angled triangles. We shall discover the secret of these 'rope-stretchers' in the investigation on pages 260 and 261.

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Chapter Ten

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- solve problems using Pythagoras' Theorem,
- determine whether a triangle is a right-angled triangle given the length of its three sides.

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10.1 Pythagoras' Theorem

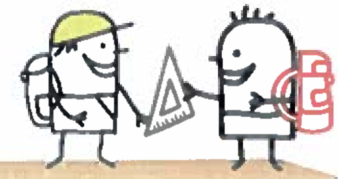


Fig. 10.1 shows a right-angled triangle ABC where $\angle C = 90°$.

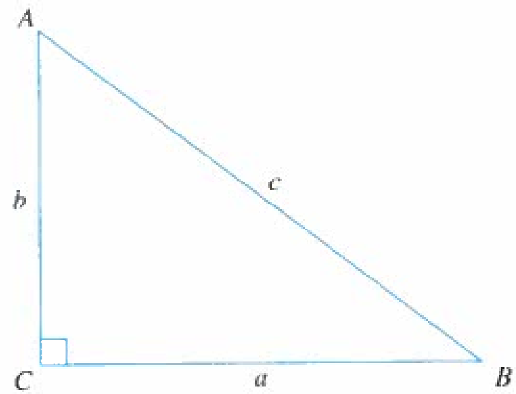


Fig. 10.1

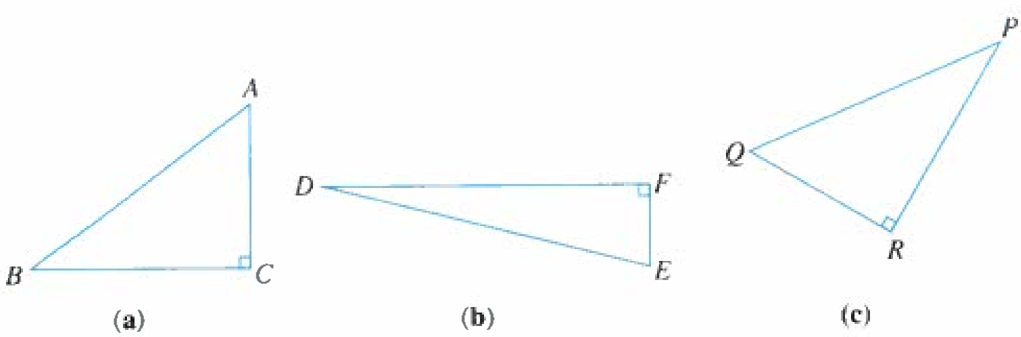
ATTENTION

We always denote the side opposite $\angle A$ as a, the side opposite $\angle B$ as b and the side opposite $\angle C$ as c.

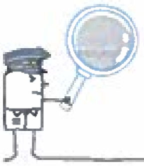
In Book 1, we have learnt that the longest side of a triangle is opposite the largest angle. In a right-angled triangle, the right angle is the largest angle. Thus in ΔABC, the side AB opposite $\angle C$ is the longest side. AB is called the **hypotenuse** of ΔABC.

PRACTISE NOW

Identify the hypotenuse of each of the following right-angled triangles.



In any right-angled triangle, there is a relationship between the lengths of the three sides of the triangle. In this chapter, we will learn about this relationship and how it can be applied to solve problems.



Investigation

Pythagoras' Theorem – The Secret of the Rope-Stretchers

Part I: Use strings to determine the relationship between the lengths of the three sides of a right-angled triangle.

For this part of the investigation, three strings of lengths 12 cm, 24 cm and 30 cm, and a ruler are required.

Fig. 10.2 shows three right-angled triangles where the lengths of the sides are *integer* values. Which side of each triangle is the hypotenuse?

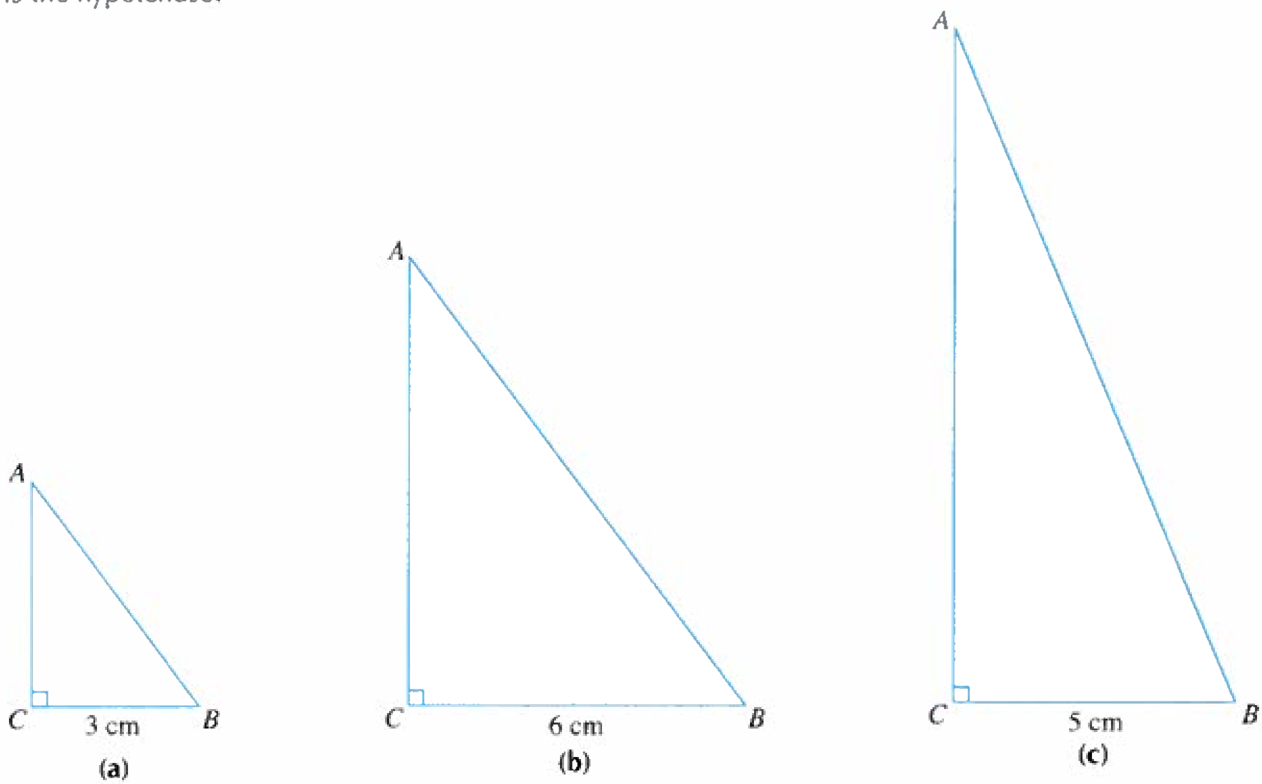


Fig. 10.2

1. Using the string of length 12 cm, form the triangle as shown in Fig. 10.2**(a)**. Measure and write down the length of *AC* and of *AB* in Table 10.1.

	BC	AC	AB	BC^2	AC^2	AB^2	$BC^2 + AC^2$
(a)	3 cm			9 cm^2			
(b)	6 cm			36 cm^2			
(c)	5 cm			25 cm^2			

Table 10.1

2. Using the string of length 24 cm, form the triangle as shown in Fig. 10.2(**b**). Measure and write down the length of *AC* and of *AB* in Table 10.1.

3. Using the string of length 30 cm, form the triangle as shown in Fig. 10.2(**c**). Measure and write down the length of *AC* and of *AB* in Table 10.1.

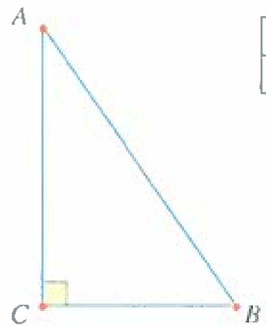
4. Complete Table 10.1. What do you notice about the value of AB^2 and that of $BC^2 + AC^2$ in Table 10.1?

Part II: Use a geometry software template to determine the relationship between the lengths of the three sides of a right-angled triangle.

For this part of the investigation, go to http://www.shinglee.com.sg/StudentResources/ and open the geometry software template 'Pythagoras' Theorem'.

Pythagoras' Theorem

What is the relationship between the value of AB^2 and that of $BC^2 + AC^2$?



BC	AC	AB	BC^2	AC^2	AB^2	$BC^2 + AC^2$
5.17 cm	7.55 cm	9.15 cm	26.72 cm²	56.99 cm²	83.71 cm²	83.71 cm²

Fig. 10.3

5. The template in Fig. 10.3 shows a right-angled triangle ABC. Which side of $\triangle ABC$ is the hypotenuse?

6. Click and move a point A, B or C to get five different right-angled triangles. Complete Table 10.2.

	BC	AC	AB	BC^2	AC^2	AB^2	$BC^2 + AC^2$
(a)							
(b)							
(c)							
(d)							
(e)							
(f)							

Table 10.2

7. What do you notice about the value of AB^2 and that of $BC^2 + AC^2$ in Table 10.2?

From the investigation, we observe that the square of the length of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the lengths of the other two sides. This result is known as **Pythagoras' Theorem**.

In a triangle ABC, if $\angle C = 90°$,

then $AB^2 = BC^2 + AC^2$,

i.e. $c^2 = a^2 + b^2$.

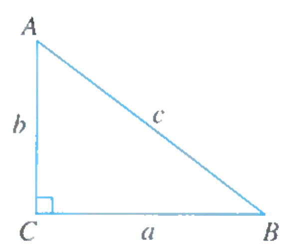


Fig. 10.4

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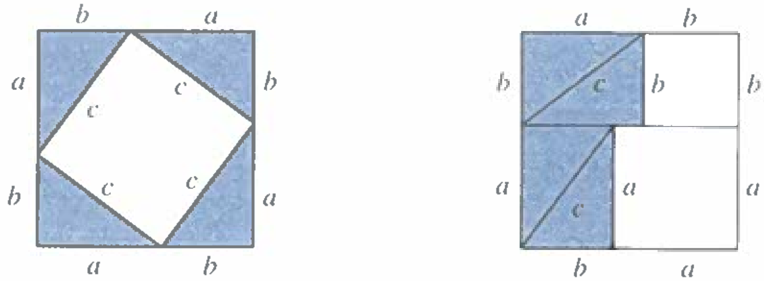
There are more than 300 proofs of Pythagoras' Theorem. We shall now take a look at one of these proofs.

Consider the figures as shown in Fig. 10.5. The eight right-angled triangles in blue are of the same size.



(a) Fig. 10.5 (b)

Fig. 10.5(**a**) and Fig. 10.5(**b**) are squares of sides $(a + b)$ units. Thus the area of the figure shown in Fig. 10.5(**a**) is equal to the area of the figure shown in Fig. 10.5(**b**). After removing four blue right-angled triangles from each figure, we obtain the figures as shown in Fig. 10.6.

(a) Fig. 10.6 (b)

Since the total area of the four right-angled triangles removed from each figure in Fig. 10.5 is the same, the area of the figure shown in Fig. 10.6(**a**) is equal to the area of the figure shown in Fig. 10.6(**b**), i.e. $c^2 = a^2 + b^2$.

INFORMATION

In mathematics, a theorem is a statement that has been proven to be true.

Internet Resources

Search on the Internet for interactive applets to prove Pythagoras' Theorem.

Performance Task

Although Pythagoras was credited for discovering the Pythagoras' Theorem in the 6^{th} century B.C., the theorem was known thousands of years ago.

As mentioned in the chapter opener, the ancient Egyptians made use of knotted ropes to form right-angled triangles for the construction of various structures such as the pyramids.

The Babylonians were known to be familiar with the Pythagorean Triple, i.e. a set of 3 positive integers a, b and c which satisfy the equation $a^2 + b^2 = c^2$.

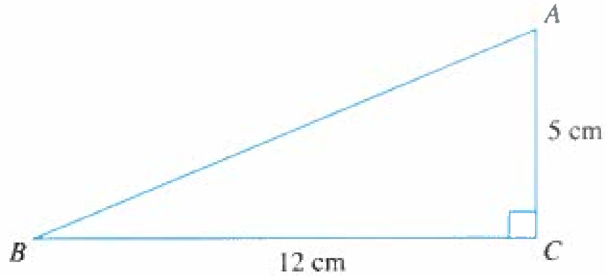
Pythagoras' Theorem was known to the ancient Chinese as Gougu Theorem. Reasoning for the Pythagoras' Theorem for the right-angled triangle of sides 3 units, 4 units and 5 units is given in a Chinese text published in the 1^{st} century B.C.

Search on the Internet to find out more about how Pythagoras' Theorem has evolved over the years and some well-known proofs of the theorem. Present your findings to the class.

Worked Example 1

(Finding the Length of the Third Side of a Right-Angled Triangle)

In $\triangle ABC$, $AC = 5$ cm, $BC = 12$ cm and $\angle C = 90°$.
Calculate the length of AB.



Solution:

In $\triangle ABC$, $\angle C = 90°$.
Using Pythagoras' Theorem,

$$AB^2 = BC^2 + AC^2$$
$$= 12^2 + 5^2$$
$$= 144 + 25$$
$$= 169$$

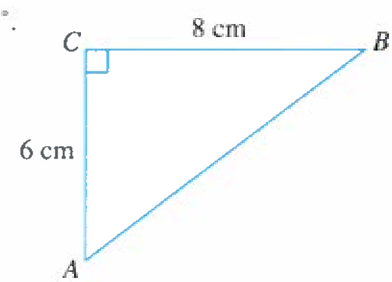
$$\therefore AB = \sqrt{169} \text{ (since } AB > 0)$$
$$= 13 \text{ cm}$$

ATTENTION

Pythagoras' Theorem can only be applied in a right-angled triangle.

PRACTISE NOW 1

1. In $\triangle ABC$, $AC = 6$ cm, $BC = 8$ cm and $\angle C = 90°$. Find the length of AB.



2. In $\triangle ABC$, $AC = 24$ cm, $BC = 7$ cm and $\angle C = 90°$. Find the length of AB.

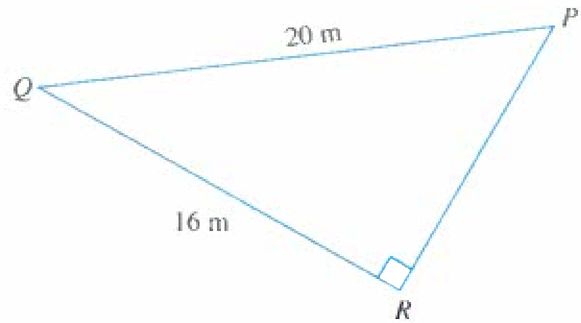
SIMILAR QUESTIONS

Exercise 10A Questions 1(a)–(d), 3–4

Worked Example 2

(Finding the Length of the Third Side of a Right-Angled Triangle)

In ΔPQR, $PQ = 20$ m, $QR = 16$ m and $\angle R = 90°$.
Calculate the length of PR.



Solution:

In ΔPQR, $\angle R = 90°$.
Using Pythagoras' Theorem,

$$\begin{aligned} PQ^2 &= QR^2 + PR^2 \\ 20^2 &= 16^2 + PR^2 \\ PR^2 &= 20^2 - 16^2 \\ &= 400 - 256 \\ &= 144 \end{aligned}$$

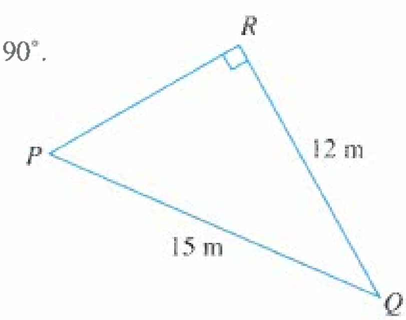
$$\therefore PR = \sqrt{144} \text{ (since } PR > 0\text{)}$$
$$= 12 \text{ m}$$

PRACTISE NOW 2

SIMILAR QUESTIONS

Exercise 10A Questions 2(a)–(d), 5–6

1. In ΔPQR, $PQ = 15$ m, $QR = 12$ m and $\angle R = 90°$.
   Find the length of PR.



2. In ΔPQR, $PQ = 35$ m, $PR = 28$ m and $\angle R = 90°$.
   Find the length of QR.

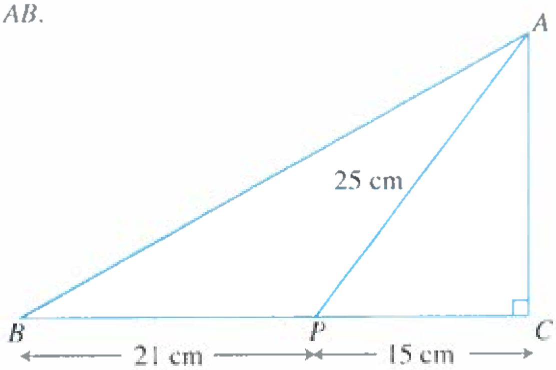
Worked Example 3

(Finding the Lengths of Unknown Sides)

In ΔABC, $\angle C = 90°$. P lies on BC such that $BP = 21$ cm, $PC = 15$ cm and $AP = 25$ cm. Calculate the length of

(i) AC,

(ii) AB.



Solution:

(i) In ΔAPC, $\angle C = 90°$.

Using Pythagoras' Theorem,

$$AP^2 = PC^2 + AC^2$$
$$25^2 = 15^2 + AC^2$$
$$AC^2 = 25^2 - 15^2$$
$$= 625 - 225$$
$$= 400$$

$\therefore AC = \sqrt{400}$ (since $AC > 0$)

$= 20$ cm

(ii) In ΔABC, $\angle C = 90°$.

Using Pythagoras' Theorem,

$$AB^2 = BC^2 + AC^2$$
$$= (21 + 15)^2 + 20^2$$
$$= 36^2 + 20^2$$
$$= 1296 + 400$$
$$= 1696$$

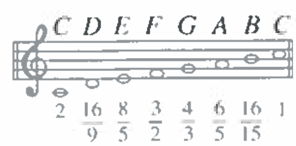
$\therefore AB = \sqrt{1696}$ (since $AB > 0$)

$= 41.2$ cm (to 3 s.f.)

Story Time

Pythagoras was a Greek philosopher and mathematician. In addition to Pythagoras' Theorem, he also discovered the following:

- The musical note produced by a vibrating string of a certain length is exactly one octave lower than the note produced by a string of the same material and half that length.
- Other notes in the musical scale can be produced by using certain fractions of the length of the string. For example, a string $\frac{4}{3}$ the length of a C-string produces the note G (one octave lower). The figure shows the various musical notes that can be produced for each given fraction of the length of a C-string.



ATTENTION

Since ΔABP is not a right-angled triangle, Pythagoras' Theorem cannot be applied in ΔABP.

PRACTISE NOW 3

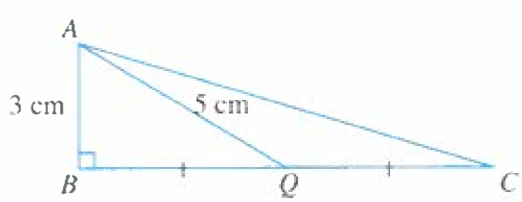
SIMILAR QUESTIONS

Exercise 10A Questions 7–8, 9(a)–(e), 10(a)–(d), 11–13

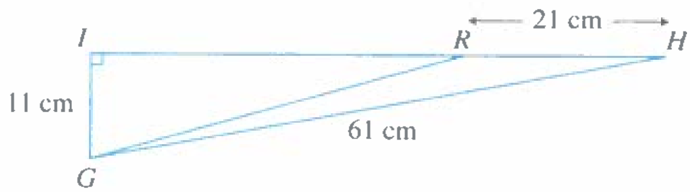
1. In ΔABC, $AB = 3$ cm and $\angle B = 90°$. Q lies on BC such that $BQ = QC$ and $AQ = 5$ cm. Find the length of

   (i) BQ,

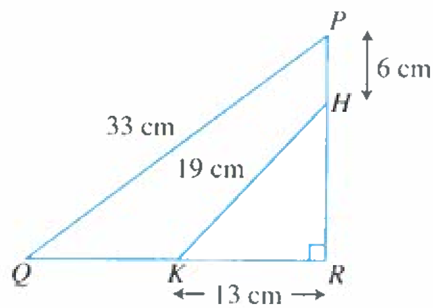
   (ii) AC.



2. In ΔGHI, $GH = 61$ cm and $GI = 11$ cm and $\angle I = 90°$. R lies on IH such that $RH = 21$ cm. Find the length of
(i) HI,
(ii) GR.



3. In ΔPQR, $PQ = 33$ cm and $\angle R = 90°$. H lies on PR such that $PH = 6$ cm and K lies on QR such that $KR = 13$ cm. Find the length of
(i) HR,
(ii) QK.

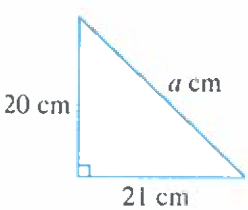


Exercise 10A

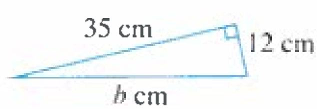
BASIC LEVEL

1. Find the value of the unknown in each of the following right-angled triangles.

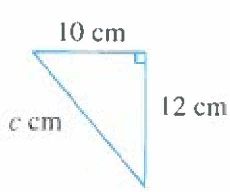
(a)



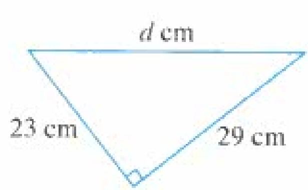
(b)



(c)

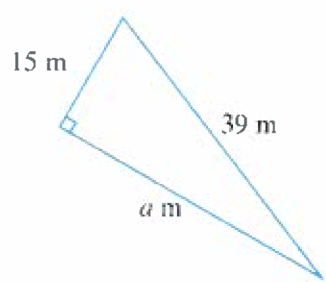


(d)

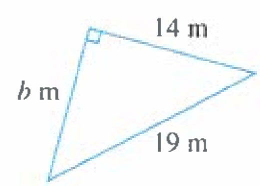


2. Find the value of the unknown in each of the following right-angled triangles.

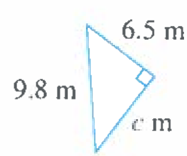
(a)



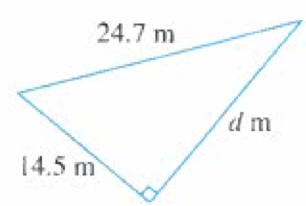
(b)



(c)



(d)



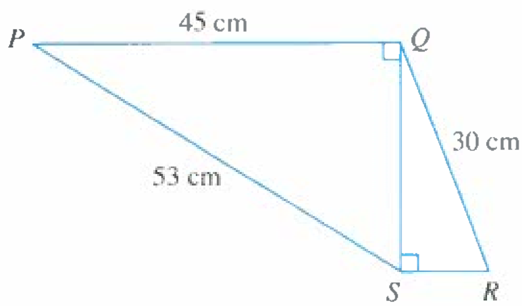
3. In ΔABC, $AB = 8$ cm, $BC = 15$ cm and $\angle B = 90°$. Find the length of AC.

4. A triangle DEF is right-angled at E, with $DE = 6.7$ m and $EF = 5.5$ m. Find the length of DF.

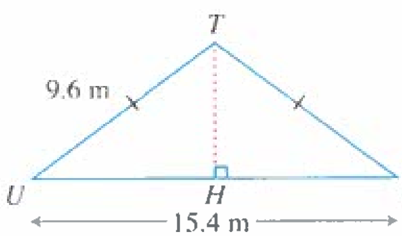
5. In ΔGHI, $GH = 33$ cm, $GI = 65$ cm and $\angle H = 90°$. Find the length of HI.

6. A triangle MNO is right-angled at N, with $NO = 11$ m and $MO = 14.2$ m. Find the length of MN.

7. In the figure, $\angle PQS = \angle QSR = 90°$. Given that $PQ = 45$ cm, $QR = 30$ cm and $PS = 53$ cm, find the length of
   (i) QS,
   (ii) SR.

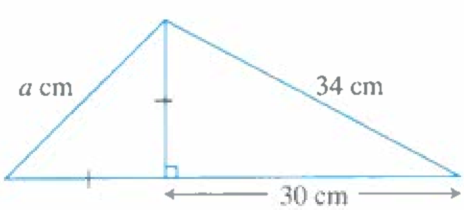


8. The figure shows an isosceles triangle TUV where $TU = TV = 9.6$ m and $UV = 15.4$ m. Find the height, TH, of the triangle.



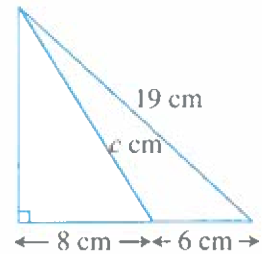
INTERMEDIATE LEVEL

9. Find the value of the unknown in each of the following figures.
   (a)

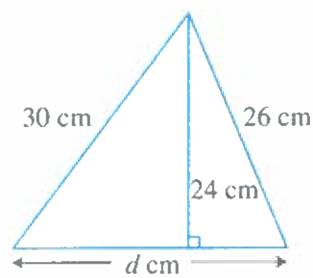


(b)

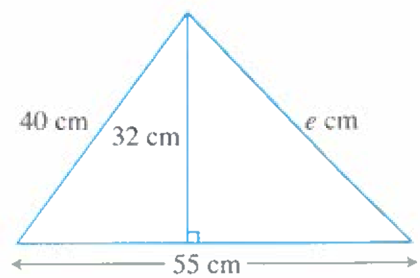
(c)



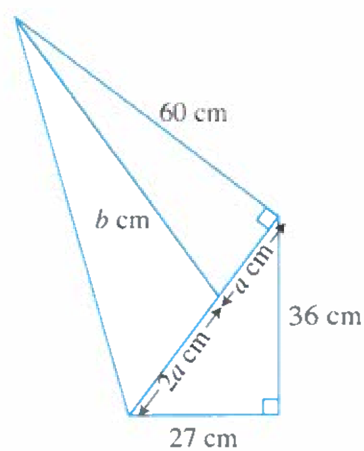
(d)



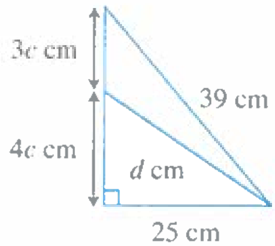
(e)



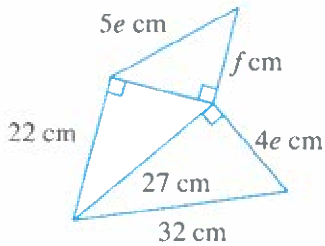
10. Find the value(s) of the unknown(s) in each of the following figures.
   (a)



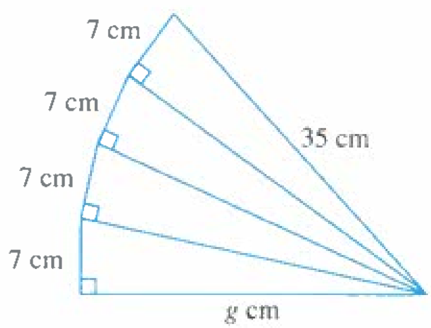
(b)



(c)



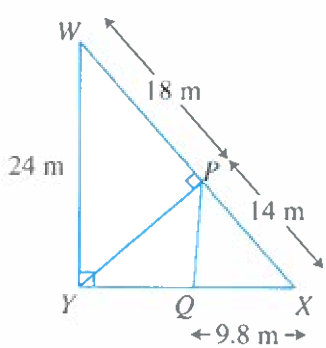
(d)



11. In ΔWXY, $WY = 24$ m and $\angle Y = 90^\circ$. P lies on WX such that YP is perpendicular to WX, $WP = 18$ m and $PX = 14$ m. Q lies on YX such that $QX = 9.8$ m. Find

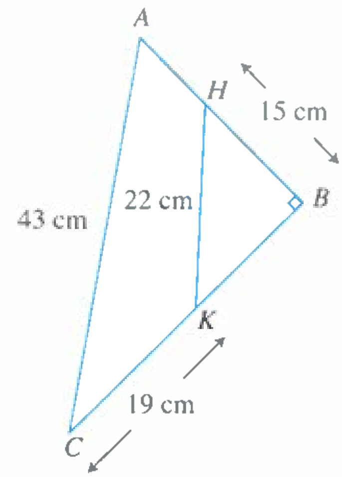
(i) the length of YQ,

(ii) the area of ΔXPY.

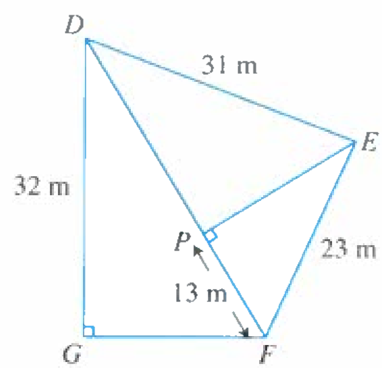


ADVANCED LEVEL

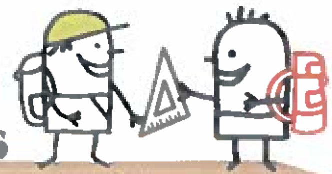
12. In ΔABC, $AC = 43$ cm and $\angle B = 90^\circ$. H lies on AB such that $HB = 15$ cm and K lies on CB such that $CK = 19$ cm. Given that $HK = 22$ cm, find the length of AH.



13. In the figure, $\angle DGF = \angle EPF = 90^\circ$. Given that $DE = 31$ m, $EF = 23$ m, $DG = 32$ m and $PF = 13$ m, find the area of the figure.



10.2 Applications of Pythagoras' Theorem in Real-World Contexts

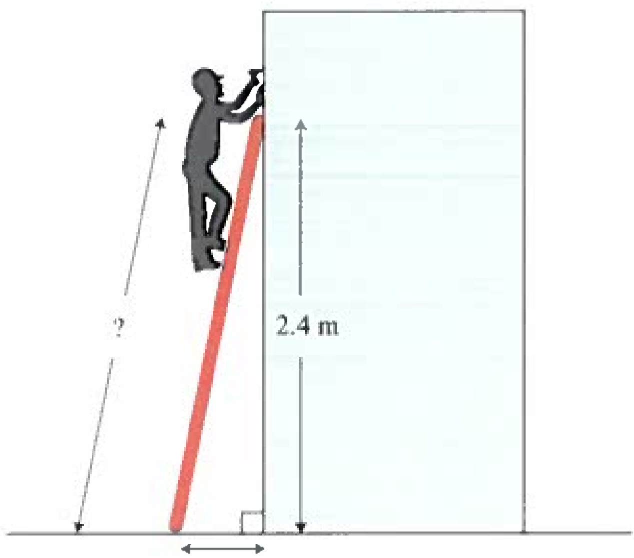


Pythagoras' Theorem is useful in fields such as civil engineering, architecture and navigation. In this section, we will learn how Pythagoras' Theorem can be applied in real-world contexts.

Worked Example 4

(Real-life Application of Pythagoras' Theorem)

A construction worker is on a ladder that is placed against a vertical wall. Given that the top of the ladder is 2.4 m above the ground and the foot of the ladder is placed 0.5 m from the wall for stability, calculate the length of the ladder.



Solution:

Let the length of the ladder be x m.
Using Pythagoras' Theorem,

$$x^2 = 2.4^2 + 0.5^2$$
$$= 5.76 + 0.25$$
$$= 6.01$$

$$\therefore x = \sqrt{6.01} \text{ (since } x > 0)$$
$$= 2.45 \text{ (to 3 s.f.)}$$

The ladder is 2.45 m long.

Story Time

Diophantus (200 – 284) posed this problem: If x, y, z and n are positive integers, under what conditions does the equation $x^n + y^n = z^n$ have a solution?
The equation $x^n + y^n = z^n$ has many solutions when $n = 2$, i.e. $x^2 + y^2 = z^2$, such as $x = 3$, $y = 4$ and $z = 5$ (Pythagorean Triples).

Fermat (1601 – 1665) wrote in the margin of his copy of the book containing this problem that he had found a proof to show that this problem does not have a solution if $n > 2$, but the margin was too small for him to write the proof. This became known as Fermat's Last Theorem. There were thousands of alleged proofs of his theorem but all of them were wrong!

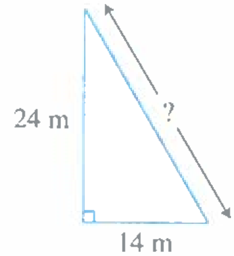
Andrew Wiles, a British mathematician, finally proved Fermat's Last Theorem in 1994 and the proof was more than 200 pages long! However, as Wiles was more than 40 years old at that time, he was not eligible for the Field's Medal, which is the highest accolade for mathematics, equivalent in prestige and recognition to the Nobel prizes, but only given to outstanding mathematicians aged 40 years old or below.

PRACTISE NOW 4

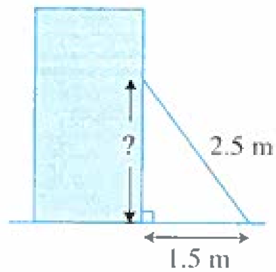
SIMILAR QUESTIONS

Exercise 10B Questions 1–5, 7

1. A vertical pole of height 24 m is supported by a taut steel cable attached from the top of the pole to a point on level ground, 14 m from the foot of the pole. Find the length of the cable.



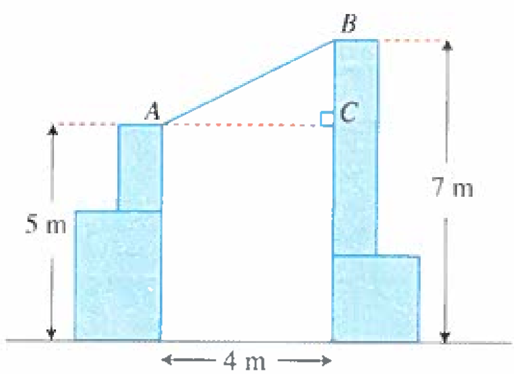
2. A ladder of length 2.5 m is placed against a vertical wall with its foot 1.5 m away from the base of the wall. How far up the wall does the ladder reach?



Worked Example 5

(Real-Life Application of Pythagoras' Theorem)

In a factory, a sliding belt, AB, used to transport products is stretched between two vertical columns 4 m apart. The heights of the columns are 5 m and 7 m. Calculate the length of the belt.



Solution:

In $\triangle ABC$, $\angle C = 90°$.

$BC = 7 - 5$
$\quad = 2$ m

Using Pythagoras' Theorem,

$$\begin{aligned} AB^2 &= BC^2 + AC^2 \\ &= 2^2 + 4^2 \\ &= 4 + 16 \\ &= 20 \end{aligned}$$

$\therefore AB = \sqrt{20}$ (since $AB > 0$)

$\quad = 4.47$ m (to 3 s.f.)

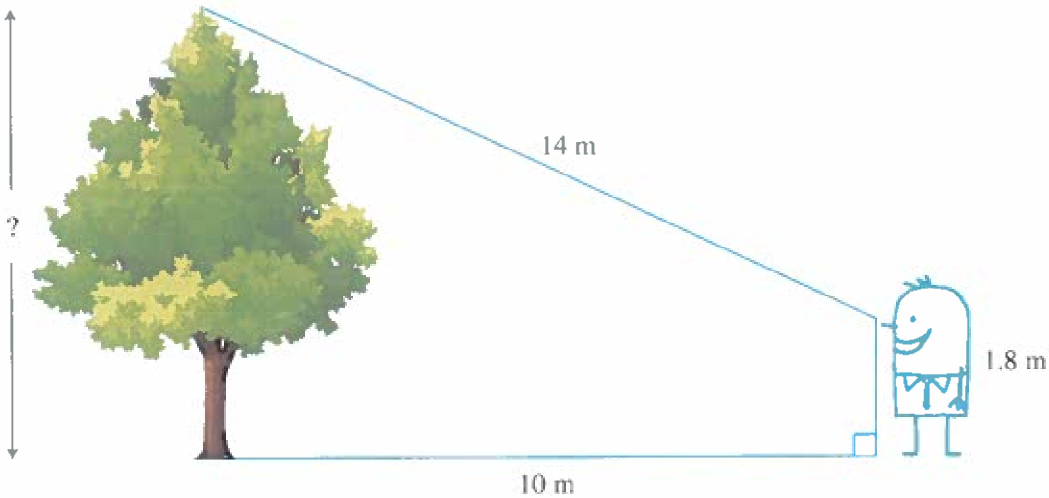
The length of the belt is 4.47 m.

PRACTISE NOW 5

SIMILAR QUESTIONS

Exercise 10B Questions 6, 8–10, 13, 15

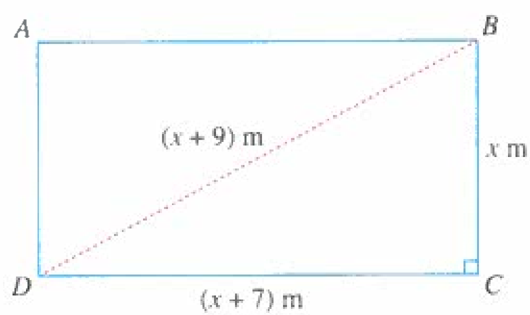
Michael is standing 10 m away from a tree. The distance of his eyes from his feet is 1.8 m. Given that the distance from his eyes to the top of the tree is 14 m, find the height of the tree.



Worked Example 6

(Real-Life Application of Pythagoras' Theorem)

A school field in the shape of a rectangle $ABCD$ has sides $(x + 7)$ m and x m, and the length of the diagonal BD is $(x + 9)$ m. Calculate the value of x.



Solution:

In ΔBCD, $\angle C = 90°$.

Using Pythagoras' Theorem,

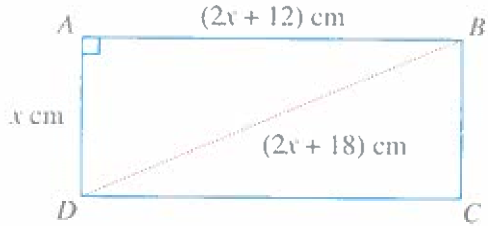
$$BD^2 = CD^2 + BC^2$$
$$(x + 9)^2 = (x + 7)^2 + x^2$$
$$x^2 + 18x + 81 = x^2 + 14x + 49 + x^2$$
$$x^2 - 4x - 32 = 0$$
$$(x + 4)(x - 8) = 0$$
$$x = -4 \text{ or } x = 8$$
$$\therefore x = 8 \text{ (since } x > 0)$$

PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 10B Questions 11–12

A dining table in the shape of a rectangle $ABCD$ has sides of length $(2x + 12)$ cm and x cm, and the length of the diagonal BD is $(2x + 18)$ cm. Find the value of x.



Worked Example 7

(Real-Life Application of Pythagoras' Theorem)

Boat A travels due East from Port P for 3 hours to reach Port Q and then travels due South for another 2 hours to reach Port R. Boat B travels due North from Port P for 2 hours to reach Port S and then travels due East for 3 hours to reach Port T. If the average speeds of Boat A and Boat B are 12 km/h and 18 km/h respectively, calculate the shortest distance between

(i) Port Q and Port S,

(ii) Port R and Port T.

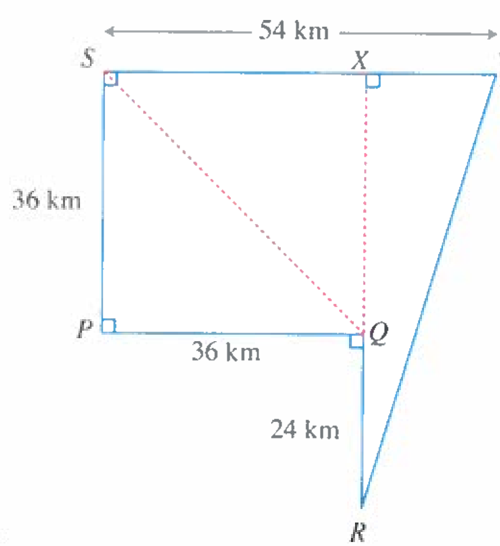
Solution:

(i) $PQ = 12 \times 3 = 36$ km

$QR = 12 \times 2 = 24$ km

$PS = 18 \times 2 = 36$ km

$ST = 18 \times 3 = 54$ km



Problem Solving Tip

You may wish to sketch a figure to help you better visualise a problem.

In ΔPQS, $\angle P = 90°$.

Using Pythagoras' Theorem,

$$\begin{aligned} QS^2 &= PQ^2 + PS^2 \\ &= 36^2 + 36^2 \\ &= 1296 + 1296 \\ &= 2592 \end{aligned}$$

$$\therefore QS = \sqrt{2592} \text{ (since } QS > 0)$$
$$= 50.9 \text{ km (to 3 s.f.)}$$

The shortest distance between Port Q and Port S is 50.9 km.

(ii) Draw a perpendicular line from Q to ST cutting ST at X.

In ΔRTX, $\angle X = 90°$.

$$\begin{aligned} TX &= 54 - 36 \\ &= 18 \text{ km} \\ RX &= 24 + 36 \\ &= 60 \text{ km} \end{aligned}$$

Using Pythagoras' Theorem,

$$\begin{aligned} RT^2 &= TX^2 + RX^2 \\ &= 18^2 + 60^2 \\ &= 324 + 3600 \\ &= 3924 \end{aligned}$$

$$\begin{aligned} \therefore RT &= \sqrt{3924} \text{ (since } RT > 0) \\ &= 62.6 \text{ km (to 3 s.f.)} \end{aligned}$$

The shortest distance between Port R and Port T is 62.6 km.

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 10B Question 14

A ship travels due South from Port A for 1.2 hours to reach Port B and then travels due East for another 1.7 hours to reach Jetty C. It then travels due South to Buoy D which is 18 km away from Jetty C. From Buoy D, it travels 38 km due West to reach Island E. If the average speed of the ship is 10 km/h, find the shortest distance between

(i) Port A and Jetty C,

(ii) Port A and Island E.

Exercise 10B

BASIC LEVEL

1. Some cables are supported by a vertical post of height 47 m. The cables are attached from the top of the post to a point on level ground, 18 m from the foot of the post. Find the length of each cable.

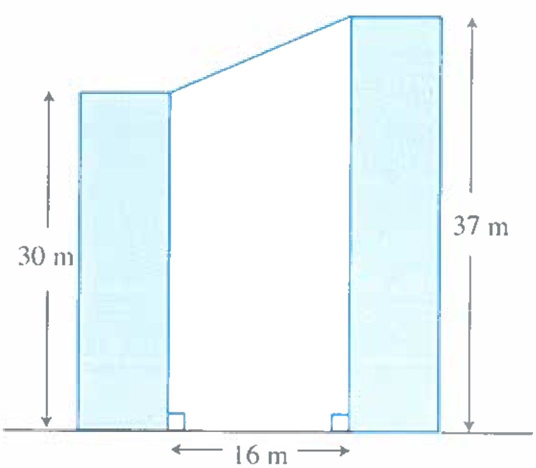
2. Each side of a square field is 50 m long. A barricade is to be placed along the diagonal of the field. Find the length of the barricade.

3. A rectangular swimming pool has a length of 50 m and a breadth of 30 m. During a swimming proficiency test, Ethan has to swim from one corner of the pool to the opposite corner. Find the distance Ethan has to swim.

4. A ladder of length 5 m is placed against a vertical wall with its foot 1.8 m away from the base of the wall. How far up the wall does the ladder reach?

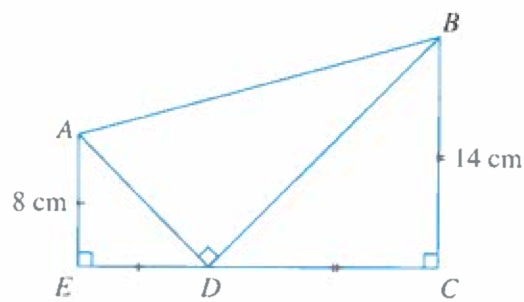
5. Mr Lee buys a 30-inch television. The height of the screen is 18 inches. Given that television screens are measured across the diagonal, i.e. the length of the diagonal is 30 inches, how wide is the screen?

6. Two vertical buildings of heights 30 m and 37 m are 16 m apart. A cable is pulled taut and attached from the top of one building to the top of the other building. Find the length of the cable.



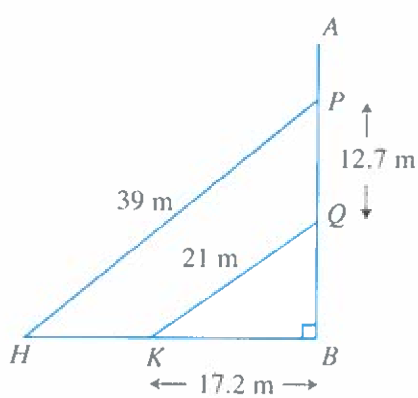
INTERMEDIATE LEVEL

7. Devi folded a sheet of paper into an envelope, as shown in the figure. In the figure, $\angle AED = \angle ADB = \angle BCD = 90°$. To seal the envelope, she has to apply glue along AD and DB. Given that $AE = ED = 8$ cm and $DC = BC = 14$ cm, find the total length of the sides along which the glue has to be applied to seal the envelope.



8. A table coaster in the shape of a rhombus has diagonals of lengths 10 cm and 24 cm. Find the length of each side of the coaster.

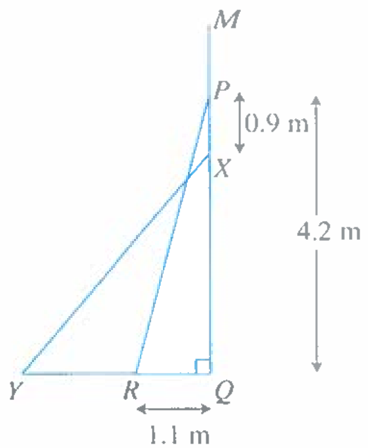
9. Two spotlights, P and Q, are mounted on a vertical pole AB. Light beams from P and Q shine to two points on the ground, H and K, respectively. Given that $PQ = 12.7$ m, $KB = 17.2$ m, $PH = 39$ m and $QK = 21$ m, find
   (i) BQ, the height above the ground at which the spotlight Q is mounted,
   (ii) HK, the distance between the projections of the light beams.



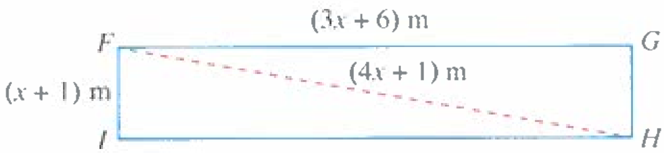
10. A straight pole PR is leaning against a vertical wall MQ.

(i) Given that $PQ = 4.2$ m and $RQ = 1.1$ m, find the length of the pole.

(ii) The upper end of the pole, P, slides down 0.9 m to a point on the wall, X. Calculate YR, the distance the lower end of the pole has slid away from its original position R.



11. A campsite in the shape of a rectangle $FGHI$ has sides $(3x + 6)$ m and $(x + 1)$ m, and the length of the diagonal FH is $(4x + 1)$ m. Find the area of the campsite.



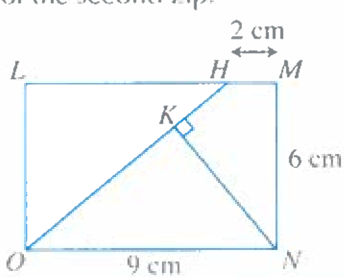
12. A folded napkin has a triangular cross section of sides x cm, $(x + 1)$ cm and $(x + 2)$ cm. If one of the angles of the triangle is 90°, find the value of x.

ADVANCED LEVEL

13. The top surface of a pouch is in the shape of a rectangle $LMNO$ with sides 9 cm and 6 cm.

(i) A zip is to be sewn along OH such that H is a point on LM and $HM = 2$ cm. Find the length of the zip.

(ii) A second zip is to be sewn along NK such that NK is the perpendicular from N to OH. Calculate the length of the second zip.



14. A courier travels due North at an average speed of 40 km/h for 6 minutes to collect a parcel, before travelling 10 km due East to deliver it. He then travels due South at an average speed of 30 km/h for 12 minutes to collect another parcel. Find the shortest distance between the courier and his starting point.

15. A designer is tasked to design two tables for children such that the tabletops are of different shapes as shown, and that the perimeter of each tabletop is 132 cm.


Square Tabletop


Round Tabletop

(a) Find

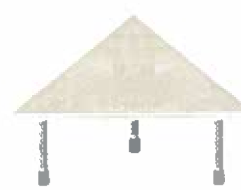
(i) the length of each side of the square tabletop,

(ii) the radius of the round tabletop.

(b) Hence, calculate the area of each tabletop.

(Take π to be $\frac{22}{7}$.)

The designer decides to design another table with a tabletop in the shape of an equilateral triangle as shown, such that the perimeter of the tabletop is also 132 cm.



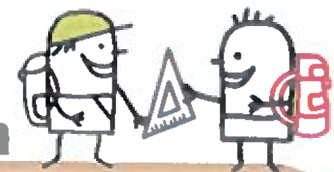
(c) Find

(i) the length of each side of the tabletop,

(ii) the area of the tabletop.

(d) If the designer wants to design a table with a tabletop in the shape of a square, a circle or an equilateral triangle, which shape should he choose such that the tabletop has the greatest area? Justify your answer.

10.3 Converse of Pythagoras' Theorem



Pythagoras' Theorem states that if an angle in a triangle is a right angle, then the square of the length of the side opposite the right angle, i.e. the hypotenuse, is equal to the sum of the squares of the lengths of the other two sides.

The **converse of Pythagoras' Theorem** is also true, i.e. if the square of the length of the *longest side* is equal to the sum of the squares of the lengths of the other two sides, then the angle opposite the *longest side* is a right angle.

In a triangle ABC, if $AB^2 = BC^2 + AC^2$,

i.e. $c^2 = a^2 + b^2$,

then $\angle C = 90^\circ$.

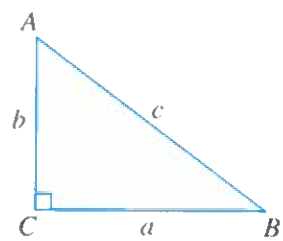


Fig. 10.7

Worked Example 8

(Determining if Triangles are Right-Angled Triangles Given the Lengths of the Sides)

Determine if each of the following triangles is a right-angled triangle. For each right-angled triangle, state the right angle.

(a) ΔABC, given that $AB = 39$ cm, $BC = 15$ cm and $AC = 36$ cm

(b) ΔPQR, given that $PQ = 28$ m, $QR = 20$ m and $PR = 19$ m

Solution:

(a) AB is the longest side of ΔABC.

$AB^2 = 39^2$
$= 1521$

$BC^2 + AC^2 = 15^2 + 36^2$
$= 225 + 1296$
$= 1521$

Since $AB^2 = BC^2 + AC^2$, then by the converse of Pythagoras' Theorem, ΔABC is a right-angled triangle where $\angle C = 90^\circ$.

(b) PQ is the longest side of ΔPQR.

$PQ^2 = 28^2$
$= 784$

$QR^2 + PR^2 = 20^2 + 19^2$
$= 400 + 361$
$= 761$

Since $PQ^2 \neq QR^2 + PR^2$, ΔPQR is not a right-angled triangle.

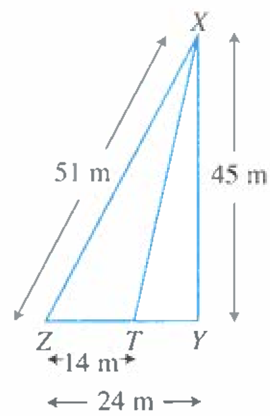
PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 10C Questions 1(a)–(d), 2–5

1. Determine if each of the following triangles is a right-angled triangle. For each right-angled triangle, state the right angle.
   (a) ΔABC, given that $AB = 12$ cm, $BC = 10$ cm and $AC = 8$ cm
   (b) ΔPQR, given that $PQ = 34$ m, $QR = 16$ m and $PR = 30$ m

2. XYZ is a plot of land such that $XY = 45$ m, $YZ = 24$ m and $XZ = 51$ m.
   (i) Show that $\angle XYZ = 90°$.
   (ii) A tree T is located on ZY such that $ZT = 14$ m. Find TX, the distance of the tree from X.



Exercise 10C

BASIC LEVEL

1. Determine if each of the following triangles is a right-angled triangle. For each right-angled triangle, state the right angle.
   (a) ΔABC, given that $AB = 16$ cm, $BC = 63$ cm and $AC = 65$ cm
   (b) ΔDEF, given that $DE = 24$ cm, $EF = 27$ cm and $DF = 21$ cm
   (c) ΔGHI, given that $GH = 7.5$ m, $HI = 7.1$ m and $GI = 2.4$ m
   (d) ΔMNO, given that $MN = \frac{5}{13}$ m, $NO = \frac{3}{13}$ m and $MO = \frac{4}{13}$ m

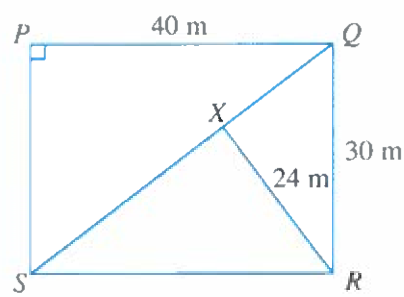
2. In ΔPQR, $PQ = 19$ cm, $QR = 24$ cm and $PR = 30$ cm. Show that ΔPQR is not a right-angled triangle.

INTERMEDIATE LEVEL

3. In ΔSTU, $ST = \frac{7}{12}$ cm, $TU = \frac{5}{6}$ cm and $SU = \frac{1}{3}$ cm. Is ΔSTU a right-angled triangle?

ADVANCED LEVEL

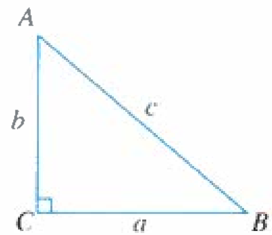
4. A rectangular grass patch, $PQRS$, has sides 40 m and 30 m. A straight path which cuts through the grass patch joins S and Q. Lamppost X is located on SQ such that $SX : XQ = 16 : 9$ and $RX = 24$ m. Jun Wei walks along the path and stops at a point which is nearest to R. Show that he stops at X.



5. The lengths of the sides, a, b and c, of a triangle are given by $a = m^2 - n^2$, $b = 2mn$ and $c = m^2 + n^2$, where m and n are positive integers and $m > n$. Show that the triangle is a right-angled triangle.

Summary

1. In ΔABC, the side AB opposite the right angle C is the longest side. AB is called the **hypotenuse** of ΔABC.



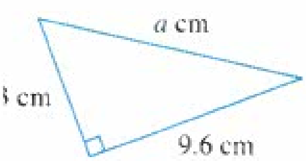
2. **Pythagoras' Theorem**
   In a triangle ABC, if $\angle C = 90°$,
   then $AB^2 = BC^2 + AC^2$,
   i.e. $c^2 = a^2 + b^2$.

3. **Converse of Pythagoras' Theorem**
   In a triangle ABC, if $AB^2 = BC^2 + AC^2$,
   i.e. $c^2 = a^2 + b^2$,
   then $\angle C = 90°$.

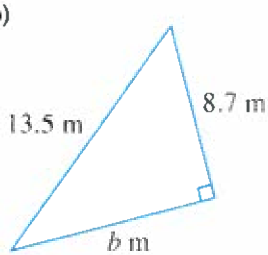
Review Exercise 10

1. Find the value of the unknown in each of the following figures.

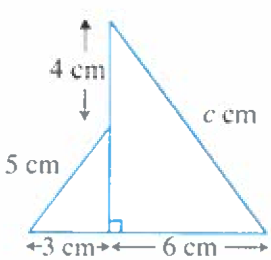
(a)



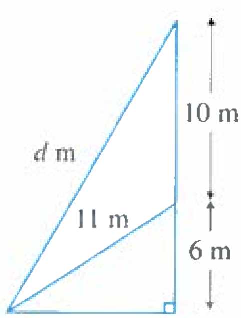
(b)



(c)

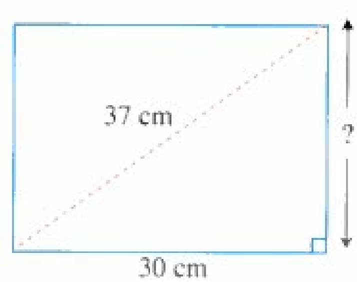


(d)



2. If the diagonal of a square is 42.5 cm long, find
   (i) the perimeter,
   (ii) the area,
   of the square.

3. Khairul's briefcase has a rectangular cross section. The length and the diagonal of the briefcase are 30 cm and 37 cm respectively. Find the height of the briefcase.

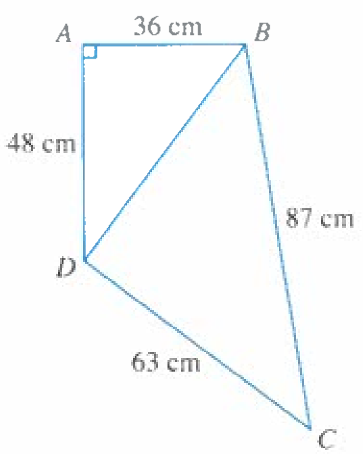


4. A rare stamp in the shape of an equilateral triangle FGH has sides 2 cm. Find the perpendicular distance from F to GH.

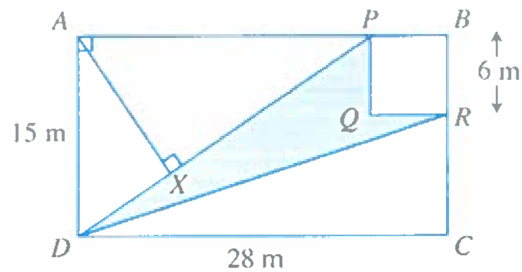
5. A piece of stained glass in the shape of a parallelogram $LMNO$ is such that the diagonal LN is at a right angle to LM. Given that $LM = 12$ cm and $MN = 15$ cm, find the area of the piece of stained glass.

6. A floor tile in the shape of a rhombus has sides of length 52 cm and a diagonal of length 48 cm. Find
   (i) the length of the other diagonal,
   (ii) the area,
   of the floor tile.

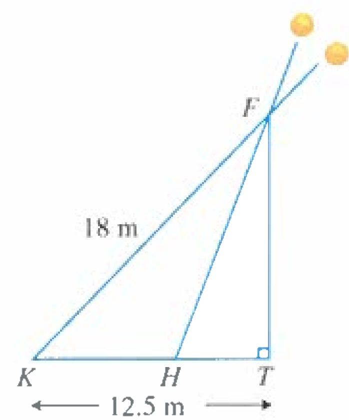
7. In the figure, $\angle BAD = 90°$.
   (i) Given that $AB = 36$ cm and $AD = 48$ cm, find the length of BD.
   (ii) Given further that $BC = 87$ cm and $CD = 63$ cm, show that $\triangle BCD$ is a right-angled triangle.



8. In the figure, $ABCD$ is a rectangle of sides 28 m and 15 m. $PBRQ$ is a square of sides 6 m.
   (i) Find the area of the shaded region $DPQR$.
   (ii) Calculate the length of DP.
   (iii) Given that X is a point on DP such that AX is perpendicular to DP, find the length of AX.



9. A vertical pole FT casts a shadow HT at a certain time in a day. It casts a shadow KT at another time on the same day. Given that $FK = 18$ m, $KT = 12.5$ m and $KH : HT = 3 : 2$, find
   (i) the height of the pole,
   (ii) the distance FH.



10. Farhan runs diagonally across a rectangular field 80 m by 60 m, from one corner to the opposite in a straight line at a speed of 7.5 m/s. Find the time taken for him to complete his run.

Challenge Yourself

1. Three positive integers, where $a < b < c$, are said to form a Pythagorean Triple if $a^2 + b^2 = c^2$. For example, 3, 4 and 5 form a Pythagorean Triple because

$$3^2 + 4^2 = 9 + 16 = 25 = 5^2.$$

   (a) In the same way, show that 6, 8 and 10 form a Pythagorean Triple.

   (b) **(i)** Find the corresponding Pythagorean Triple when $a = 12$ and $b = 16$.

   (ii) State a Pythagorean Triple where $c = 25$.

   (c) **(i)** Simplify $(3n)^2 + (4n)^2$.

   (ii) Hence, find the corresponding Pythagorean Triple when $c = 35$.

   (d) Consider $1 + 2n + n^2 = (1 + n)^2$.

   If $1 + 2n$ is a perfect square, i.e. $1 + 2n = k^2$, then we have $k^2 + n^2 = (1 + n)^2$.
   Then k, n and $1 + n$ form a Pythagorean Triple.

   For example, when $n = 12$, $1 + 2n = 1 + 2 \times 12 = 1 + 24 = 25 = 5^2$, then 5, 12 and 13 form a Pythagorean Triple.

   (i) In the same way, find the corresponding Pythagorean Triple when $n = 24$.

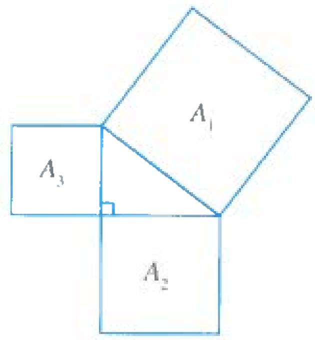
   (ii) Are we able to obtain a Pythagorean Triple in the same way when $1 + 2n = 42$? Explain your answer.

   (iii) Find the corresponding Pythagorean Triple when $k = 9$.

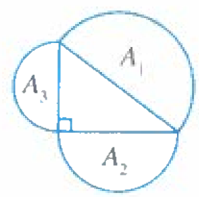
2. Given a triangle ABC where $BC^2 = 370$ units^2, $AB^2 = 116$ units^2 and $AC^2 = 74$ units^2, find the area of the triangle.

   Hint: $370 = 9^2 + 17^2$, $116 = 4^2 + 10^2$, $74 = 5^2 + 7^2$

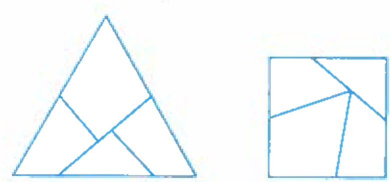
3. Pythagoras' Theorem is usually stated in terms of the lengths of the sides of a right-angled triangle, i.e. $c^2 = a^2 + b^2$. However, it can also be stated in terms of the areas of squares, i.e. $A_1 = A_2 + A_3$.



   If the areas refer to the areas of semicircles instead of the areas of squares, does the relationship still hold true? Is A_1 still equal to $A_2 + A_3$? Explain your answer.



4. Congruent figures have exactly the same shape and size. Similar figures have the same shape but not necessarily the same size. What about figures with the same size but of different shapes? An interesting example is the Haberdasher's Puzzle: how do we cut an equilateral triangle into 4 pieces and rearrange them to form a square? The figure shows a solution of the puzzle by Henry Dudeney (1857 – 1930).

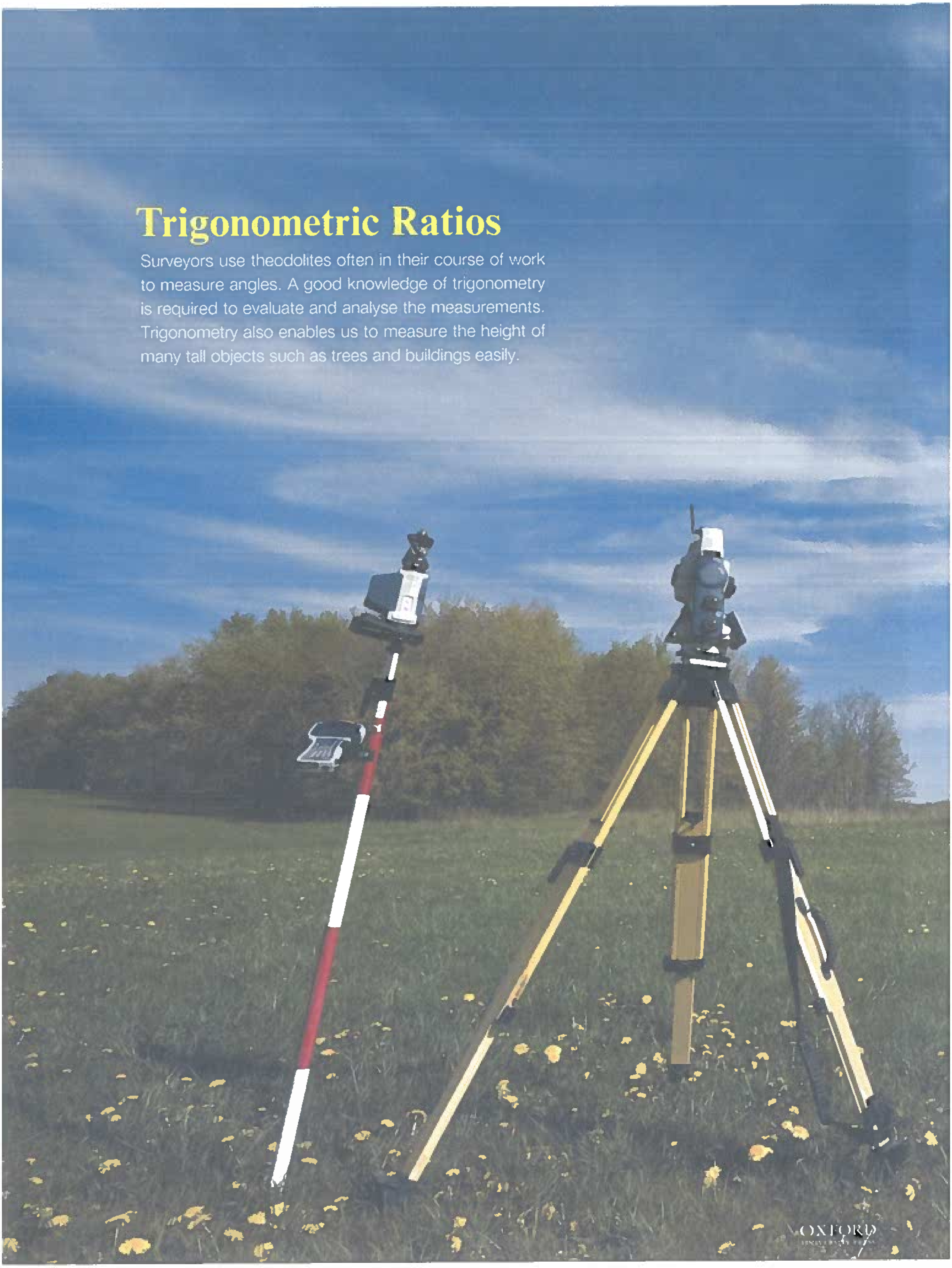


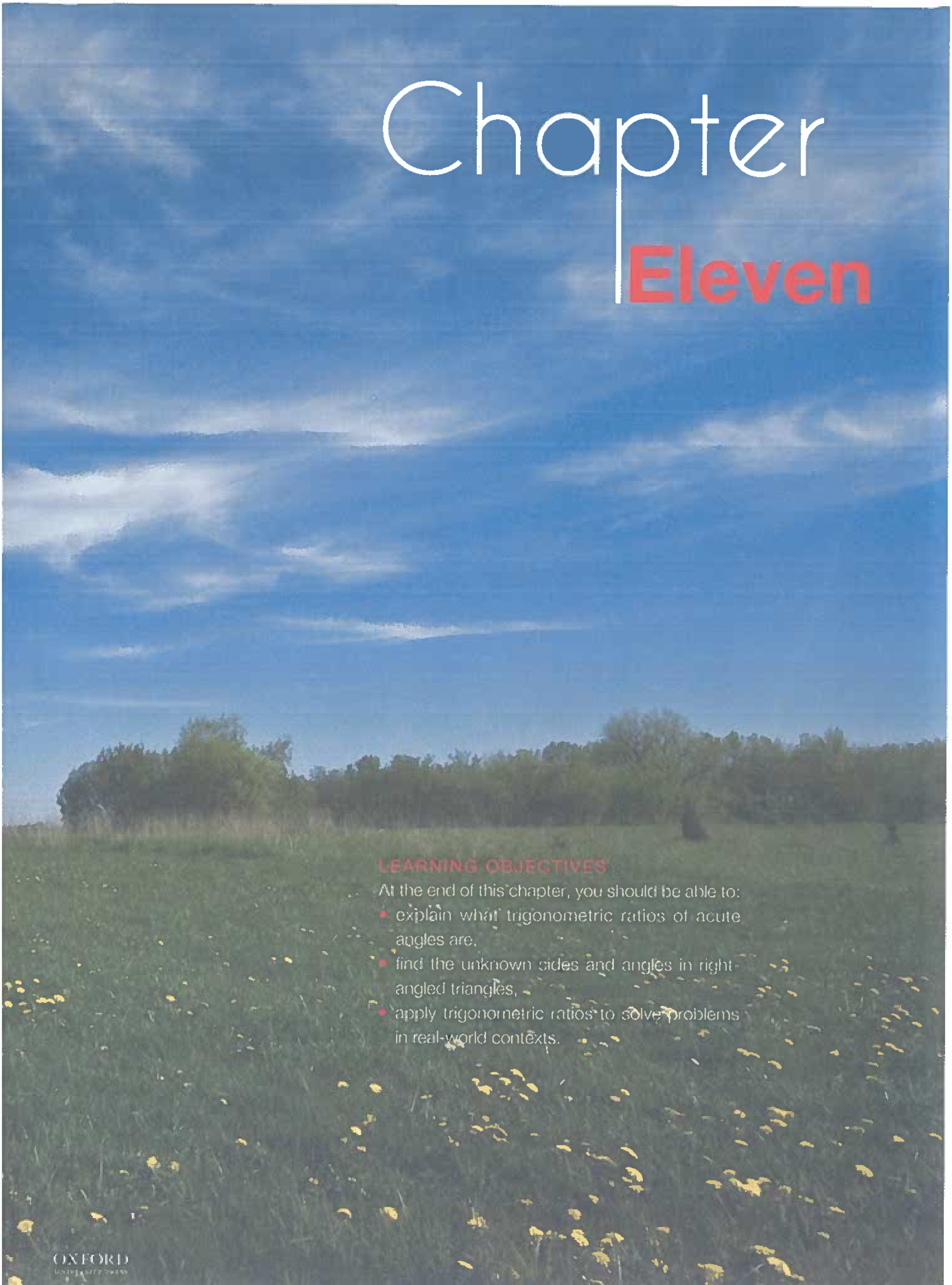
   (i) If the length of each side of the square formed is 3 cm, find the length of each side of the equilateral triangle.

   (ii) Is it possible for the value of the length of each side of the equilateral triangle and of the square to be an integer? Explain your answer.

Trigonometric Ratios

Surveyors use theodolites often in their course of work to measure angles. A good knowledge of trigonometry is required to evaluate and analyse the measurements. Trigonometry also enables us to measure the height of many tall objects such as trees and buildings easily.





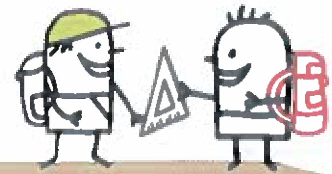
Chapter Eleven

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- explain what trigonometric ratios of acute angles are,
- find the unknown sides and angles in right-angled triangles,
- apply trigonometric ratios to solve problems in real-world contexts.

11.1 Trigonometric Ratios



Trigonometric Ratios

In this chapter, we will learn a new branch of mathematics known as trigonometry.

Investigation

Trigonometric Ratios

1. Draw and label a triangle ABC on a piece of paper where $\angle A = 30°$ and $\angle C = 90°$. You can choose any length for AC but try to draw the triangle as big as possible.
2. Measure and write down the lengths of AB, BC and AC.
3. Find the ratios $\frac{BC}{AB}$, $\frac{AC}{AB}$ and $\frac{BC}{AC}$, giving each of your answers correct to 2 significant figures.
4. Based on the triangles which you and your classmate have drawn, answer the following questions.
   (a) Is your classmate's triangle in Step 1 congruent or similar to yours? Explain your answer.
   (b) Compare the three ratios obtained by your classmate in Step 3 with yours. What can you say about the values?
   (c) What is the condition for two right-angled triangles to have the same set of ratios $\frac{BC}{AB}$, $\frac{AC}{AB}$ and $\frac{BC}{AC}$?
5. Fig. 11.1 shows a right-angled triangle ABC where $\angle C = 90°$.

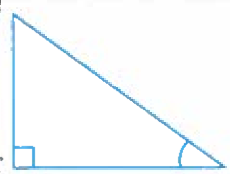


Fig. 11.1

   (a) We have learnt in Chapter 10 that the longest side of a right-angled triangle is called the **hypotenuse**. Label the hypotenuse in Fig. 11.1.
   (b) With respect to $\angle A$ in Fig. 11.1, the side opposite $\angle A$ is called the **opposite** side. Label the opposite side in Fig. 11.1.
   (c) With respect to $\angle A$ in Fig. 11.1, there are two sides adjacent to $\angle A$. One of them is the hypotenuse. The other side is called the **adjacent** side. Label the adjacent side in Fig. 11.1.
6. In terms of the hypotenuse (hyp), the opposite side (opp) and the adjacent side (adj), the ratio $\frac{BC}{AB}$ can be rewritten as $\frac{\text{opp}}{\text{hyp}}$. Write down an expression for $\frac{AC}{AB}$ and $\frac{BC}{AC}$ respectively.

From the investigation, we observe that for any two *similar* right-angled triangles, the three ratios in Step 3 and Step 6 are always *equal*.

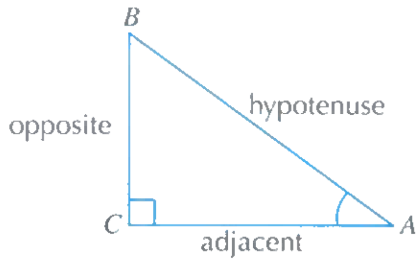


Fig. 11.2

In a triangle ABC, if $\angle C = 90°$,

then $\frac{BC}{AB} = \frac{\text{opp}}{\text{hyp}}$ is called the **sine** of $\angle A$, or $\sin A = \frac{\text{opp}}{\text{hyp}}$,

$\frac{AC}{AB} = \frac{\text{adj}}{\text{hyp}}$ is called the **cosine** of $\angle A$, or $\cos A = \frac{\text{adj}}{\text{hyp}}$,

$\frac{BC}{AC} = \frac{\text{opp}}{\text{adj}}$ is called the **tangent** of $\angle A$, or $\tan A = \frac{\text{opp}}{\text{adj}}$.

These three ratios are known as **trigonometric ratios** and they are numbers *without any units* because they are ratios of one length to another length.

Also, since ΔABC is a right-angled triangle, $\angle A$ is an acute angle, i.e. $0° < \angle A < 90°$. Hence, the definitions of trigonometric ratios *only apply to acute angles in a right-angled triangle*.

Problem Solving Tip

To remember the three trigonometric ratios, you may wish to use the mnemonic '**TOA CAH SOH**', where

- **TOA** represents $\tan = \frac{\text{opp}}{\text{adj}}$,
- **CAH** represents $\cos = \frac{\text{adj}}{\text{hyp}}$,
- **SOH** represents $\sin = \frac{\text{opp}}{\text{hyp}}$.

Thinking Time

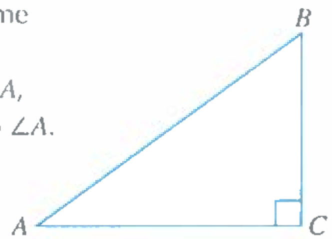
Consider ΔXYZ where $\angle X = 50°$ and $\angle Y = 90°$. Are the trigonometric ratios of $\angle X$ the same as those of $\angle A$ in the previous investigation, i.e. is $\sin 50° = \sin 30°$? Is $\cos 50° = \cos 30°$? Is $\tan 50° = \tan 30°$? In other words, do trigonometric ratios depend on the value of the angle? Explain your answer.

PRACTISE NOW

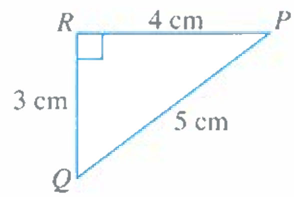
SIMILAR QUESTIONS

Exercise 11A Questions 1(a)–(b), 2(a)–(b), 3(a)–(b)

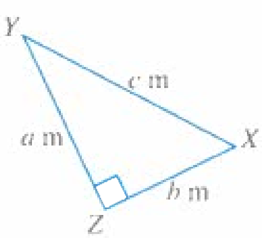
1. In ΔABC, $\angle C = 90°$. Name
   (i) the hypotenuse,
   (ii) the side opposite $\angle A$,
   (iii) the side adjacent to $\angle A$.



2. In ΔPQR, $PR = 4$ cm, $RQ = 3$ cm, $PQ = 5$ cm and $\angle R = 90°$. State the value of
   (i) $\sin P$, (ii) $\cos P$, (iii) $\tan P$,
   (iv) $\sin Q$, (v) $\cos Q$, (vi) $\tan Q$.



3. In ΔXYZ, $XY = c$ m, $YZ = a$ m, $XZ = b$ m and $\angle Z = 90°$. Write down an expression for
   (i) $\sin X$, (ii) $\cos X$, (iii) $\tan X$,
   (iv) $\sin Y$, (v) $\cos Y$, (vi) $\tan Y$,
   in terms of a, b and/or c.



Finding Trigonometric Ratios Given Angles

Fig. 11.3 shows a right-angled triangle where $\angle A = 25°$ and $\angle C = 90°$.

Fig. 11.3

If we measure the length of AB and of BC in Fig. 11.3, and find the ratio $\frac{BC}{AB}$, the value that we obtain is approximately 0.42.

INFORMATION

Trigonometric ratios are used to describe many natural phenomena. They are used in the study of acoustics, X-rays, light, etc. Do you know of other uses of trigonometric ratios?

Fig. 11.4 shows a right-angled triangle where $\angle A = 35°$ and $\angle C = 90°$.

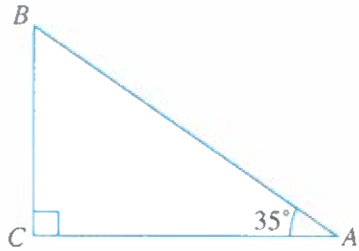


Fig. 11.4

If we measure the length of AB and of BC in Fig. 11.4, and find the ratio $\frac{BC}{AB}$, the value that we obtain is approximately 0.57.

Since the ratio $\frac{BC}{AB}$ is equal to the trigonometric ratio sin A, by repeating this method with other right-angled triangles where $0° < A < 90°$, we will be able to obtain a table of values for the sine ratios.

We can repeat this for $\frac{AC}{AB}$ and $\frac{BC}{AC}$ for different values of A where $0° < A < 90°$ to obtain tables of values for the cosine and the tangent ratios respectively.

Use of Calculators

In the olden days, trigonometric ratios were computed manually and presented in tables. Table 11.1 shows the trigonometric ratios of the angles 50°, 51° and 52°.

Angle	Sine	Cosine	Tangent
50°	0.766	0.643	1.19
51°	0.777	0.629	1.23
52°	0.788	0.616	1.28

Table 11.1

It is troublesome to find the value of sin 50° by drawing a triangle and measuring the lengths of the sides to find the sine ratio as we have done earlier. There may also be large errors for angles near to 0° or 90°. Fortunately, with the advance of technology, we can now use a calculator to find the trigonometric ratios of any angle.

As we are computing the values of angles in degrees, remember to set the mode of your calculator to 'DEG'. Refer to your calculator manual to find out how you can do so.

In Book 2, we will only learn the trigonometric ratios of angles between 0° and 90°.

Worked Example 1

(Use of a Calculator to Evaluate Trigonometric Ratios)

Use a calculator to evaluate each of the following.

(a) $\sin 32°$ (b) $\cos 15.3°$

(c) $\tan 25.96°$ (d) $2 \sin 37° + 5 \tan 56°$

(e) $\dfrac{3}{\cos 48.1°}$ (f) $\dfrac{\cos 57°}{\sin 46.5° + \tan 26.4°}$

Solution:

(a) Sequence of calculator keys:

sin 3 2 =

$\therefore \sin 32° = 0.530$ (to 3 s.f.)

(b) Sequence of calculator keys:

cos 1 5 • 3 =

$\therefore \cos 15.3° = 0.965$ (to 3 s.f.)

(c) Sequence of calculator keys:

tan 2 5 • 9 6 =

$\therefore \tan 25.96° = 0.487$ (to 3 s.f.)

(d) Sequence of calculator keys:

2 sin 3 7 + 5 tan 5 6 =

$\therefore 2 \sin 37° + 5 \tan 56° = 8.62$ (to 3 s.f.)

(e) Sequence of calculator keys:

3 ÷ cos 4 8 • 1 =

$\therefore \dfrac{3}{\cos 48.1°} = 4.49$ (to 3 s.f.)

(f) Sequence of calculator keys:

cos 5 7 ÷ (sin 4 6 • 5 + tan 2 6 • 4) =

$\therefore \dfrac{\cos 57°}{\sin 46.5° + \tan 26.4°} = 0.446$ (to 3 s.f.)

ATTENTION

The sequence of calculator keys in worked example 1 follows the 'Direct Algebraic Logic (DAL)' type of Scientific calculator. Refer to your calculator manual if you are using a different type of calculator.

PRACTISE NOW 1

Use a calculator to evaluate each of the following.

(a) $\cos 24°$ (b) $\tan 74.6°$

(c) $\sin 72.15°$ (d) $3 \sin 48° + 2 \cos 39°$

(e) $\dfrac{5}{\tan 18.3°}$ (f) $\dfrac{\tan 48.3° - \sin 28.7°}{\cos 15° + \cos 35°}$

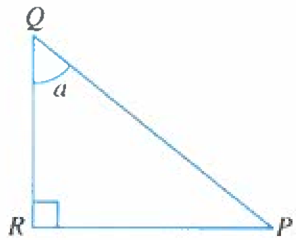
SIMILAR QUESTIONS

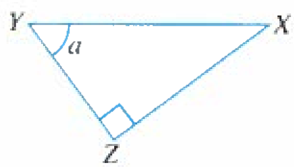
Exercise 11A Questions 4(a)–(f), 5(a)–(h)

Exercise 11A

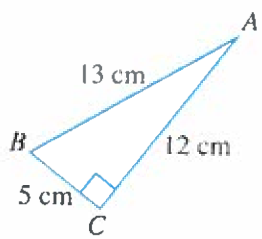
BASIC LEVEL

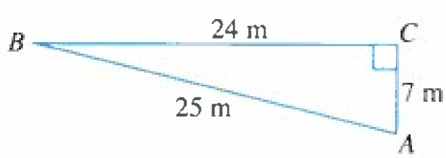
1. For each of the following right-angled triangles, name
   (i) the hypotenuse,
   (ii) the side opposite $\angle a$,
   (iii) the side adjacent to $\angle a$.

   (a)


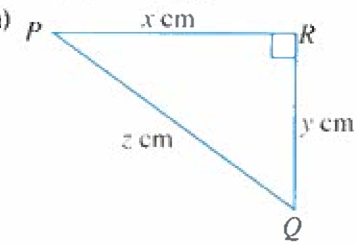
   (b)


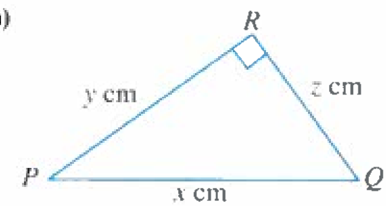
2. For each of the following right-angled triangles, state the value of
   (i) $\sin A$, (ii) $\cos A$,
   (iii) $\tan A$, (iv) $\sin B$,
   (v) $\cos B$, (vi) $\tan B$.

   (a)


   (b)


3. For each of the following right-angled triangles, write down an expression for
   (i) $\sin P$, (ii) $\cos P$,
   (iii) $\tan P$, (iv) $\sin Q$,
   (v) $\cos Q$, (vi) $\tan Q$,
   in terms of x, y and/or z.

   (a)


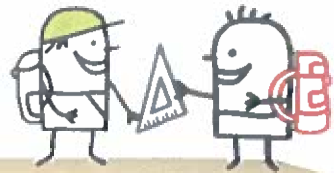
   (b)


4. Use a calculator to evaluate each of the following.
   (a) $\tan 47^\circ$ (b) $\sin 75.3^\circ$
   (c) $\cos 30.19^\circ$ (d) $\sin 35^\circ + \cos 49^\circ$
   (e) $2 \cos 42.3^\circ + 3 \sin 16.8^\circ$
   (f) $\sin 71.6^\circ \times \tan 16.7^\circ$

INTERMEDIATE LEVEL

5. Use a calculator to evaluate each of the following.
   (a) $\dfrac{5 \tan 61.4^\circ}{2 \cos 10.3^\circ}$ (b) $\dfrac{4(\sin 22.5^\circ)^2}{\cos 67.5^\circ}$
   (c) $\dfrac{\tan 15^\circ + \cos 33^\circ}{\sin 78.4^\circ}$ (d) $\dfrac{\tan 47.9^\circ}{\cos 84^\circ - \sin 63^\circ}$
   (e) $\dfrac{\cos 67^\circ + \sin 89^\circ}{\tan 63.4^\circ \times \cos 15.5^\circ}$ (f) $\dfrac{\sin 24.6^\circ + \cos 62.1^\circ}{\tan 21^\circ + \cos 14^\circ}$
   (g) $\dfrac{\sin 57^\circ - \cos 73^\circ}{\tan 15.3^\circ \times \sin 83.4^\circ}$ (h) $\dfrac{\cos 24.7^\circ \times \sin 35.1^\circ}{\tan 57^\circ - \cos 15^\circ}$

11.2 Applications of Trigonometric Ratios to Find Unknown Sides of Right-Angled Triangles



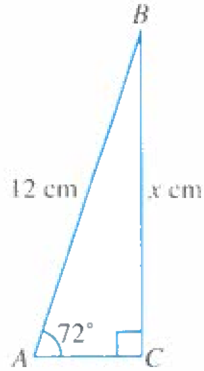
Worked Examples 2 to 5 illustrate how trigonometric ratios are used to find the lengths of the unknown sides of right-angled triangles.

Worked Example 2

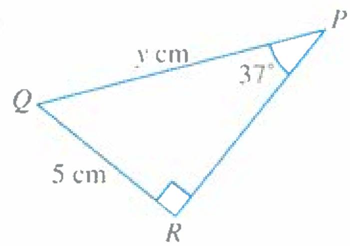
(Use of Sine Ratio)

Calculate the value of the unknown in each of the following right-angled triangles.

(a)



(b)



Solution:

(a) $\sin \angle BAC = \frac{\text{opp}}{\text{hyp}} = \frac{BC}{AB}$

$\sin 72° = \frac{x}{12}$

$\therefore x = 12 \sin 72°$

$= 11.4$ (to 3 s.f.)

(b) $\sin \angle QPR = \frac{\text{opp}}{\text{hyp}} = \frac{QR}{PQ}$

$\sin 37° = \frac{5}{y}$

$\therefore y = \frac{5}{\sin 37°}$

$= 8.31$ (to 3 s.f.)

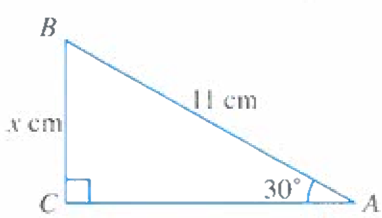
PRACTISE NOW 2

SIMILAR QUESTIONS

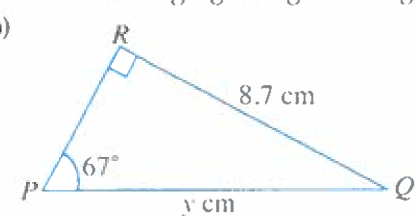
Exercise 11B Questions 1(a)–(h)

Find the value of the unknown in each of the following right-angled triangles.

(a)



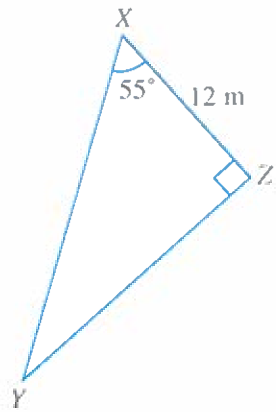
(b)



Worked Example 3

(Use of Cosine Ratio)

In ΔXYZ, $\angle Z = 90°$. Given that $\angle X = 55°$ and $XZ = 12$ m, calculate the length of XY.



Solution:

$$\cos \angle YXZ = \frac{\text{adj}}{\text{hyp}} = \frac{XZ}{XY}$$

$$\cos 55° = \frac{12}{XY}$$

$$\therefore XY = \frac{12}{\cos 55°}$$

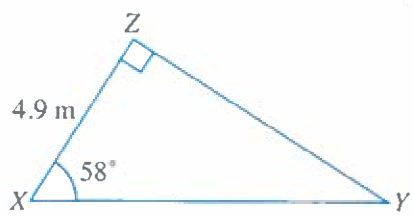
$$= 20.9 \text{ m (to 3 s.f.)}$$

PRACTISE NOW 3

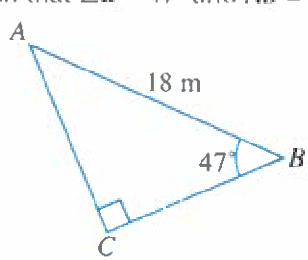
SIMILAR QUESTIONS

Exercise 11B Questions 2(a)–(b)

1. In ΔXYZ, $\angle Z = 90°$. Given that $\angle X = 58°$ and $XZ = 4.9$ m, find the length of XY.



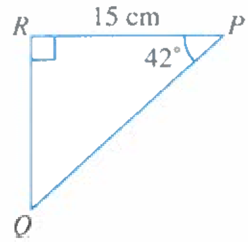
2. In ΔABC, $\angle C = 90°$. Given that $\angle B = 47°$ and $AB = 18$ m, find the length of BC.



Worked Example 4

(Use of Tangent Ratio)

In ΔPQR, $\angle R = 90°$. Given that $\angle P = 42°$ and $PR = 15$ cm, calculate the length of QR.



Solution:

$\tan \angle QPR = \frac{\text{opp}}{\text{adj}} = \frac{QR}{PR}$

$\tan 42° = \frac{QR}{15}$

$\therefore QR = 15 \tan 42°$

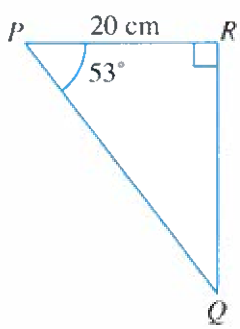
$= 13.5$ cm (to 3 s.f.)

PRACTISE NOW 4

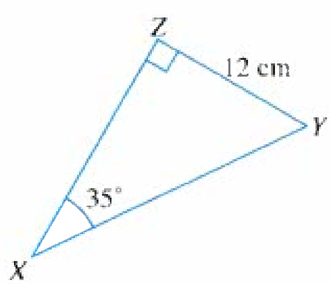
SIMILAR QUESTIONS

Exercise 11B Questions 3(a)–(b)

1. In ΔPQR, $\angle R = 90°$. Given that $\angle P = 53°$ and $PR = 20$ cm, find the length of QR.



2. In ΔXYZ, $\angle Z = 90°$. Given that $\angle X = 35°$ and $YZ = 12$ cm, find the length of XZ.

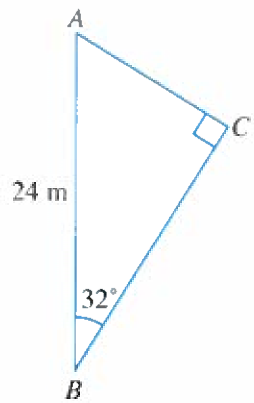


Worked Example 5

(Finding the Lengths of Unknown Sides of a Right-Angled Triangle)

In ΔABC, $\angle C = 90°$. Given that $\angle B = 32°$ and $AB = 24$ m, calculate the length of

(i) AC, (ii) BC.



Solution:

(i) $\sin \angle ABC = \frac{\text{opp}}{\text{hyp}} = \frac{AC}{AB}$

$\sin 32° = \frac{AC}{24}$

$\therefore AC = 24 \sin 32°$

$= 12.7$ m (to 3 s.f.)

(ii) **Method 1:**

$\cos \angle ABC = \frac{\text{adj}}{\text{hyp}} = \frac{BC}{AB}$

$\cos 32° = \frac{BC}{24}$

$\therefore BC = 24 \cos 32°$

$= 20.4$ m (to 3 s.f.)

Method 2:

In ΔABC, $\angle C = 90°$.

Using Pythagoras' Theorem,

$AB^2 = AC^2 + BC^2$

$24^2 = 12.72^2 + BC^2$ (Use $AC = 12.72$ (4 s.f.) instead of 12.7 or directly use $AC = 24 \sin 32°$.)

$BC^2 = 24^2 - 12.72^2$

$= 576 - 161.7984$

$= 414.2016$

$\therefore BC = \sqrt{414.2016}$ (since $BC > 0$)

$= 20.4$ m (to 3 s.f.)

Which method do you prefer and why?

RECALL

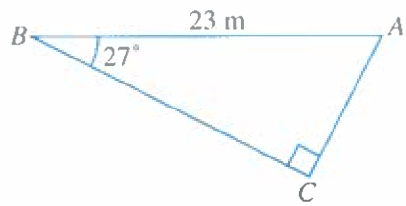
In order for the final answer to be accurate to three significant figures, any intermediate working must be correct to at least four significant figures.

PRACTISE NOW 5

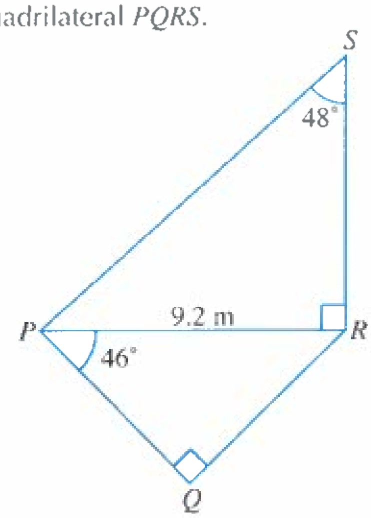
SIMILAR QUESTIONS

Exercise 11B Questions 4(a)–4(d), 5–8

1. In $\triangle ABC$, $\angle C = 90°$. Given that $\angle B = 27°$ and $AB = 23$ m, find the length of
   (i) BC,
   (ii) AC.



2. The figure shows a quadrilateral $PQRS$ where $\angle PQR = \angle PRS = 90°$. Given that $PR = 9.2$ m, $\angle QPR = 46°$ and $\angle PSR = 48°$, find
   (i) the length of PQ,
   (ii) the length of QR,
   (iii) the length of PS,
   (iv) the length of RS,
   (v) the perimeter of the quadrilateral $PQRS$,
   (vi) the area of the quadrilateral $PQRS$.

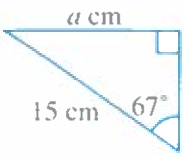


Exercise 11B

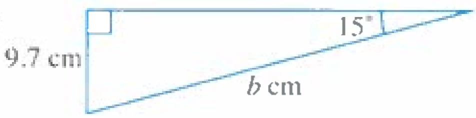
BASIC LEVEL

1. Find the value of the unknown in each of the following right-angled triangles.

(a)

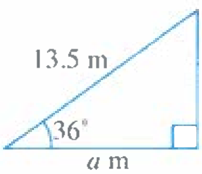


(b)

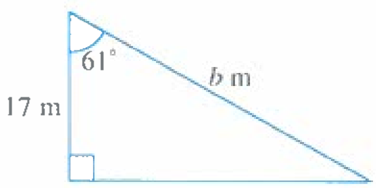


2. Find the value of the unknown in each of the following right-angled triangles.

(a)

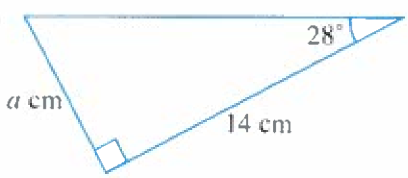


(b)

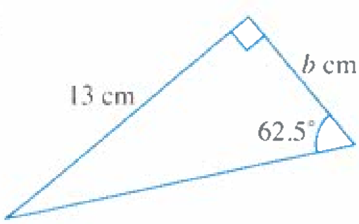


3. Find the value of the unknown in each of the following right-angled triangles.

(a)

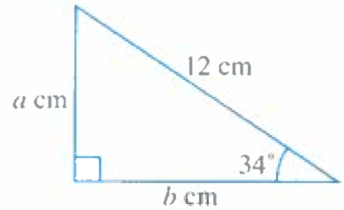


(b)

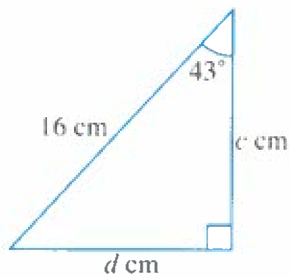


4. Find the values of the unknowns in each of the following right-angled triangles.

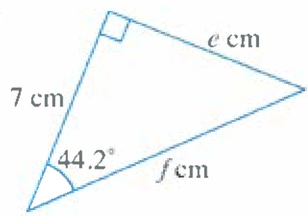
(a)



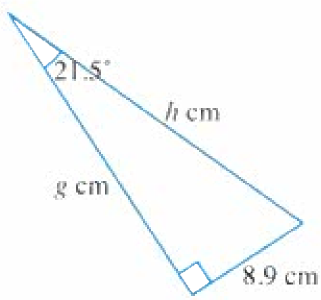
(b)



(c)

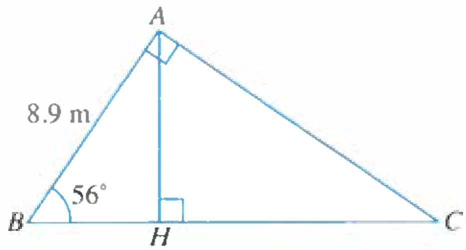


(d)

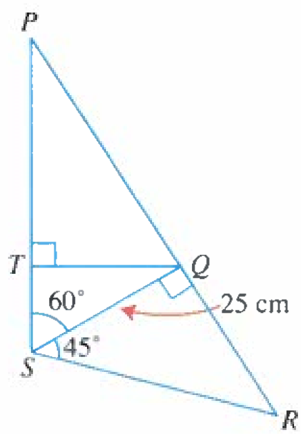


INTERMEDIATE LEVEL

5. In ΔABC, $AB = 8.9$ m, $\angle BAC = 90°$ and $\angle ABC = 56°$. H lies on BC such that AH is perpendicular to BC. Find the length of
   (i) AH,
   (ii) HC.

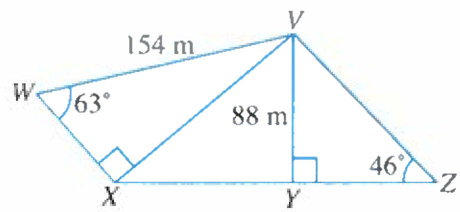


6. In the figure, $QS = 25$ cm, $\angle QSR = 45°$ and $\angle QST = 60°$. Find the length of
   (i) TQ,
   (ii) PT,
   (iii) PR.



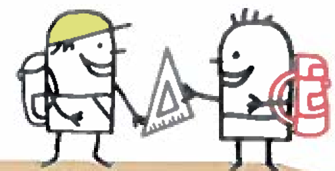
ADVANCED LEVEL

7. A figure $VWXYZ$ is made up of three right-angled triangles. Given that $VW = 154$ m, $VY = 88$ m, $\angle VXW = \angle VYZ = 90°$, $\angle VWX = 63°$, $\angle VZY = 46°$, find
   (i) the perimeter,
   (ii) the area,
   of the figure.



8. If y is inversely proportional to $(\tan x)^2$ and $y = 2$ when $x = 30°$, find the value of y when this value of x is doubled.

11.3 Applications of Trigonometric Ratios to Find Unknown Angles in Right-Angled Triangles



Finding Angles Given Trigonometric Ratios

In Section 11.1, we have learnt how to find trigonometric ratios when we are given an acute angle in a right-angled triangle. We will now learn how to find an acute angle in a right-angled triangle when we are given its trigonometric ratio.

If we are given that $\sin A = 0.45$, how do we find the acute angle A?

Since the value of the trigonometric ratio $\sin A$ is 0.45, we can write $\sin A = \frac{45}{100}, \frac{9}{20}$ or $\frac{4.5}{10}$, etc.

Consider the trigonometric ratio $\sin A = \frac{4.5}{10}$. Recall that since $\sin A = \frac{\text{opp}}{\text{hyp}}$, then in a right-angled triangle, the length of the side opposite $\angle A$ is 4.5 cm and the length of the hypotenuse is 10 cm. Fig. 11.5 shows a right-angled triangle ABC where $\angle C = 90°$, $BC = 4.5$ cm and $AB = 10$ cm.

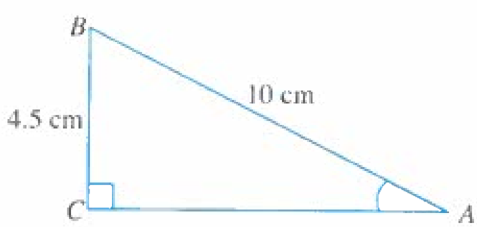


Fig. 11.5

Using a protractor, we measure $\angle A$ to be 27°.

Use of a Calculator to Find Angles Given Trigonometric Ratios

We can also use a calculator to find $\angle A$ where $\sin A = 0.45$.
Since we are computing the values of angles in degrees, remember to set the mode of your calculator to 'DEG'.

Sequence of calculator keys:

2nd F	sin	0	•	4	5	=

$\therefore \angle A = \sin^{-1}(0.45)$
$= 26.7°$ (to 1 d.p.)

ATTENTION

The sequence of calculator keys follows the 'Direct Algebraic Logic (DAL)' type of Scientific calculator. Refer to the manual of your calculator if you are using a different type of calculator.

Unless stated otherwise in a question, we leave angles in degrees correct to one decimal place.

PRACTISE NOW

Use a calculator to find each of the following angles, given its trigonometric ratio.
(a) $\sin A = 0.78$ **(b)** $\cos B = 0.35$ **(c)** $\tan C = 1.23$

SIMILAR QUESTIONS

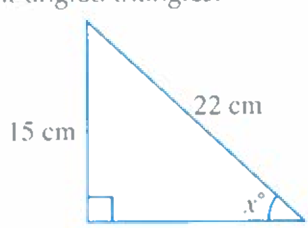
Exercise 11C Questions 1(a)–(c)

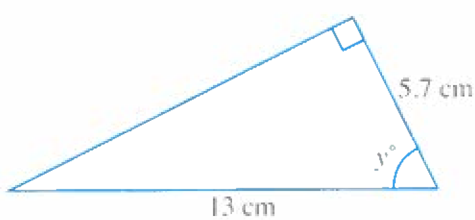
Applications of Trigonometric Ratios to Find Unknown Angles in Right-Angled Triangles

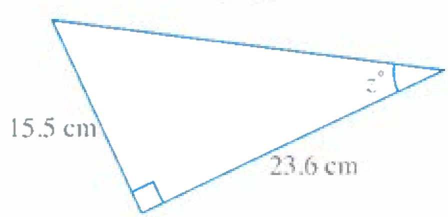
In Section 11.2, we have learnt how to use trigonometric ratios to find the lengths of the unknown sides of right-angled triangles. Now, we will learn how to use trigonometric ratios to find the unknown angles in right-angled triangles.

Worked Example 6

(Finding Unknown Angles in Right-Angled Triangles)
Calculate the value of the unknown in each of the following right-angled triangles.

(a)


(b)


(c)


Solution:

(a) $\sin x^\circ = \frac{15}{22}$

$\therefore x^\circ = \sin^{-1}\left(\frac{15}{22}\right)$

$= 43.0^\circ$ (to 1 d.p.)

$\therefore x = 43.0$

(b) $\cos y^\circ = \frac{5.7}{13}$

$\therefore y^\circ = \cos^{-1}\left(\frac{5.7}{13}\right)$

$= 64.0^\circ$ (to 1 d.p.)

$\therefore y = 64.0$

(c) $\tan z^\circ = \frac{15.5}{23.6}$

$\therefore z^\circ = \tan^{-1}\left(\frac{15.5}{23.6}\right)$

$= 33.3^\circ$ (to 1 d.p.)

$\therefore z = 33.3$

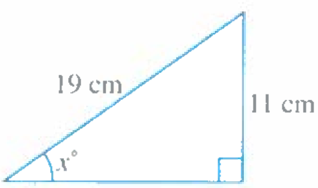
PRACTISE NOW 6

SIMILAR QUESTIONS

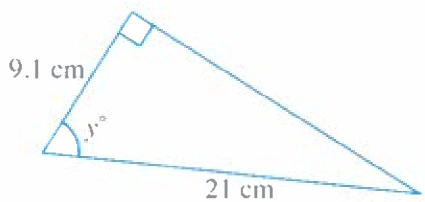
Exercise 11C Questions 2(a)–(i)

Find the value of the unknown in each of the following right-angled triangles.

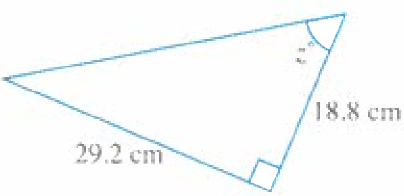
(a)



(b)



(c)

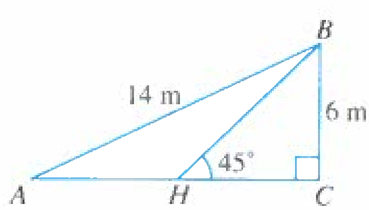


Worked Example 7

(Using Trigonometric Ratios to Find an Unknown Angle and the Length of an Unknown Side)

In ΔABC, $AB = 14$ m, $BC = 6$ m and $\angle ACB = 90^\circ$. H lies on AC such that $\angle BHC = 45^\circ$. Calculate

(i) $\angle BAC$, **(ii)** the length of AH.



Solution:

(i) In ΔABC,

$\sin \angle BAC = \frac{6}{14}$

$\therefore \angle BAC = \sin^{-1}\left(\frac{6}{14}\right)$

$= 25.4°$ (to 1 d.p.)

(ii) In ΔBHC,

$\tan 45° = \frac{6}{HC}$

$\therefore HC = \frac{6}{\tan 45°}$

$= 6$ m

In ΔABC, $\angle ACB = 90°$.

Using Pythagoras' Theorem,

$AB^2 = AC^2 + BC^2$

$14^2 = AC^2 + 6^2$

$AC^2 = 14^2 - 6^2$

$= 196 - 36$

$= 160$

$\therefore AC = \sqrt{160}$ (since $AC > 0$)

$= 12.65$ m (to 4 s.f.)

$\therefore AH = AC - HC$

$= 12.65 - 6$

$= 6.65$ (to 3 s.f.)

PRACTISE NOW 7

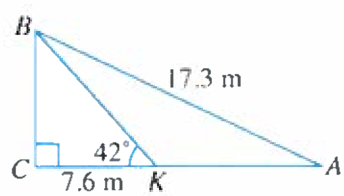
SIMILAR QUESTIONS

Exercise 11C Questions 3–8

1. In ΔABC, $AB = 17.3$ m and $\angle ACB = 90°$. K lies on CA such that $CK = 7.6$ m and $\angle BKC = 42°$. Find

(i) $\angle BAC$,

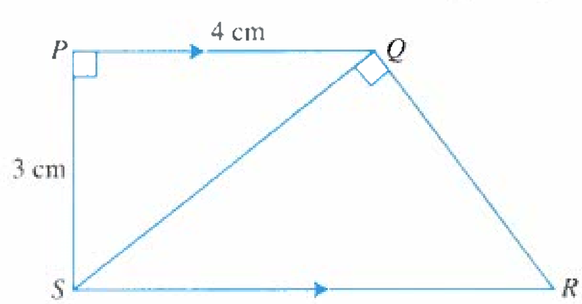
(ii) the length of KA.

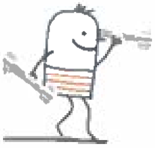


2. The figure shows a trapezium $PQRS$ in which PQ is parallel to SR. Given that $PQ = 4$ cm and $PS = 3$ cm, find

(i) $\angle PQS$,

(ii) the length of QR.





Exercise 11C

BASIC LEVEL

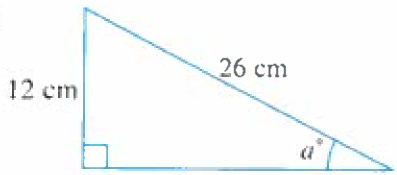
1. Use a calculator to find each of the following angles, given its trigonometric ratio.

(a) $\sin A = 0.527$ **(b)** $\cos B = 0.725$

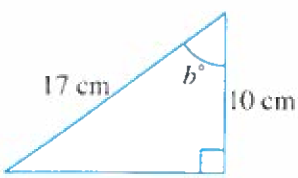
(c) $\tan C = 2.56$

2. Find the value of the unknown in each of the following right-angled triangles.

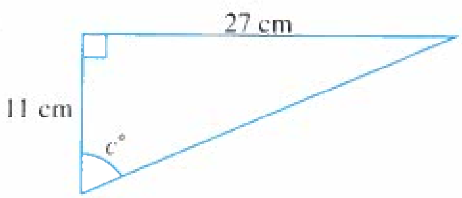
(a)



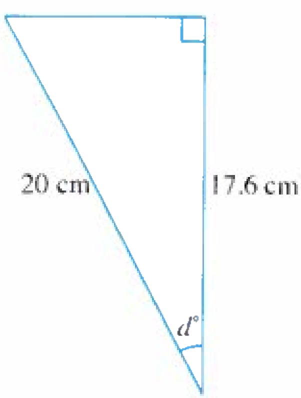
(b)



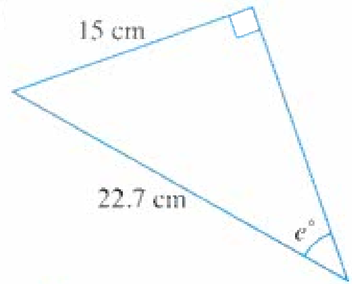
(c)



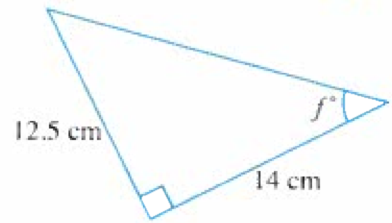
(d)



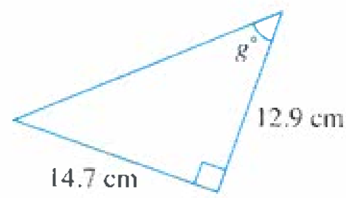
(e)



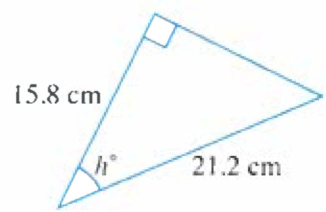
(f)



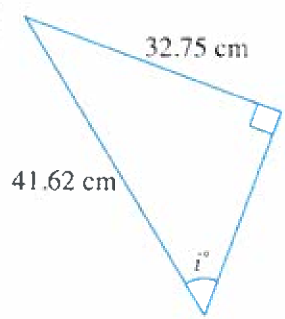
(g)



(h)

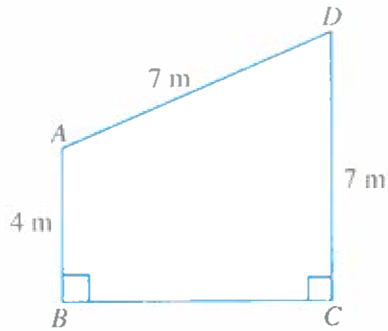


(i)

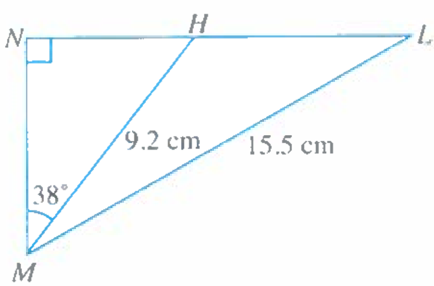


INTERMEDIATE LEVEL

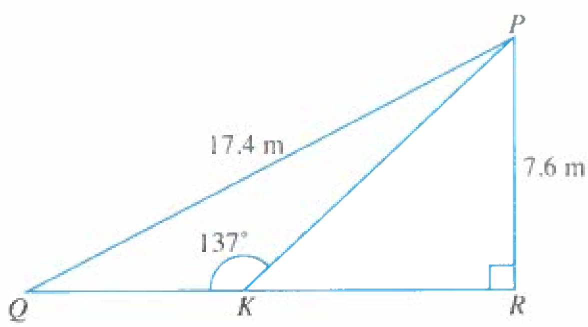
3. The figure shows a quadrilateral $ABCD$ where $\angle ABC = \angle BCD = 90°$. Given that $AB = 4$ m and $DA = DC = 7$ m, find
   (i) $\angle ADC$, (ii) the length of BC.



4. In ΔLMN, $LM = 15.5$ cm and $\angle LNM = 90°$. H lies on NL such that $HM = 9.2$ cm and $\angle HMN = 38°$. Find
   (i) $\angle MLN$, (ii) the length of HL.

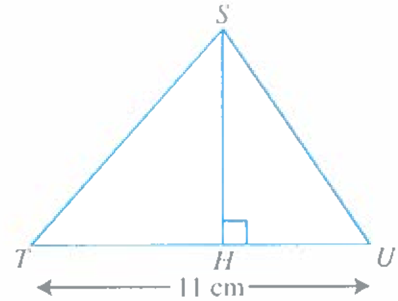


5. In ΔPQR, $PQ = 17.4$ m and $PR = 7.6$ m. K lies on QR such that $\angle PKQ = 137°$. Find
   (i) $\angle QPK$, (ii) the length of QK.

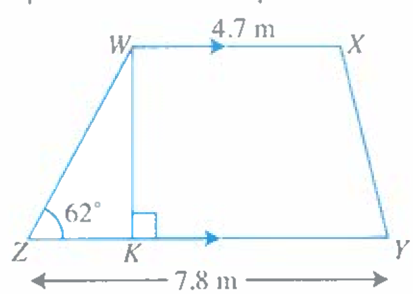


ADVANCED LEVEL

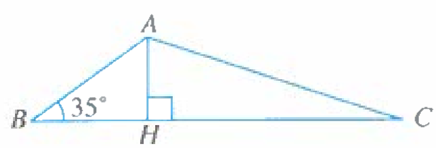
6. The figure shows a triangle STU where $TU = 11$ cm. H lies on TU such that the length of TH is 120% of the length of HU and $\angle SHU = 90°$. Given that the area of ΔSTH is 21 cm², find $\angle TSU$.



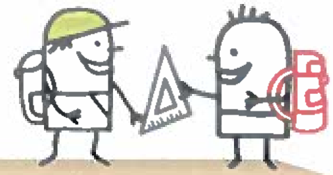
7. The figure shows a trapezium $WXYZ$ in which WX is parallel to ZY. It is given that $WX = 4.7$ m, $ZY = 7.8$ m and $\angle WZY = 62°$. K lies on ZY such that the ratio of the length of WK to the length of ZY is 6 : 13. Find
   (i) $\angle XYZ$,
   (ii) the perimeter of the trapezium $WXYZ$.



8. The figure shows a triangle ABC where $\angle ABC = 35°$. H lies on BC such that the length of HC is twice that of BH. Find $\angle ACB$.



11.4 Applications of Trigonometric Ratios in Real-World Contexts



Trigonometry is commonly used to find the heights of buildings and mountains, the distance of the shore from a point in the sea and the distance between celestial bodies, etc. In this section, we will learn how to apply trigonometric ratios to solve problems in real-world contexts.

Investigation

Using a Clinometer to Find the Height of an Object

In this investigation, we will learn how to use trigonometry to find the height of a tree or a building.

1. Fig. 11.6 shows an instrument known as a **clinometer**.

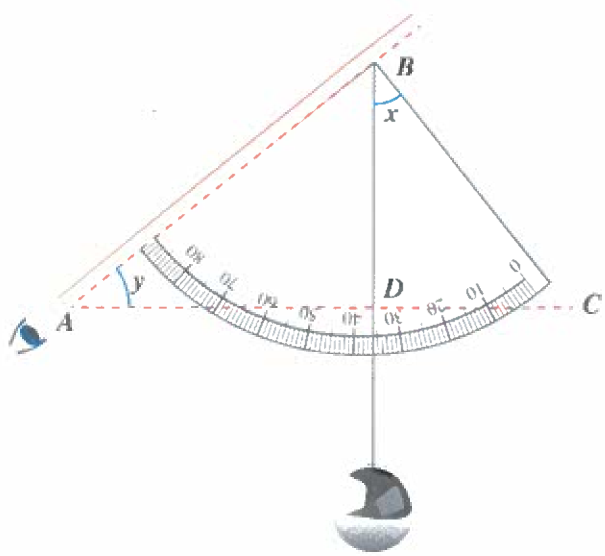


Fig. 11.6

If we look at the top of a tree or a building through the straw, the instrument will give us $\angle y$, which is also equal to $\angle x$. Explain clearly why $\angle y = \angle x$. $\angle y$ is called the angle of elevation.

2. Follow Steps (i) – (iii) to make a clinometer.

(i) Photocopy the protractor in Fig 11.7 and paste it on a piece of cardboard.

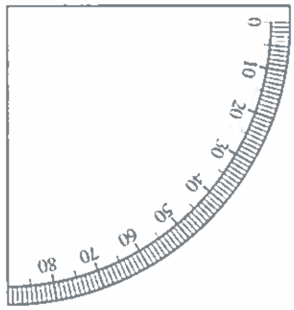


Fig. 11.7

(ii) Make a hole in the cardboard, close to the corner of the protractor.

(iii) Using a straw, a piece of thread, a ball bearing (or any suitable weight) and some sticky tape, assemble the clinometer as shown in Fig. 11.6. Ensure that the straw lies along the 90° mark along the line AB, and that the thread touches the straw at the corner B of the protractor as shown in Fig. 11.6.

3. Follow Steps (i) – (iii) to use the clinometer which you have made in Step 2 to find the height of an object.

(i) Stand at a spot that is about 10 m to 20 m from an object such as a tree as shown in Fig. 11.8. Measure this distance using a measuring tape.

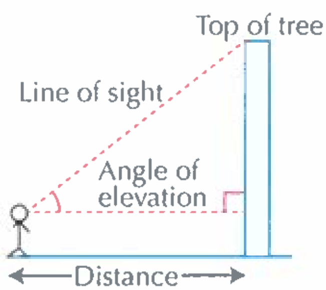


Fig. 11.8

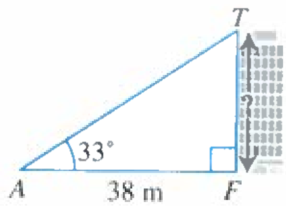
(ii) Look at the top of the object through the straw of the clinometer. Ask your classmate to read the angle of elevation from your clinometer.

(iii) Use a trigonometric ratio to find the height of the object, showing your working clearly. Note that the clinometer is at a certain height above the ground.

Worked Example 8

(Finding the Height of a Building)

A point A on level ground is 38 m away from the foot F of a building TF. Given that AT makes an angle of 33° with the horizontal, calculate the height of the building.



Solution:

$\tan 33° = \frac{TF}{38}$

$\therefore TF = 38 \tan 33°$

$= 24.7$ m (to 3 s.f.)

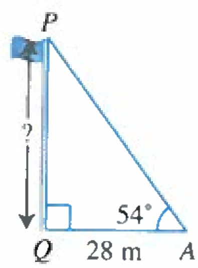
The height of the building is 24.7 m.

PRACTISE NOW 8

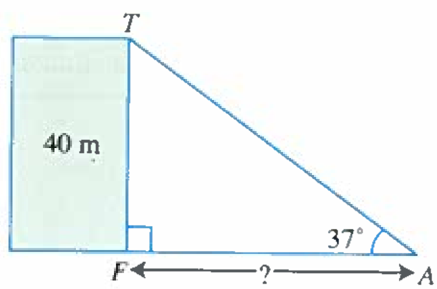
SIMILAR QUESTIONS

Exercise 11D Questions 1–3, 7, 13

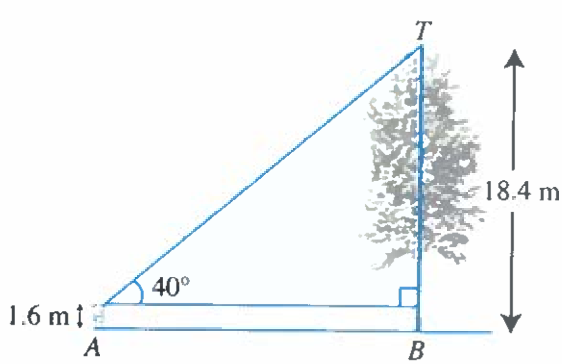
1. A point A on level ground is 28 m away from the foot Q of a flagpole PQ. Given that AP makes an angle of 54° with the horizontal, find the height of the flagpole.



2. The height of a tower TF is 40 m. Given that A is a point on level ground such that AT makes an angle of 37° with the horizontal, find the distance FA.



3. Devi uses a clinometer to measure the height of a tree TB. From where she stands, the angle of elevation of the tree is 40° and the height of the tree is calculated to be 18.4 m. Given that the clinometer is 1.6 m above the ground, find the distance AB between Devi and the foot of the tree.



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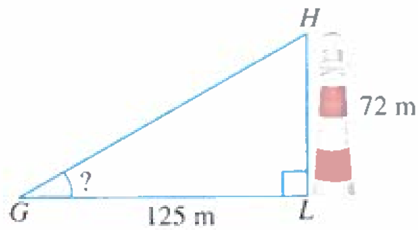
Worked Example 9

(Real-life Application of Trigonometric Ratios)

A lighthouse HL is 72 m tall. Given that a point G on level ground is 125 m away from the foot L of the lighthouse, calculate $\angle HGL$.



Solution:

$\tan \angle HGL = \frac{72}{125}$

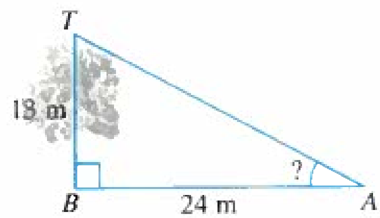
$\therefore \angle HGL = 29.9°$ (to 1 d.p.)

PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 11D Question 4

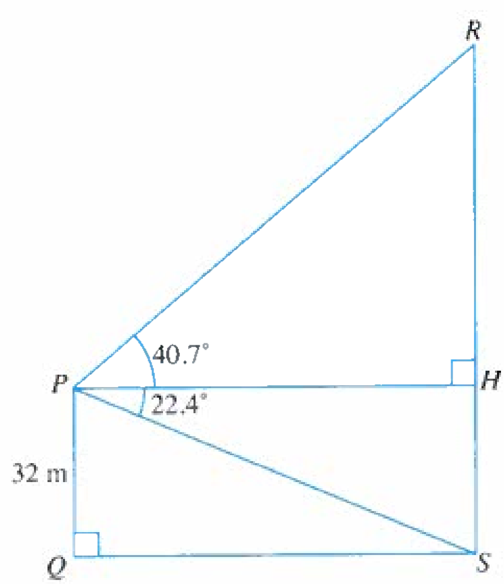
A tree TB is 13 m tall. Given that a point A on level ground is 24 m away from the foot B of the tree, find $\angle TAB$.



Worked Example 10

(Real-Life Application of Trigonometric Ratios)

The height of a building PQ is 32 m. RHS is another building and PH is a horizontal sky bridge linking the buildings. Given that $\angle RPH = 40.7°$ and $\angle HPS = 22.4°$, calculate the height of the building RHS.



Solution:

$HS = PQ = 32$ m

In ΔPSH,

$$\tan 22.4^\circ = \frac{32}{PH}$$

$$\therefore PH = \frac{32}{\tan 22.4^\circ}$$

$$= 77.64 \text{ m (to 4 s.f.)}$$

In ΔPRH,

$$\tan 40.7^\circ = \frac{RH}{PH}$$

$$\therefore RH = PH \tan 40.7^\circ$$

$$= 77.64 \tan 40.7^\circ$$

$$= 66.78 \text{ m (to 4 s.f.)}$$

$$\therefore RS = RH + HS$$

$$= 66.78 + 32$$

$$= 98.8 \text{ m (to 3 s.f.)}$$

The height of the mast is 98.8 m.

RECALL

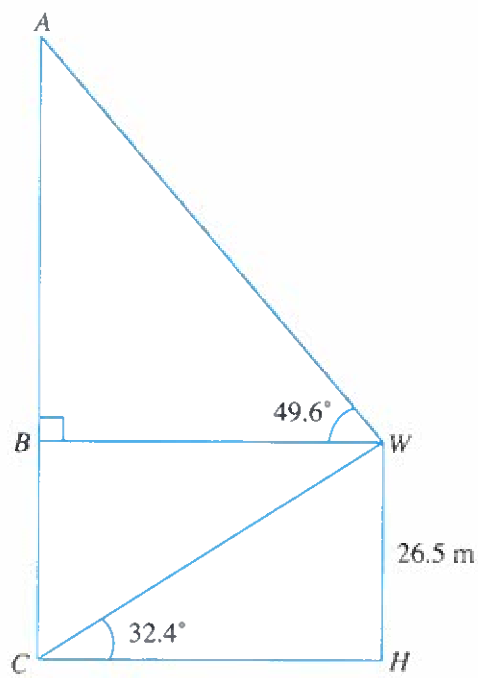
In order for the final answer to be accurate to three significant figures, any intermediate working must be correct to at least four significant figures.

PRACTISE NOW 10

SIMILAR QUESTIONS

Exercise 11D Question 8

The height of a warehouse WH is 26.5 m. ABC is a vertical mast in front of the warehouse. BW is a horizontal cable attached to the mast from the top of the warehouse. Given that $\angle AWB = 49.6^\circ$ and $\angle WCH = 32.4^\circ$, find the height of the mast.



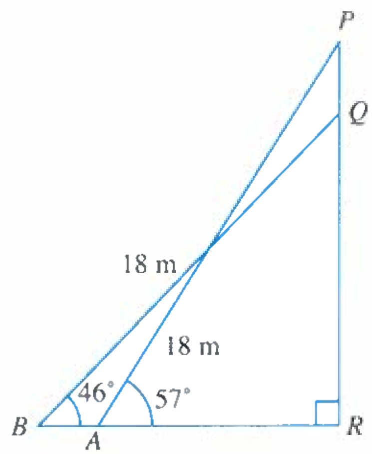
Worked Example 11

(Real-Life Application of Trigonometric Ratios)

When a ladder of length 18 m leans against the top edge of a window of a building, it forms an angle of 57° with the ground. When the ladder leans against the lower edge of the same window, it forms an angle of 46° with the ground. Calculate the height of the window.

Solution:

In the figure, AP and BQ represent the two positions of the ladder and PQ represents the window.



In ΔAPR,

$\sin 57° = \frac{PR}{18}$

$\therefore PR = 18 \sin 57°$

$= 15.10$ m (to 4 s.f.)

In ΔBQR,

$\sin 46° = \frac{QR}{18}$

$\therefore QR = 18 \sin 46°$

$= 12.95$ m (to 4 s.f.)

$\therefore PQ = PR - QR$

$= 15.10 - 12.95$

$= 2.15$ m (to 3 s.f.)

The height of the window is 2.15 m.

PRACTISE NOW 11

When a straight pole of length 14.5 m leans against the top edge of a signboard on a building, it forms an angle of 53° with the ground. When the pole leans against the lower edge of the same signboard, it forms an angle of 42° with the ground. Find the height of the signboard.

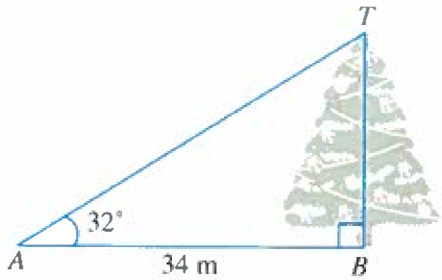
SIMILAR QUESTIONS

Exercise 11D Questions 5–6, 9–12

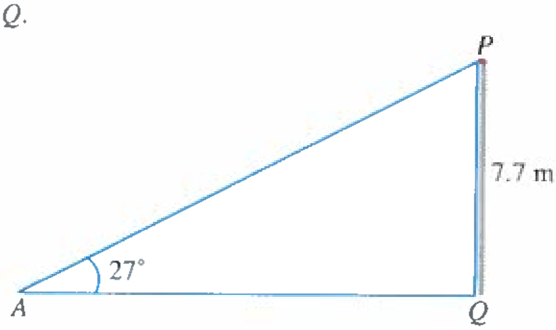
Exercise 11D

BASIC LEVEL

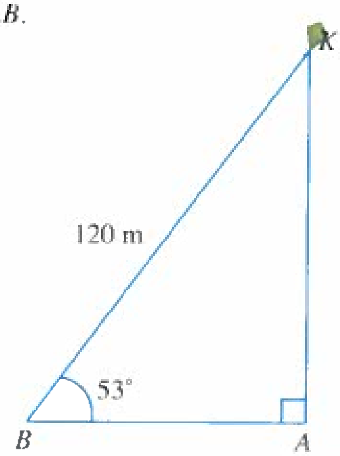
1. A point A on level ground is 34 m away from the foot B of a Christmas tree TB. Given that AT makes an angle of 32° with the horizontal, find the height of the Christmas tree.



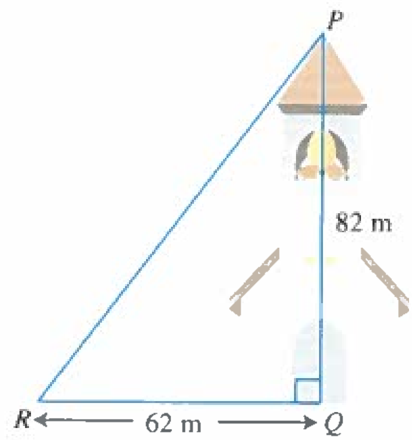
2. The height of a vertical post PQ is 7.7 m. Given that A is a point on level ground such that AP makes an angle of 27° with the horizontal, find the distance AQ.



3. Huixin standing at B is flying a kite. The kite is vertically above A. The string BK of length 120 m, attached to the kite, makes an angle of 53° to the horizontal. Assuming the string is taut, find the distance AB.

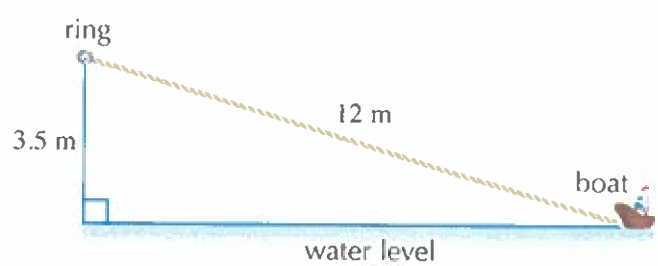


4. At a certain time of a day, a church spire PQ, 82 m high, casts a shadow RQ, 62 m long. Find $\angle PRQ$.



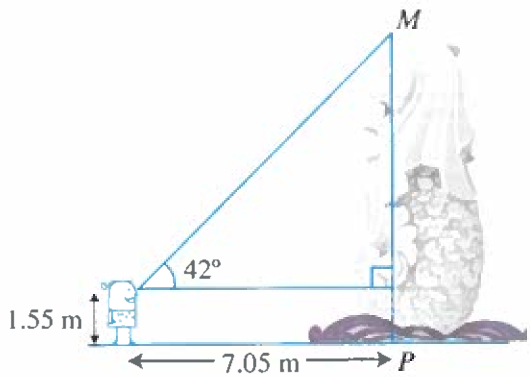
5. A ladder of length 5 m leans against a nail on a vertical wall. It forms an angle of 60° with the ground. Find
   (i) the height of the nail above the ground,
   (ii) the distance of the foot of the ladder from the base of the wall.

6. A boat is tied to a rope of length 12 m which is attached to a ring that is 3.5 m above the water. Assuming that the rope is taut, find the angle it makes with the water.

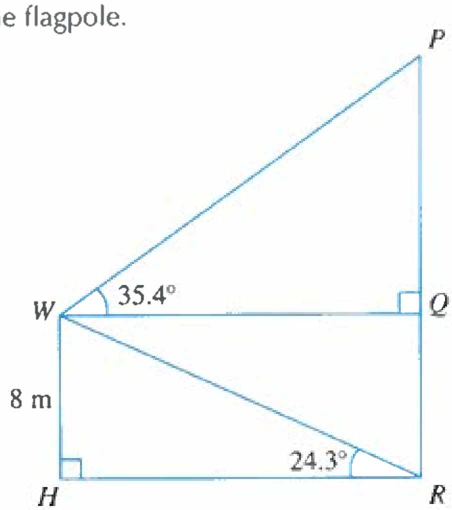


INTERMEDIATE LEVEL

7. Lixin is standing 7.05 m away from the statue of the Merlion MP at Merlion Park, Singapore. The height of her eyes from her feet is 1.55 m. Given that the angle of elevation of the top of the statue from her eyes is 42°, find the height of the statue.

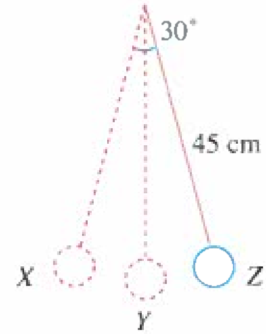


8. The height of a building WH is 8 m. PQR is a flagpole and WQ is a horizontal cable connected to the top of the building and to the flagpole. Given that $\angle PWQ = 35.4°$ and $\angle WRH = 24.3°$, find the height of the flagpole.



9. A plank of length 4 m rests against a wall 1.8 m high such that 1.2 m of the plank lies beyond the wall. Find the angle the plank makes with the wall.

10. A pendulum of length 45 cm swings backwards and forwards from X to Z passing through Y, the middle point of oscillation. The angle between the pendulum at X and at Z is 30°. Find the height in which the pendulum bob rises above Y.



11. When a ladder of length 2.5 m leans against the top edge of a window of a building, it forms an angle of 55° with the ground. When the ladder leans against the lower edge of the same window, it forms an angle of 38° with the ground. Find the height of the window, giving your answer in centimetres.

12. A crane stands on level ground. It is represented by a vertical tower AB of height 18 m and a jib AC of length 36 m. A vertical cable hangs from C and is attached to a load at D. The jib is inclined at an angle of 35° to the horizontal line AH.

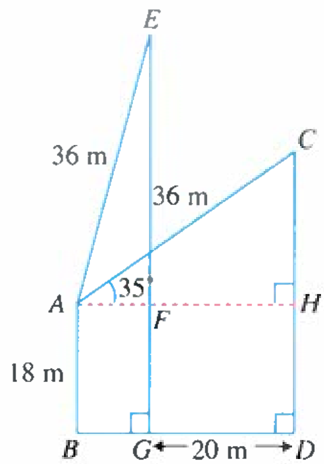
(i) Find CD.

The load is lifted from D and the jib is rotated in a vertical plane about A. When the jib is at the position AE, the load is lowered and placed on the level ground at the point G, which is vertically below E. The line EG cuts the line AH at F and GD = 20 m.

Find

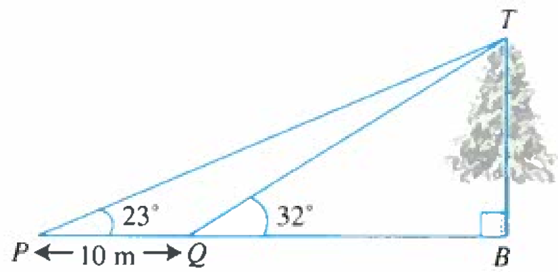
(ii) EF,

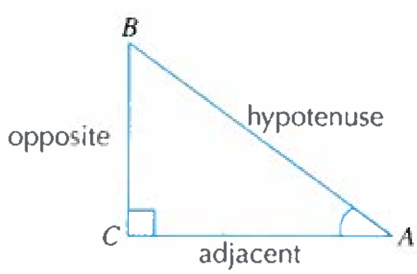
(iii) the angle in which the jib has rotated in the vertical plane about A.



ADVANCED LEVEL

13. Two points P and Q, 10 m apart on level ground, are due West of the foot B of a tree TB. Given that $\angle TPB = 23°$ and $\angle TQB = 32°$, find the height of the tree.





In a triangle ABC, if $\angle C = 90°$,

then $\frac{BC}{AB} = \frac{\text{opp}}{\text{hyp}}$ is called the **sine** of $\angle A$, or $\sin A = \frac{\text{opp}}{\text{hyp}}$,

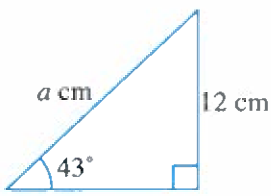
$\frac{AC}{AB} = \frac{\text{adj}}{\text{hyp}}$ is called the **cosine** of $\angle A$, or $\cos A = \frac{\text{adj}}{\text{hyp}}$,

$\frac{BC}{AC} = \frac{\text{opp}}{\text{adj}}$ is called the **tangent** of $\angle A$, or $\tan A = \frac{\text{opp}}{\text{adj}}$.

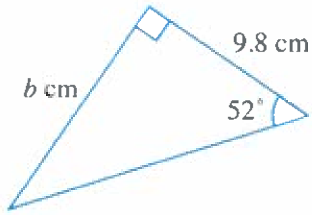
Review Exercise 11

1. Find the value of the unknown in each of the following right-angled triangles.

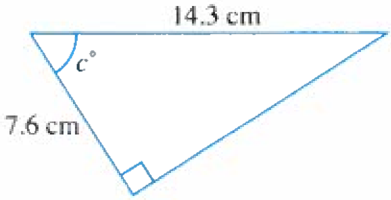
(a)



(b)



(c)

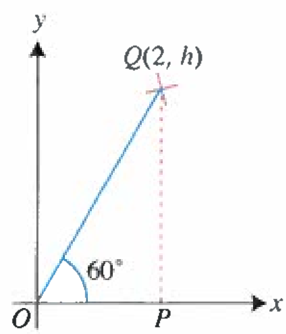


2. In ΔABC, $AB = 24$ m and $\angle ABC = 90°$. Given that $\sin \angle ACB = \frac{3}{5}$, find

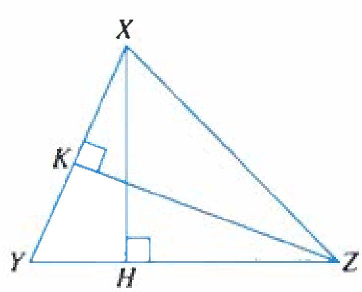
(i) the length of AC, **(ii)** the length of BC,
(iii) the value of $\cos \angle ACB + \tan \angle BAC$.

3. The coordinates of the point Q are $(2, h)$ and $\angle POQ = 60°$. Find

(i) the length of OQ, **(ii)** the value of h.



4. In the figure, XH and ZK are perpendicular to YZ and XY respectively. Given that the ratio of the length of XH to the length of ZK is 3 : 4, find the value of $\frac{\sin X}{\sin Z}$.



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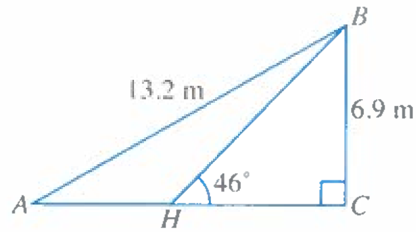
5. The length and the breadth of a rectangle are 19.2 cm and 12.4 cm respectively. Find the angle between each diagonal and the breadth of the rectangle.

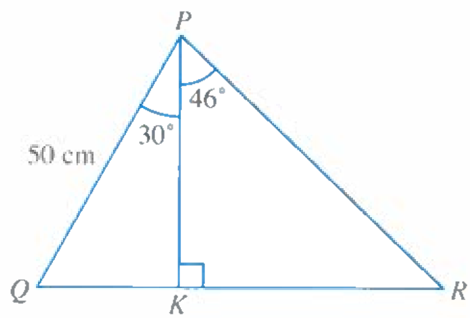
6. In ΔABC, $AB = 13.2$ m, $BC = 6.9$ m and $\angle ACB = 90^\circ$. H lies on AC such that $\angle BHC = 46^\circ$. Find

(i) $\angle ABH$, (ii) the length of AH.



7. The figure shows a triangle PQR where $PQ = 50$ cm. K lies on QR such that $\angle QPK = 30^\circ$ and $\angle RPK = 46^\circ$. Find the length of

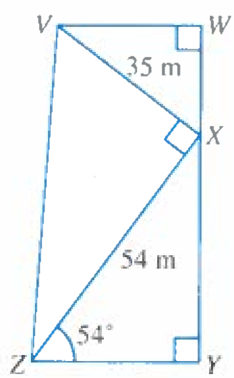
(i) QK, (ii) QR.



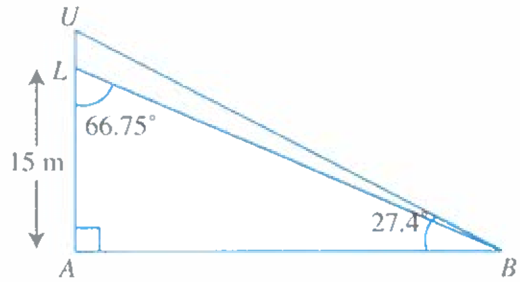
8. A figure $VWXYZ$ is made up of three right-angled triangles and WXY is a straight line. Given that $VX = 35$ m, $XZ = 54$ m, $\angle VWX = \angle VXZ = \angle XYZ = 90^\circ$ and $\angle XZY = 54^\circ$, find

(i) $\angle VZX$, (ii) the length of XY,

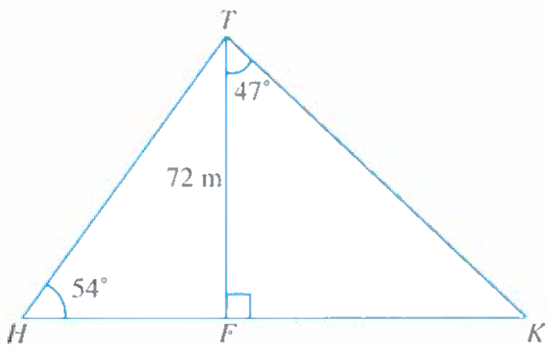
(iii) the length of VW.



9. The lower edge L of a window UL in a house is 15 m vertically above a point A on level ground. B is another point on level ground such that BU makes an angle of 27.4° with the horizontal and $\angle ALB = 66.75°$. Find the height of the window.



10. H is a point on level ground due West of a building TF of height 72 m while K is a point which is in line with H and due East of the building. Given that HT makes an angle of 54° with the horizontal and $\angle FTK = 47°$, find the distance HK.

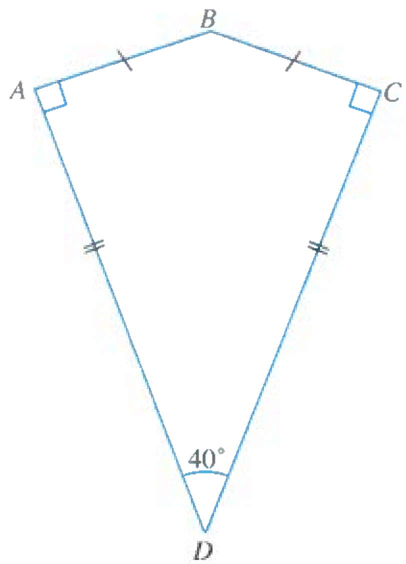


11. An aerial mast is supported by four cables. The cables are attached from the top of the mast to points on level ground, 57 m away from the foot of the mast. Given that each cable forms an angle of 32° with the ground, find the height of the mast.

12. A ladder of length 6.5 m leans against a vertical wall and touches a window sill. It forms an angle of 61° with the ground. Find
   (i) the height of the window sill above the ground,
   (ii) the distance of the foot of the ladder from the base of the wall.

Challenge Yourself

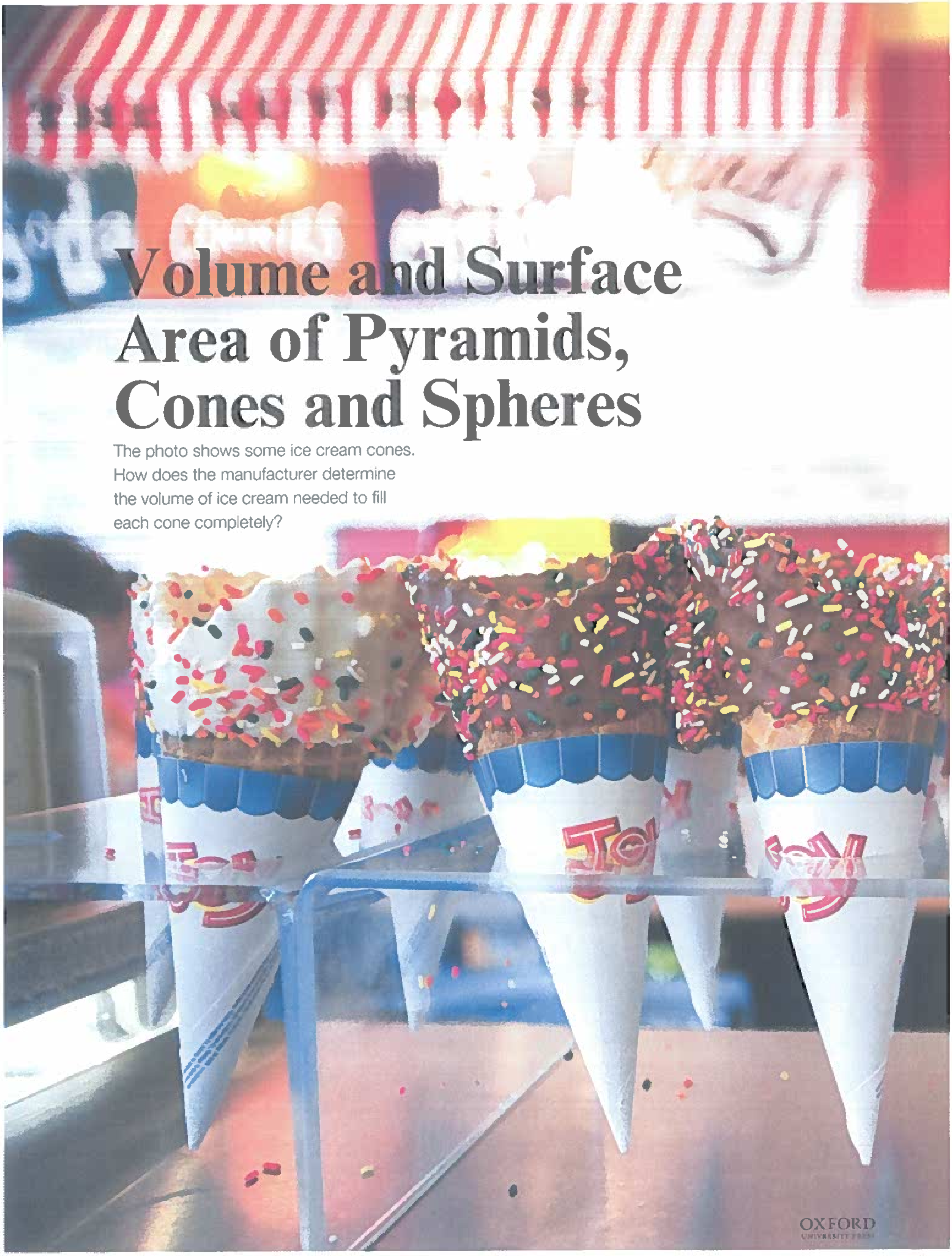
1. The figure shows a kite $ABCD$ where $\angle ADC = 40°$. Given that the area of the kite is 900 units^2, find its perimeter.



2. If $0° < x < 90°$, state the range of values of x for which $\sin x < \cos x$. Explain your answer clearly.

Volume and Surface Area of Pyramids, Cones and Spheres

The photo shows some ice cream cones. How does the manufacturer determine the volume of ice cream needed to fill each cone completely?



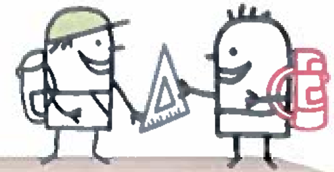
Chapter Twelve

LEARNING OBJECTIVES

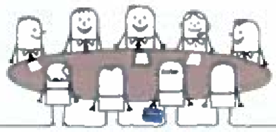
At the end of this chapter, you should be able to:

- identify and sketch pyramids, cones and spheres,
- draw and use nets of pyramids and cones to visualise their surface area,
- use formulae to calculate the volume and the surface area of pyramids, cones and spheres,
- solve problems involving the volume and the surface area of composite solids made up of pyramids, cones, spheres, prisms and cylinders.

12.1 Volume and Surface Area of Pyramids



In Book 1, we have learnt about prisms. In this section, we will learn about pyramids.



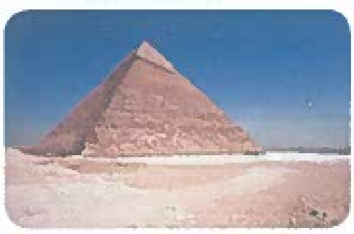
Class Discussion

What are Pyramids?

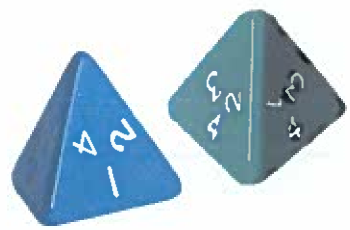
The photo in Fig. 12.1**(a)** shows the Great Pyramid built by the Egyptians in Giza, Egypt around the year 2560 BC. The photo in Fig. 12.1**(b)** shows two tetrahedral dice, each with 4 triangular faces. These are real-life examples of pyramids.

INFORMATION

When a tetrahedral die with four sides numbered 1, 2, 3 and 4 is tossed, the score is the number on the side facing downwards. However, since it is difficult to read this number, the dice are numbered as shown in Fig. 12.1**(b)**. The score is the number closest to the base. What is the score on each of the dice in Fig. 12.1**(b)**?



(a) The Great Pyramid



(b) Tetrahedral Dice

Fig. 12.1

Discuss each of the following questions with your classmates.

1. What are some common features of these pyramids?
2. What do you notice about the *slanted faces* of these pyramids?
3. What do you notice about the *bases* of these pyramids?
4. What is the difference between a vertex and the apex of a pyramid?
5. What do you notice about the cross sections of a pyramid? Are they uniform?
6. Now, look at the photos in Fig. 12.2. Are they pyramids? Do they have the same features as the pyramids in Fig. 12.1?

ATTENTION

The *base* of a pyramid is also considered a face of the pyramid, but it is different from the *slanted faces* of the pyramid.

(a) Food Pyramid

(b) Human Pyramid

(c) Rice Dumpling

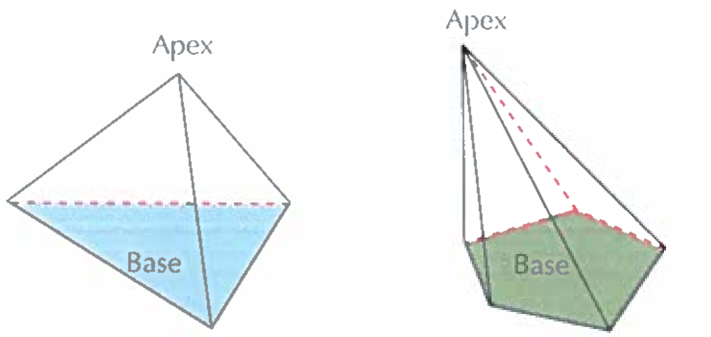
Fig. 12.2

7. Give three more real-life examples of pyramids.

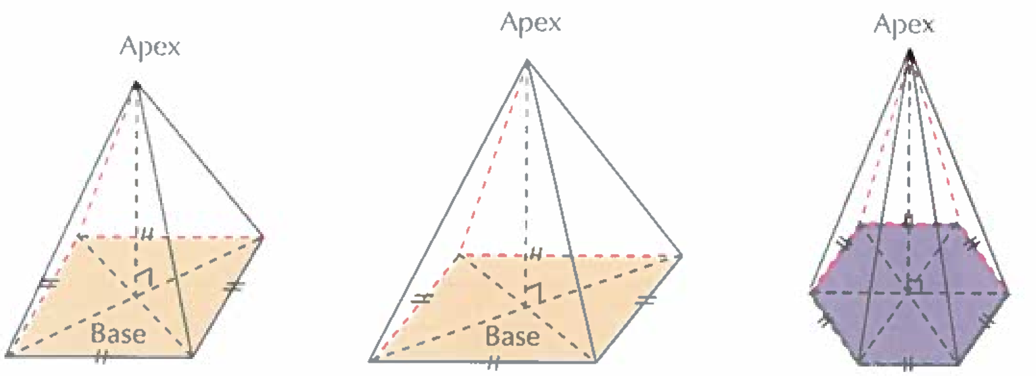
Types of Pyramids

A pyramid is a solid in which one of the faces is a **polygonal base** and the other **slanted faces** are triangles joined to the edges (or sides) of the base. The corner points of a pyramid are known as **vertices** (singular: vertex) and the vertex where all the slanted faces meet is called the **apex,** which is opposite the base.

Fig. 12.3 shows some examples of pyramids with different bases. The base of each pyramid is shaded. A pyramid is named after its polygonal base. Can you name the last two pyramids? Write your answers in the spaces provided in Fig. 12.3.



(a) Triangular Pyramid **(b)** Pentagonal Pyramid



(c) Square Pyramid **(d)** ________ Pyramid **(e)** ________ Pyramid

Fig. 12.3

For the square pyramid in Fig. 12.3**(c)**, do you notice that the apex is *vertically above the centre of the base?* Such a pyramid is called a **right pyramid**. Can you identify other right pyramids in Fig. 12.3?

INFORMATION

For the pyramid in Fig. 12.3**(e)**, do you notice that it is a right pyramid and the base is a regular hexagon? Such a right pyramid with a regular polygonal base is known as a **regular pyramid**.

Recall that a *regular polygon* is a polygon with *all sides equal* and *all angles equal*. Since a *regular quadrilateral* has to be a square and not a rectangle, then a rectangular pyramid can never be a regular pyramid. However, are all square pyramids regular pyramids?

The **perpendicular height** (or simply the **height**) of a pyramid is the perpendicular distance from the apex to the base of the pyramid (see Fig. 12.4). A **slant height** of a pyramid is the distance from the apex to the midpoint of an edge (or side) of the base. The edges that join the apex to the vertices (or corner points) of the base are called the **slant edges**.

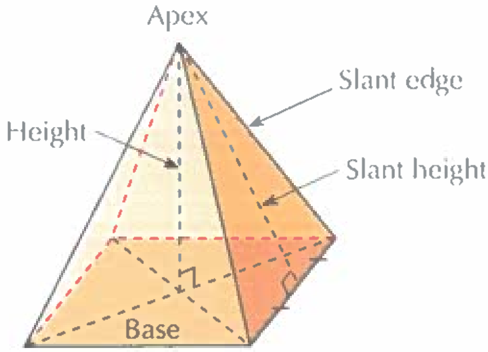
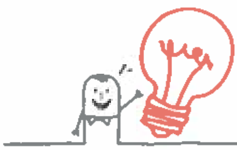


Fig. 12.4

Use 6 identical matchsticks to form 4 equilateral triangles without breaking the matchsticks.



Thinking Time

What can we say about the slant edges, slant heights and slant faces of regular pyramids?

In this section, we will study only right pyramids. Therefore, the term 'pyramid' is used to refer to a right pyramid. Also, the term 'height' is used to refer to the perpendicular height unless otherwise stated.

Journal Writing

In Book 1, we have learnt about prisms. Write down three differences between a prism and a pyramid.

Volume of Pyramids

In Book 1, we have learnt:

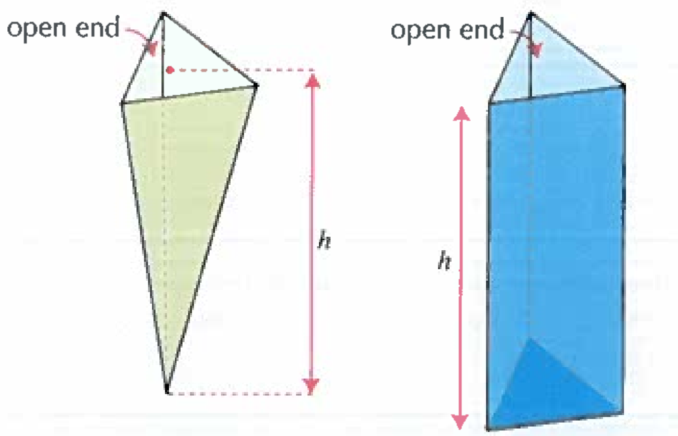
Volume of prism = area of cross section × height
= base area × height

We will now learn how to find the volume of a pyramid.

Investigation

Volume of Pyramids

Consider an open triangular pyramid and an open triangular prism with the *same base* and the *same height* (see Fig. 12.5). If we fill the entire pyramid with sand before pouring it into the prism, how many times will it take to fill the prism completely?



(a) Triangular pyramid

(b) Triangular prism

Fig. 12.5

INFORMATION

An open pyramid refers to a pyramid that is open at its base while an open prism refers to a prism that is open at one end.

Fig. 12.6 shows the **net** of a triangular pyramid which can be photocopied and pasted on a piece of cardboard before cutting it out and folding along the dotted lines to obtain an open pyramid as shown in Fig. 12.5**(a)**.

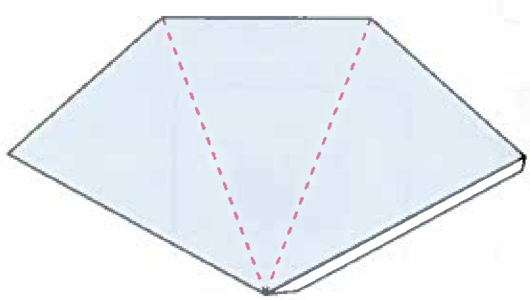


Fig. 12.6

ATTENTION

- The cardboard is necessary. If we use only paper, the pyramid will be distorted due to the mass of the sand and the result will be inaccurate.
- The tabs are not part of the net as they are for gluing purposes.

Similarly, use the net of the triangular prism in Fig. 12.7 to make an open prism as shown in Fig. 12.5(**b**).

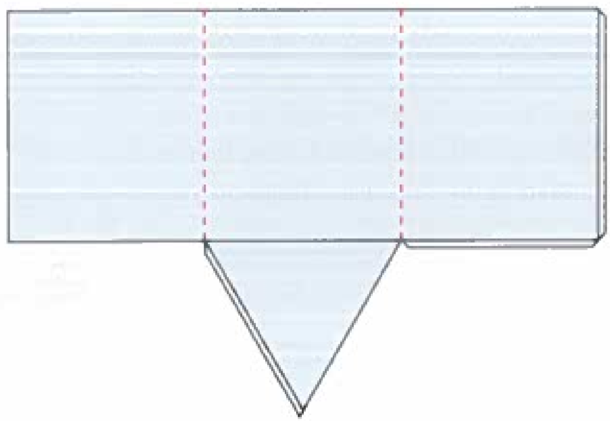


Fig. 12.7

Internet Resources

Search on the Internet for 'Dynamic Paper' for more templates of nets.

Notice that both the pyramid in Fig. 12.6 and the prism in Fig. 12.7 have the *same base* and the *same height*.

Fill the entire pyramid with sand such that the sand is level. Pour the sand into the prism. How many times must you do this until the prism is completely filled? Do you get the same result as your classmates? This suggests that:

Volume of pyramid = ________ × volume of corresponding prism

You may want to repeat the experiment in the investigation for different pyramids and their *corresponding prisms* (i.e. prisms with the same base and height). Fig. 12.8 shows a series of photos where sand is poured from a square pyramid into a *square prism* with the *same base* and the *same height*. The process is repeated until the prism is completely full. It shows that it takes 3 times the volume of a pyramid to fill the prism completely.

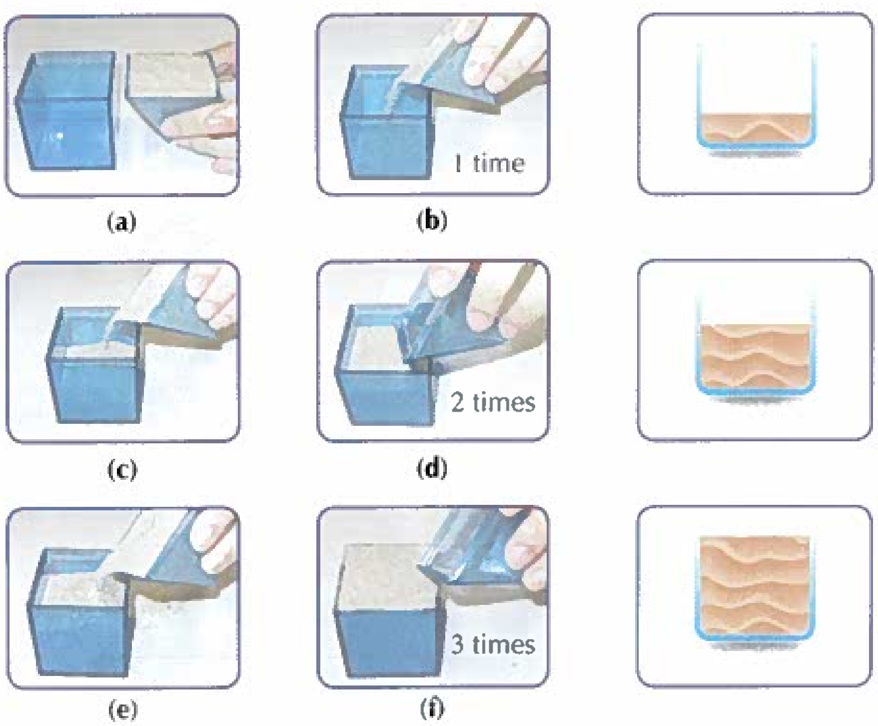


Fig. 12.8

This suggests that:

$$\text{Volume of pyramid} = \frac{1}{3} \times \text{volume of } \textit{corresponding} \text{ prism}$$
$$= \frac{1}{3} \times \text{base area} \times \text{height}$$

ATTENTION

The *corresponding* prism of a pyramid has the same base and the same height as the pyramid.

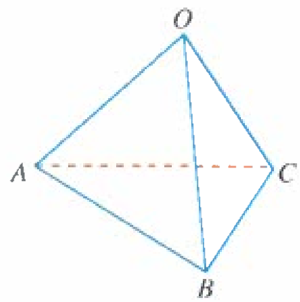
Worked Example 1

(Finding the Volume of a Pyramid, Given the Base Area and Height)

OABC is a triangular pyramid with a base area of 25 cm^2 and a height of 8 cm. Find the volume of the triangular pyramid.

Solution:

$$\text{Volume of triangular pyramid} = \frac{1}{3} \times \text{base area} \times \text{height}$$
$$= \frac{1}{3} \times 25 \times 8$$
$$= 66\frac{2}{3} \text{ cm}^3$$



INFORMATION

Another name for a triangular pyramid is 'tetrahedron'. If all the edges are of the same length, it is called a regular tetrahedron.

PRACTISE NOW 1

SIMILAR QUESTIONS

Exercise 12A Questions 1–3, 8

1. A triangular pyramid has a base area of 36 cm^2 and a height of 7 cm. Find the volume of the triangular pyramid.
2. The height of the Great Pyramid of Egypt is 146 m and its base is a square of sides 229 m. Find the volume of the pyramid, leaving your answer correct to 3 significant figures.

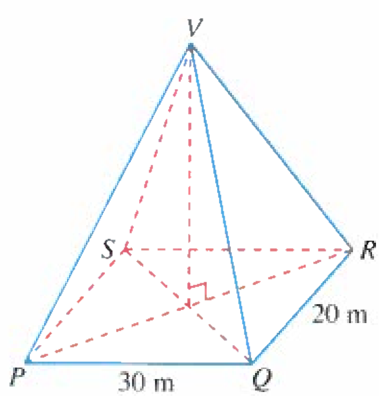
Worked Example 2

(Finding the Height of a Pyramid, Given the Volume and Dimensions of Base)

VPQRS is a rectangular pyramid where *PQ* = 30 m and *QR* = 20 m. Given that the volume of the pyramid is 7000 m^3, find its height.

Solution:

$$\text{Volume of pyramid} = \frac{1}{3} \times \text{base area} \times \text{height}$$
$$7000 = \frac{1}{3} \times (30 \times 20) \times \text{height}$$
$$7000 = 200 \times \text{height}$$
$$\therefore \text{Height} = 35 \text{ m}$$



PRACTISE NOW 2

SIMILAR QUESTIONS

Exercise 12A Questions 4–6, 9–10

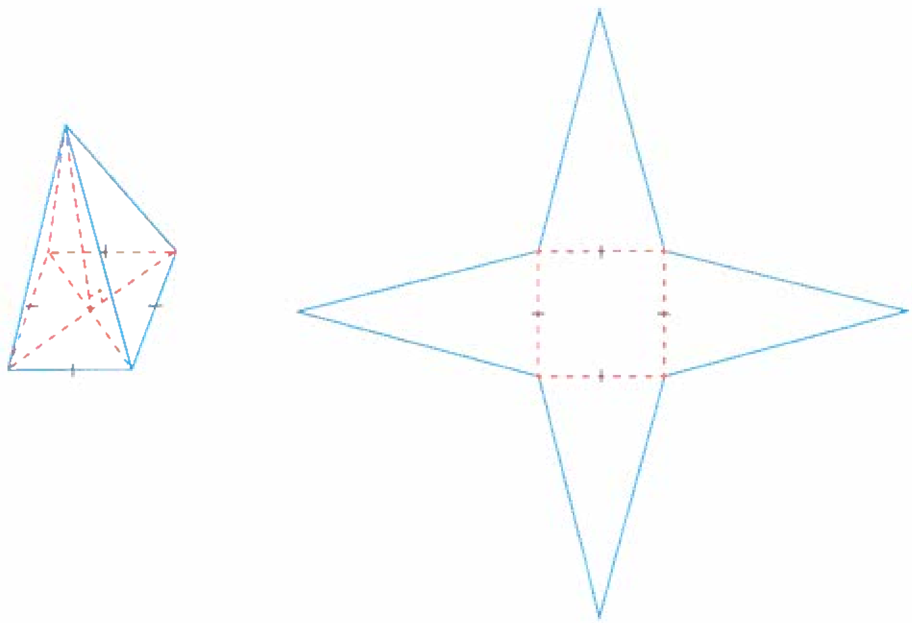
The volume of a pyramid with a square base of length 5 m is 75 m^3. Find its height.

Surface Area of Pyramids

Fig. 12.9(**a**) shows a pyramid with a square base and Fig. 12.9(**b**) shows its net. If we fold the net along the dotted lines, we will obtain the pyramid. How do we find the total surface area of the pyramid?

INFORMATION

The word 'surface' is pronounced as sur-fis, not sur-face.



(a) **(b)**

Fig. 12.9

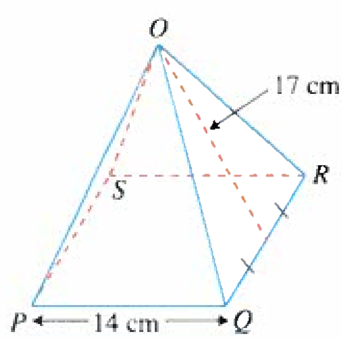
From the net, it can be seen that:

Total surface area of pyramid = total area of all faces

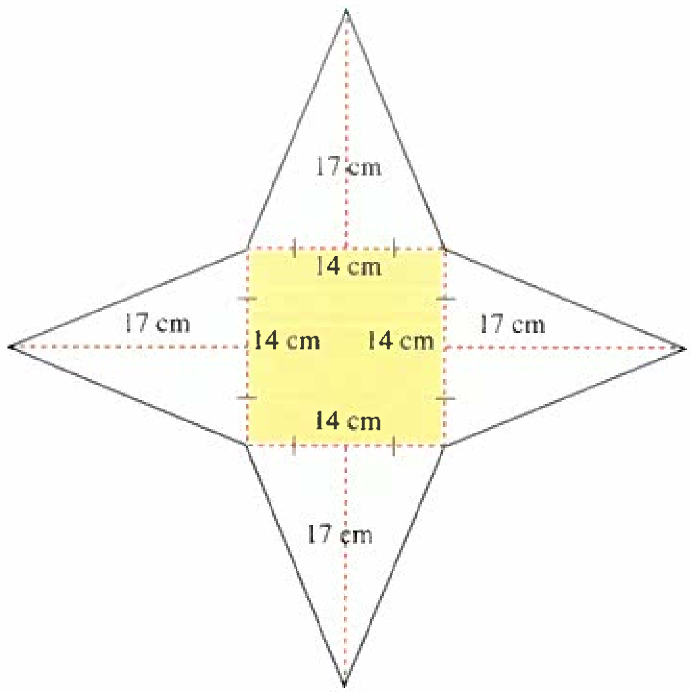
Worked Example 3

(Finding the Total Surface Area of a Pyramid, Given the Slant Height and Dimensions of Base)

OPQRS is a pyramid with a square base of length 14 cm. Given that the slant height of the pyramid is 17 cm, draw its net and hence, find its total surface area.



Solution:



Area of each triangular face $= \frac{1}{2} \times 14 \times 17$

$= 119 \text{ cm}^2$

Area of square base $= 14 \times 14$

$= 196 \text{ cm}^2$

$\therefore$ Total surface area of pyramid $= 4 \times$ area of each triangular face + area of square base

$= 4 \times 119 + 196$

$= 672 \text{ cm}^2$

When we are required to draw the net of a solid, we do *not* draw the tabs as they are for gluing purposes (see Fig. 12.6 and Fig. 12.7). We just draw the net without the tabs (see Fig. 12.9(**b**)).

PRACTISE NOW 3

A pyramid has a square base of length 12 m. Given that the slant height of the pyramid is 15 m, draw its net and hence, find its total surface area.

SIMILAR QUESTIONS

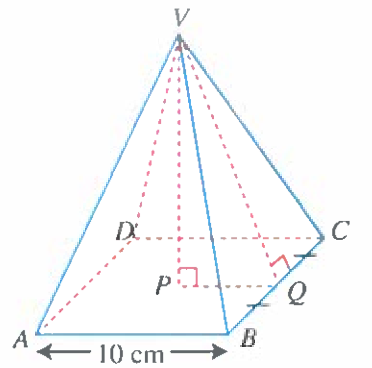
Exercise 12A Question 7

Worked Example 4

(Finding the Volume of a Pyramid, Given the Total Surface Area and Dimensions of Base)

A pyramid has a square base of length 10 cm and a total surface area of 272 cm². Find the volume of the pyramid.

Solution:



Total surface area of pyramid = 4 × area of each triangular face + area of square base

$$\text{Area of each triangular face} = \frac{\text{Total surface area of pyramid} - \text{area of square base}}{4}$$

$$= \frac{272 - 10 \times 10}{4}$$

$$= \frac{172}{4}$$

$$= 43 \text{ cm}^2$$

$$\text{Area of } \triangle VBC = \frac{1}{2} \times 10 \times VQ = 43$$

$$5 \times VQ = 43$$

$$VQ = 8.6 \text{ cm}$$

$$PQ = \frac{1}{2} \times AB$$

$$= \frac{1}{2} \times 10$$

$$= 5 \text{ cm}$$

In $\triangle VPQ$, $\angle P = 90°$.

Using Pythagoras' Theorem,

$$VQ^2 = VP^2 + PQ^2$$

$$8.6^2 = VP^2 + 5^2$$

$$VP^2 = 8.6^2 - 5^2$$

$$= 73.96 - 25$$

$$= 48.96$$

$$\therefore VP = \sqrt{48.96} \text{ (since } VP > 0)$$

$$= 6.997 \text{ cm (to 4 s.f.)}$$

$$\therefore \text{Volume of pyramid} = \frac{1}{3} \times \text{base area} \times \text{height}$$

$$= \frac{1}{3} \times 10 \times 10 \times 6.997$$

$$= 233 \text{ cm}^3 \text{ (to 3 s.f.)}$$

Problem Solving Tip

Pólya's 4-Stage Problem Solving Model

Stage 1 (Understand the problem):

You may wish to sketch the pyramid for better visualisation.

Stage 2 (Think of a plan):

To find the volume of the pyramid, according to the formula, we need to know its base area and its height. We are able to find its base area easily but how do we find its height? Which information have we not used?

Do you notice that there is a right-angled triangle in the figure? What topic comes into your mind when you see right-angled triangles?

Stage 3 (Carry out the plan): See solution.

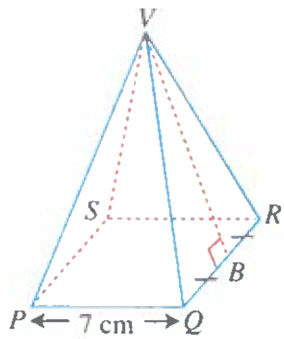
Stage 4 (Look back): Check your solution.

RECALL

In order for the final answer to be accurate to three significant figures, any intermediate working must be correct to at least four significant figures.

PRACTISE NOW 4

VPQRS is a pyramid where the length of each side of its square base is 7 cm. Given that the total surface area of the pyramid is 161 cm², find its
(i) slant height *VB*,
(ii) volume.



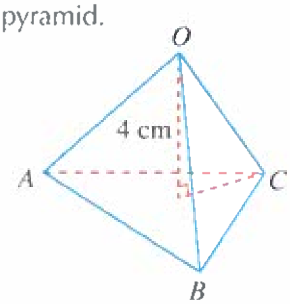
SIMILAR QUESTIONS

Exercise 12A Questions 11–16

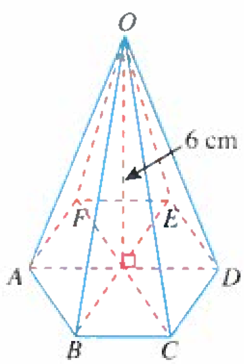
Exercise 12A

BASIC LEVEL

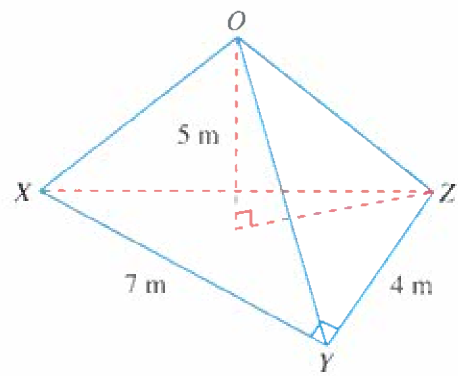
1. *OABC* is a triangular pyramid with a base area of 15 cm² and a height of 4 cm. Find the volume of the triangular pyramid.



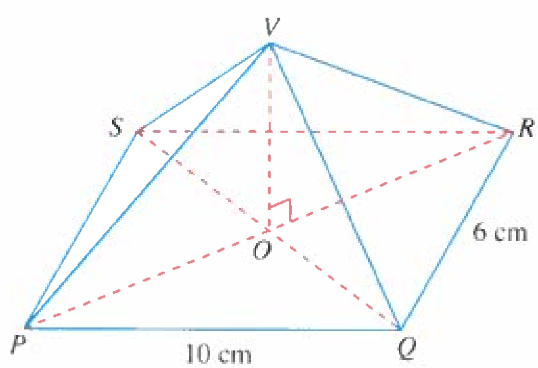
2. *OABCDEF* is a hexagonal pyramid with a base area of 23 cm² and a height of 6 cm. Find the volume of the pyramid.



3. *OXYZ* is a pyramid whose base is a right-angled triangle where *XY* = 7 m and *YZ* = 4 m. Given that the height of the pyramid is 5 m, find its volume.



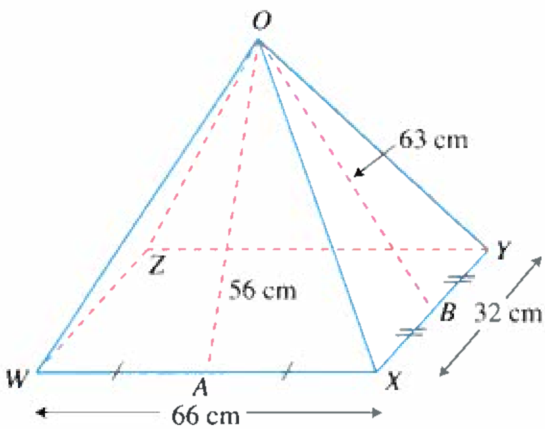
4. *VPQRS* is a rectangular pyramid where *PQ* = 10 cm and *QR* = 6 cm. Given that the volume of the pyramid is 100 cm³, find its height *VO*.



5. A pyramid with a triangular base has a volume of 50 cm^3. If the base and the height of the triangular base are 5 cm and 8 cm respectively, find the height of the pyramid.

6. The volume of a square pyramid with a height of 12 m is 100 m^3. Find the length of its square base.

7. *OWXYZ* is a rectangular pyramid where *WX* = 66 cm and *XY* = 32 cm. Given that the slant heights *OA* and *OB* of the pyramid are 56 cm and 63 cm respectively, draw its net and hence, find its total surface area.

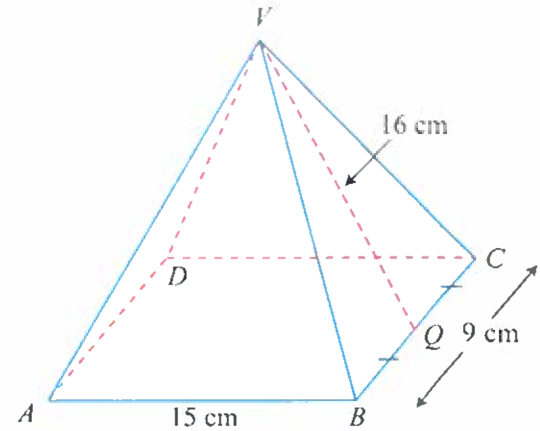


INTERMEDIATE LEVEL

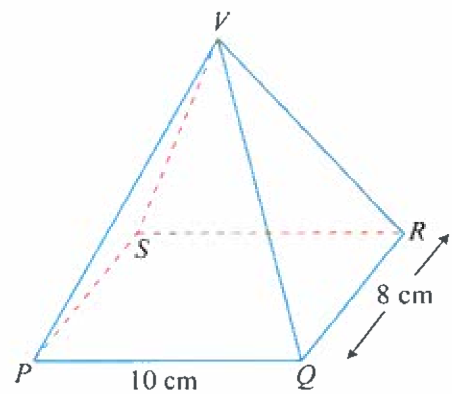
8. A glass paper weight is in the shape of a solid pyramid with a square base of length 6 cm and a height of 7 cm. Given that the density of the glass is 3.1 g/cm^3, find the mass of four identical paper weights.

9. A solid pentagonal pyramid has a mass of 500 g. It is made of a material with a density of 6 g/cm^3. Given that the base area of the pyramid is 30 cm^2, find its height.

10. *VABCD* is a pyramid with a rectangular base of sides 15 cm by 9 cm. Given that the slant height *VQ* of the pyramid is 16 cm, find its
(i) height,
(ii) volume.



11. *VPQRS* is a pyramid with a rectangular base of sides 10 cm by 8 cm. Given that the volume of the pyramid is 180 cm^3, find its
(i) height,
(ii) total surface area.

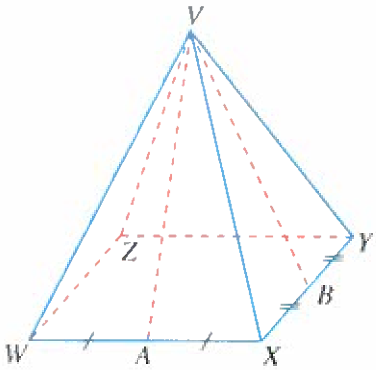


12. A pyramid has a rectangular base of sides 16 m by 14 m. Given that the volume of the pyramid is 700 m^3, find its
(i) height,
(ii) total surface area.

ADVANCED LEVEL

13. A solid pyramid has a rectangular base of sides 15 cm by 10 cm and a height of 20 cm. It is placed inside an open cubical tank of sides 30 cm. The tank is then completely filled with water. If the pyramid is removed, what will be the depth of the remaining water in the tank?

14. *VWXYZ* is a rectangular pyramid where *WX* is longer than *XY*. Is the slant height *VA* longer or shorter than the slant height *VB*? Explain your answer.

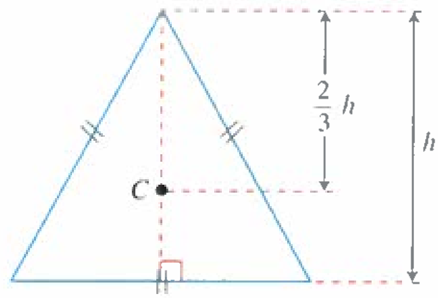


15. The length of each edge of a regular tetrahedron, whose faces are identical equilateral triangles, is 8 cm. Find its

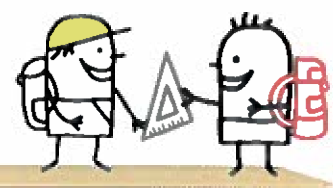
(i) slant height,

(ii) volume.

Hint: All sides are equal. The centre *C* of an equilateral triangle is $\frac{2}{3}$ of its height *h*.



12.2 Volume and Surface Area of Cones



In Book 1, we have learnt about cylinders. In this section, we will learn about cones.

Class Discussion

What are Cones?

Fig. 12.10 shows some real-life examples of cones.

(a) Ice-Cream Cone

(b) Party Hat

Fig. 12.10

Discuss each of the following questions with your classmates.

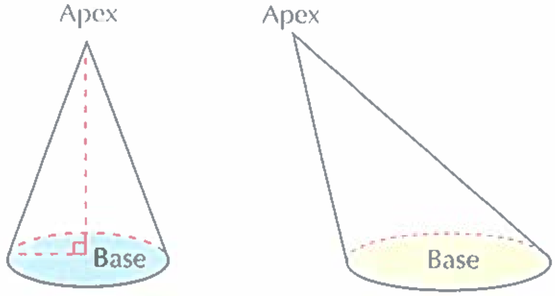
1. What are some common features of these cones?
2. What are some similarities and differences between a cone and a cylinder?
3. What are some similarities and differences between a cone and a pyramid?
4. Give three more real-life examples of cones.

Types of Cones

A cone is a solid in which the base is bounded by a **simple closed curve** and the curved surface tapers into a point called the **apex**, which is opposite the base. If the base is circular, the cone is known as a **circular cone**. Fig. 12.11 shows some examples of cones with different bases.

INFORMATION

A simple closed curve refers to a closed curve that does not intersect itself.



(a) Circular Base **(b)** Elliptical Base

Fig. 12.11

In Fig. 12.11**(a)**, the apex is *vertically above the centre of the circular base*. What do you think such a circular cone is called?

The **perpendicular height** (or simply the **height**) of a cone is the perpendicular distance from the apex to the base of the cone (see Fig. 12.12). The **slant height** of a right circular cone is the distance from the apex to the circumference of the base.

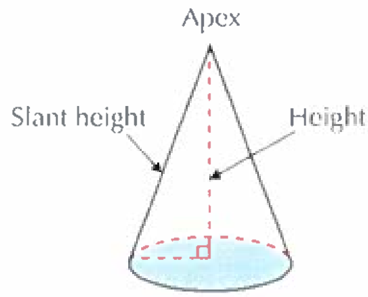


Fig. 12.12

In this section, we will study only right circular cones. Therefore, the term 'cone' is used to refer to a right circular cone. Also, the term 'height' is used to refer to the perpendicular height unless otherwise stated. The radius of the circular base is also known as the **base radius**.

Journal Writing

One of your classmates is absent from school and he does not understand what a cone is. Write in your journal how you would explain to him some features of a cone and how a cone is different from a cylinder or a pyramid. Include some appropriate diagrams where necessary.

Volume of Cones

In Book 1, we have learnt:

Volume of cylinder = area of cross section × height = base area × height = $\pi r^2 h$,

where r is the base radius and h is the height of the cylinder.

Now, we will learn how to find the volume of a cone by comparing a cone with a pyramid that has a regular polygonal base.

Investigation

Comparison between a Cone and a Pyramid

1. Fig. 12.13 shows **(a)** a regular pentagon, **(b)** a regular hexagon, **(c)** a regular 12-gon, and **(d)** a regular 16-gon inside a circle respectively.

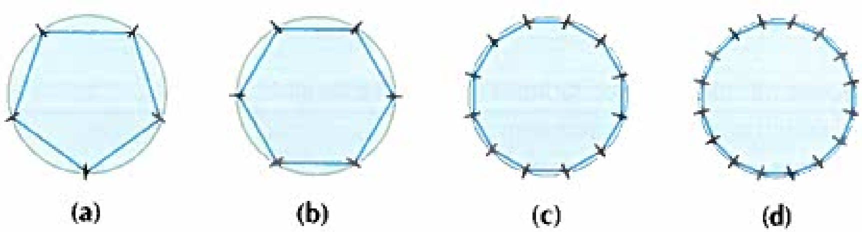


Fig. 12.13

If the number of sides of a *regular* polygon is increased indefinitely, what will the polygon become?

2. Fig. 12.14 shows a sequence of regular pyramids, i.e. right pyramids with regular polygonal bases.

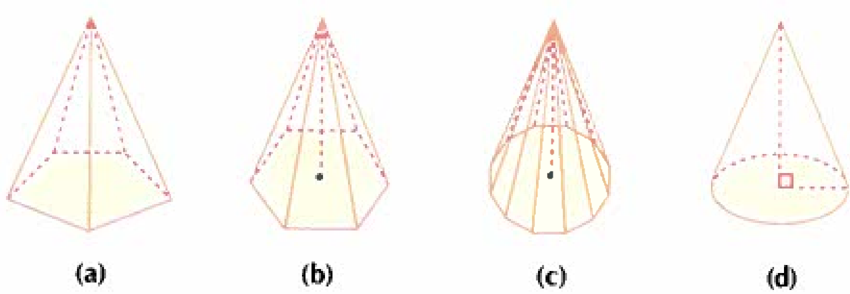


Fig. 12.14

If the number of sides of the regular polygonal base of a pyramid is increased indefinitely, what will the pyramid become?

In many ways, a cone is *like* a pyramid. However, a cone is not a pyramid because the base of a pyramid must be a polygon but the base of a cone is a circle. Although a regular polygon can become a circle if its number of sides is increased indefinitely, a polygon must have a *finite* number of sides and so a circle is *not* a polygon.

Since a cone is like a pyramid (see Fig. 12.14), by analogy, the formula for the volume of a cone should be the same as the formula for the volume of a pyramid. We have:

$$\text{Volume of cone} = \frac{1}{3} \times \text{base area} \times \text{height}$$
$$= \frac{1}{3}\pi r^2 h,$$

where r is the base radius and h is the height of the cone.

Thinking Time

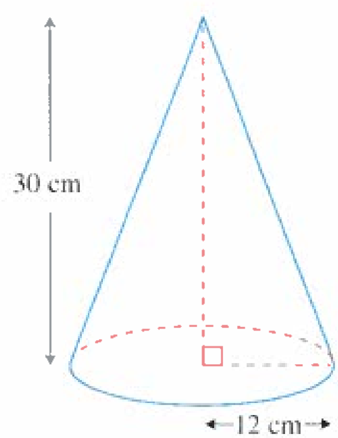
If a cone and a cylinder have the *same base* and the *same height*, what is the relationship between the volume of the cone and the volume of this corresponding cylinder?

Worked Example 5

(Finding the Volume of a Cone, Given the Base Radius and Height)

A cone has a circular base of radius 12 cm and a height of 30 cm. Find the volume of the cone.

Solution:



RECALL

If the question does not specify the value of π, we use the value of π stored in the calculator.

$$\text{Volume of cone} = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times \pi \times 12^2 \times 30$$
$$= 1440\pi$$
$$= 4520 \text{ cm}^3 \text{ (to 3 s.f.)}$$

Problem Solving Tip

For accuracy, we should work in terms of π in the intermediate steps.

PRACTISE NOW 5

SIMILAR QUESTIONS

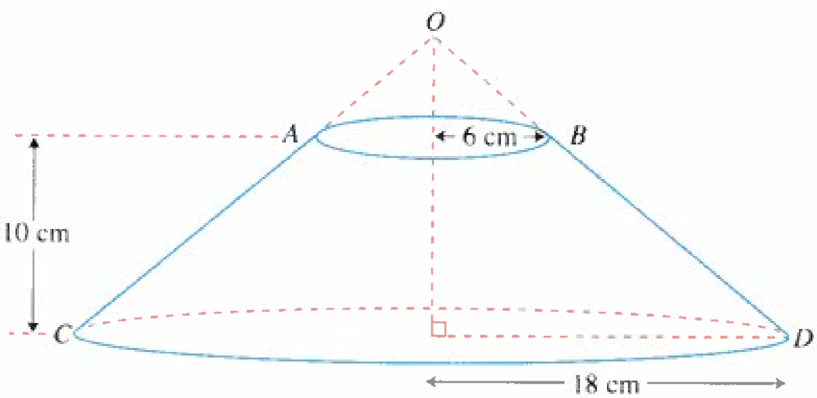
Exercise 12B Questions 1(a)–(d), 2–4, 9–10

1. A cone has a circular base of radius 8 cm and a height of 17 cm. Find the volume of the cone.
2. The volume of a cone with a circular base of radius 6 m is 84π m³. Find its height.

Worked Example 6

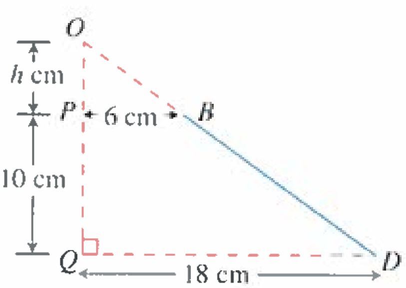
(Finding the Volume of a Frustum)

The figure shows a frustum which is obtained by removing the smaller cone *OAB* with a base radius of 6 cm from the bigger cone *OCD* that has a base radius of 18 cm. Given that the height of the frustum is 10 cm, find its volume.



Solution:

Let the height of the smaller cone be h cm.
Then the height of the bigger cone is $(h + 10)$ cm.



Since ΔOPB is similar to ΔOQD,

$$\frac{OP}{OQ} = \frac{PB}{QD}$$

$$\frac{h}{h+10} = \frac{6}{18}$$

$$\frac{h}{h+10} = \frac{1}{3}$$

$$3h = h + 10$$

$$2h = 10$$

$$h = 5$$

$\therefore$ Height of bigger cone $= 5 + 10$

$= 15$ cm

∴ Volume of frustum = volume of bigger cone − volume of smaller cone

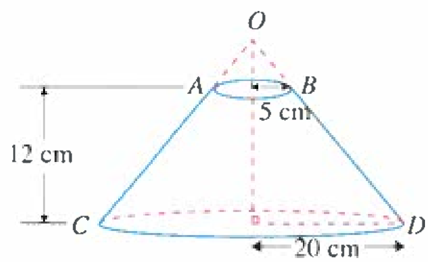
$$= \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(R^2 H - r^2 h)$$
$$= \frac{1}{3}\pi(18^2 \times 15 - 6^2 \times 5)$$
$$= \frac{1}{3}\pi(4680)$$
$$= 1560\pi$$
$$= 4900 \text{ cm}^3 \text{ (to 3 s.f.)}$$

PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 12B Question 16

The figure shows a frustum which is obtained by removing the smaller cone *OAB* with a base radius of 5 cm from the bigger cone *OCD* that has a base radius of 20 cm. Given that the height of the frustum is 12 cm, find its volume.



Surface Area of Cones

As a cone has a curved surface, it is difficult to use the flat slant faces of a pyramid as an analogy to find a formula for the curved surface area of a cone. Instead, we will use the same approach as finding the formula for the area of a circle which we have learnt in primary school.

Investigation

Curved Surface Area of Cones

Consider the cone as shown in Fig. 12.15**(a)**.

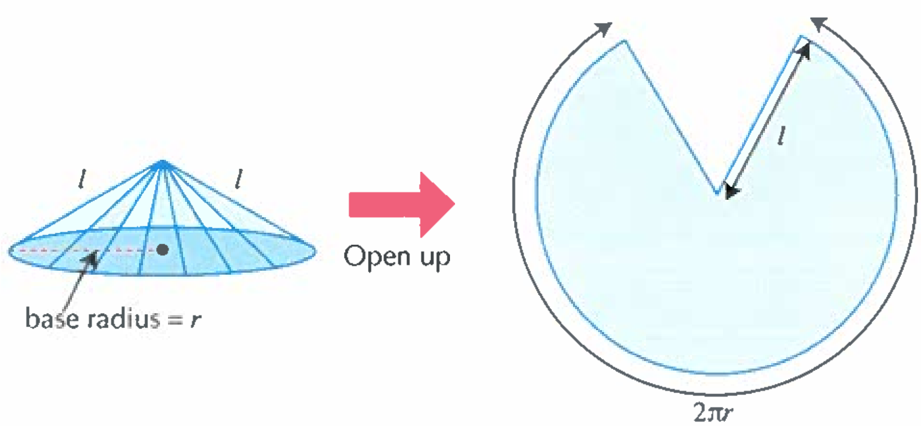


Fig. 12.15(a)

Unfold the curved surface of the cone to become the sector shown in Fig. 12.15(**b**). Divide the sector into 44 smaller sectors and arrange them to form the shape as shown in Fig.12.15(**b**).

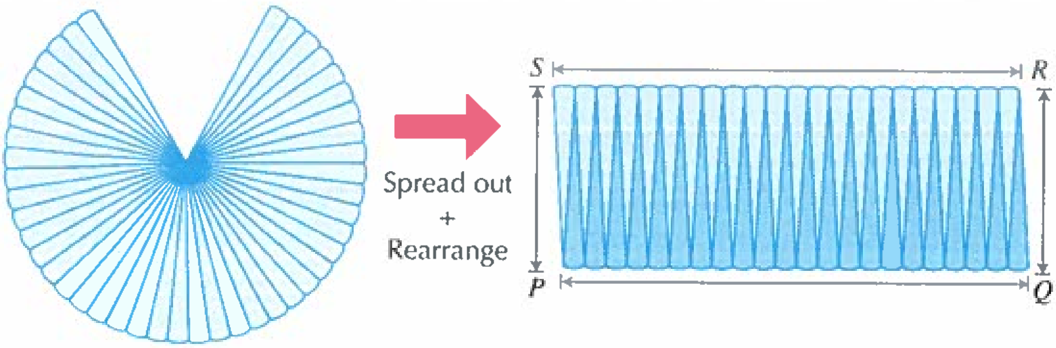


Fig. 12.15(b)

If the number of sectors is increased indefinitely, then the shape in Fig. 12.15(**b**) will become a __________ *PQRS*.

Since $PQ + RS$ = circumference of the base circle in Fig. 12.15(**a**), then the length of the rectangle is PQ = __________.

Since PS = slant height of the cone in Fig. 12.15(**a**), then the breadth of the rectangle is PS = __________.

∴ Curved surface area of cone = area of rectangle

= __________ × __________

= __________

From the investigation, we have:

Curved surface area of cone = πrl,

where r is the base radius and l is the slant height of the cone.

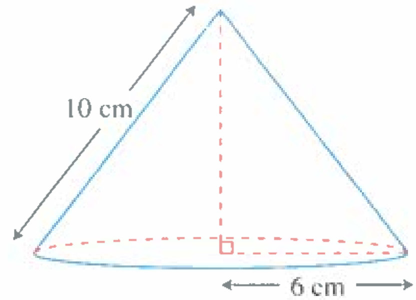
Thinking Time

What is the total surface area of a *solid* cone?

Worked Example 7

(Finding the Total Surface Area of a Cone, Given the Base Radius and Slant Height)

A cone has a circular base of radius 6 cm and a slant height of 10 cm. Find the total surface area of the cone.



Solution:

Total surface area of cone $= \pi rl + \pi r^2$

$= \pi \times 6 \times 10 + \pi \times 6^2$

$= 60\pi + 36\pi$

$= 96\pi$

$= 302$ cm^2 (to 3 s.f.)

PRACTISE NOW 7

1. A cone has a circular base of radius 9 cm and a slant height of 5 cm. Find the total surface area of the cone.
2. A cone has a circular base of radius 8 m. Given that the total surface area of the cone is 350 m^2, find its slant height. (Take π to be 3.142.)

SIMILAR QUESTIONS

Exercise 12B Questions 5(a)–(c), 6–8, 11–12

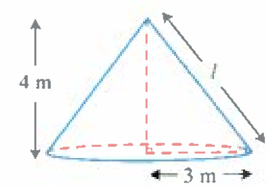
Worked Example 8

(Finding the Curved Surface Area of a Cone, Given the Base Radius and Height)

A cone has a circular base of radius 3 m and a height of 4 m. Find the curved surface area of the cone.

Problem Solving Tip

A sketch is helpful.



To find the curved surface area of the cone, according to the formula, we need to know its slant height, but it is not given. However, we notice that there is a right-angled triangle in the figure and so we can make use of Pythagoras' Theorem.

Solution:

Let the slant height of the cone be l m.

Using Pythagoras' Theorem, $l = \sqrt{3^2 + 4^2}$

$= 5$

$\therefore$ Curved surface area of cone $= \pi rl$

$= \pi \times 3 \times 5$

$= 15\pi$

$= 47.1$ m^2 (to 3 s.f.)

PRACTISE NOW 8

1. A cone has a circular base of radius 8 m and a height of 15 m. Find the curved surface area of the cone.
2. A cone has a circular base of radius 7 cm and a slant height of 12 cm. Find the volume of the cone.

SIMILAR QUESTIONS

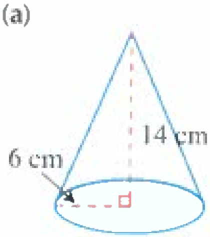
Exercise 12B Questions 13–15, 17–18

Exercise 12B

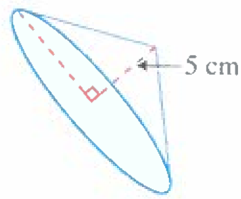
BASIC LEVEL

1. Find the volume of each of the following cones.

(a)

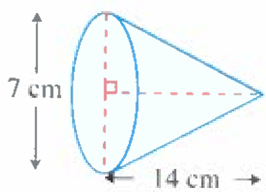


(b)

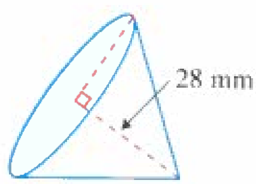


Area of base = 154 cm^2

(c)



(d)



Circumference of base = 132 mm

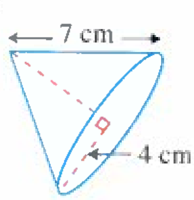
2. The volume of a cone with a circular base of radius 8 cm is 320π cm^3. Find its height.

3. A cone has a base area of 20 m^2 and a volume of 160 m^3. Find the height of the cone.

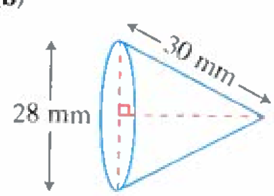
4. A cone has a height of 14 cm and a volume of 132 cm^3. Find the radius of the circular base. (Take π to be $\frac{22}{7}$.)

5. Find the total surface area of each of the following cones.

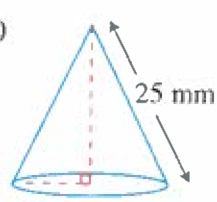
(a)



(b)



(c)



Circumference of base = 132 mm

6. A cone has a circular base of radius 6 mm. Given that the curved surface area of the cone is 84π mm^2, find its slant height.

7. A cone has a circular base of radius 15 cm. Given that the total surface area of the cone is 1000 cm^2, find its slant height. (Take π to be 3.142.)

8. An open cone has a slant height of 5 m and a curved surface area of 251 m^2. Find the radius of the circular base.

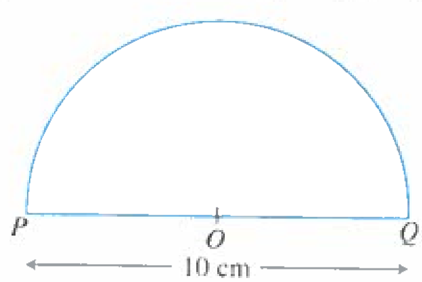
INTERMEDIATE LEVEL

9. A conical funnel of diameter 23.2 cm and depth 42 cm contains water filled to the brim. The water is poured into a cylindrical tin of diameter 16.2 cm. If the tin must contain all the water, find its least possible height.

10. A conical block of silver has a height of 16 cm and a base radius of 12 cm. The silver is melted to form coins $\frac{1}{6}$ cm thick and $1\frac{1}{2}$ cm in diameter. Find the number of coins that can be made.

11. An open cone has a circular base of radius 10 cm and a slant height of 20 cm. Draw the net of the cone and label its dimensions.

12. The semicircle shown is folded to form a right circular cone so that the arc PQ becomes the circumference of the base. Given that the diameter of the semicircle, PQ, is 10 cm, find
   (i) the diameter of the base,
   (ii) the curved surface area of the cone.



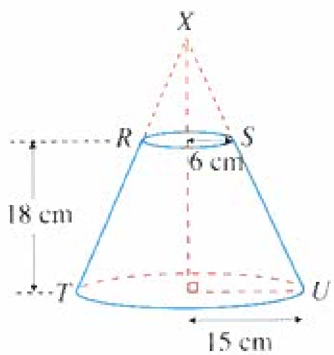
13. A cone has a circular base of radius 5 cm and a height of 12 cm. Find the curved surface area of the cone.

14. A cone has a circular base of radius 8 cm and a slant height of 20 cm. Find the volume of the cone.

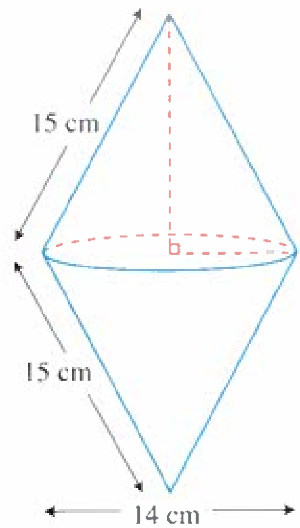
15. A circular cone has a height of 17 mm and a slant height of 21 mm. Find
(i) the volume,
(ii) the total surface area,
of the cone.

ADVANCED LEVEL

16. The figure shows a frustum which is obtained by removing the smaller cone *XRS* with a base radius of 6 cm from the bigger cone *XTU* that has a base radius of 15 cm. Given that the height of the frustum is 18 cm, find its volume.



17. A solid is made up of two identical cones, each with a base diameter of 14 cm and a slant height of 15 cm. Find
(i) the total surface area,
(ii) the volume,
of the solid.



18. A cone has a circular base of radius 13.5 m. Given that the total surface area of the cone is 1240 m^2, find its volume.

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12.3 Volume and Surface Area of Spheres

Fig. 12.16 shows some real-life examples of spheres.

(a) Soccer Ball (b) Crystal Ball (c) Globe

Fig. 12.16

Internet Resources

Map projection is a method of representing the surface of a *sphere* on a *plane*. Is it possible to draw the map of all the countries in the world on a plane accurately without distorting the shape or the area? Search on the Internet to find out more about conformal (or area-preserving) map projections.

Thinking Time

What is a **hemisphere**? What are some real-life examples of hemispheres?

Volume of Spheres

Class Discussion

Is the King's Crown Made of Pure Gold?

Archimedes is one of the three greatest mathematicians of all time. He lived from 287 B.C. to 212 B.C. in Greece. One day, the King asked Archimedes to find out whether his crown was made of pure gold or whether the goldsmith had cheated. The real problem lies in finding the volume of the crown. Archimedes thought for a long time but still had no idea. He then decided to take a break by taking a bath. As he stepped into the bathtub, the water overflowed. Archimedes was so excited by this discovery that he dashed out into the street shouting 'Eureka!' which means 'I have found it!', but had forgotten that he was unclothed.

Archimedes realised that a sinking solid displaces an amount of water that is equal to the volume of the solid. To find the volume of the crown, we can fill up the Eureka can (named as a result of the above story) as shown in Fig. 12.17**(a)** with water until it overflows before putting in the crown. The volume of water displaced into the container as shown in Fig. 12.17 **(b)** is equal to the volume of the crown.

Eureka can Container

(a) Before

Eureka can Container

(b) After

Fig. 12.17

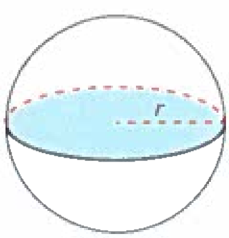
Suppose Archimedes found out that the volume of water displaced was 714 cm^3 and the mass of the crown was 11.6 kg. Given that the density of gold is 19.3 g/cm^3, determine whether the crown was made of pure gold. Do you arrive at the same conclusion as your classmates?

Investigation

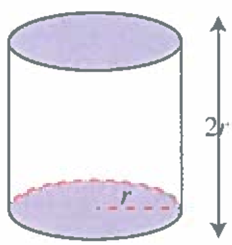
Volume of Spheres

Part I: Archimedes' Discovery of the Volume of Sphere

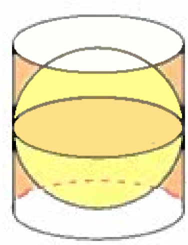
Using the displacement method mentioned in the class discussion, Archimedes discovered a formula to find the volume of a sphere. Fig. 12.18(**a**) shows a sphere of radius r and Fig. 12.18(**b**) shows its related circular cylinder of base radius r and height $2r$. Archimedes filled the cylinder with water and placed the sphere inside it (see Fig. 12.18(**c**)).



(a) Sphere



(b) Related Cylinder



(c) Sphere inside related Cylinder

Fig. 12.18

Archimedes found out that the volume of water displaced was equal to $\frac{2}{3}$ of the volume of the cylinder. This was one of his greatest discoveries.

Part II: Work out the Volume of Sphere

Volume of cylinder $= \pi r^2 h$

$= \pi \times r^2 \times 2r$

$=$ ____________

Volume of sphere $= \frac{2}{3} \times$ volume of cylinder

$= \frac{2}{3} \times$ ____________

$=$ ____________

From the investigation, we have:

> Volume of sphere $= \frac{4}{3}\pi r^3$,

where r is the radius of the sphere.

Worked Example 9

(Finding the Volume of a Sphere, Given the Radius)

A ball bearing (which is spherical in shape) has a radius of 0.3 cm. Find

(i) the volume of the ball bearing,

(ii) the mass of 6000 identical ball bearings if they are made of steel of density 7.85 g/cm^3.

Solution:

(i) Volume of ball bearing $= \frac{4}{3}\pi r^3$

$= \frac{4}{3} \times \pi \times 0.3^3$

$= 0.036\pi$

$= 0.113$ cm^3 (to 3 s.f.)

(ii) Mass of 6000 ball bearings = volume of 6000 ball bearings × density

$= 6000 \times 0.036\pi \times 7.85$

$= 5330$ g (to 3 s.f.)

PRACTISE NOW 9

1. A lead ball bearing has a diameter of 0.4 cm. Find the mass of 5000 identical lead ball bearings if the mass of 1 cm^3 of lead is 11.3 g.

2. A basketball has a volume of 5600 cm^3. Find its radius.

SIMILAR QUESTIONS

Exercise 12C Questions 1(a)–(c), 2(a)–(f), 7–11, 15

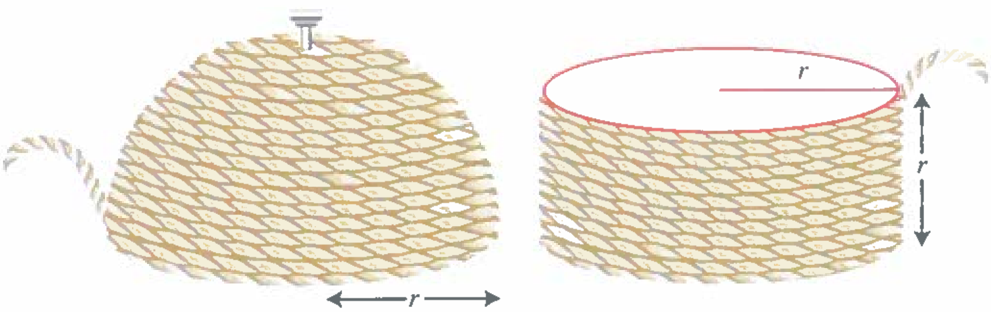
Surface Area of Spheres

Investigation

Surface Area of Spheres

Part I: Archimedes' Discovery of the Surface Area of Sphere

Archimedes also discovered a formula to find the surface area of a sphere. Fig. 12.19(**a**) shows a hemisphere of radius r. One end of a piece of twine is stuck in the centre of the curved surface of the hemisphere by a pin before the twine is coiled around the curved surface completely. Fig. 12.19(**b**) shows a circular cylinder of base radius r and height r. One end of a piece of twine is stuck at the bottom of the curved surface of the cylinder by a pin before the twine is coiled around the curved surface completely.



(a) Hemisphere **(b)** Cylinder

Fig. 12.19

Archimedes found out that the two pieces of twine were of the same length.

Length of second piece of twine $= 2\pi rh$

$= 2\pi \times r \times r$

= ____________

Curved surface area of sphere = 2 × length of first piece of twine

= 2 × length of second piece of twine

= 2 × ____________

= ____________

Part II: Simple Experiment to Find the Surface Area of a Sphere

1. Take an orange and cut it into halves. Place one half flat on a sheet of paper and draw a circle around it. The area of the circle is πr^2, where r is the radius of the orange, assuming that the orange is spherical.
2. Repeat step **1** and draw a few more circles.
3. Peel the skin of both halves of the orange, tear into small pieces and cover as many circles as completely as possible with the skin.
4. How many circles are covered completely with the orange skin?
5. What do you think is the surface area of the orange (spherical in shape)?

In Part II, about 4 circles are completely covered by the orange skin.
From the investigation, we have:

Surface area of sphere = $4\pi r^2$,

where r is the radius of the sphere.

Thinking Time

What is the total surface area of a solid hemisphere?

Worked Example 10

(Finding the Surface Area of a Sphere, Given the Radius)

A solid sphere has a diameter of 14 cm. Find its surface area.

Solution:

Radius of sphere = $14 \div 2$
= 7 cm

Surface area of sphere = $4\pi r^2$
= $4 \times \pi \times 7^2$
= 196π
= 616 cm^2 (to 3 s.f.)

PRACTISE NOW 10

A solid sphere has a diameter of 25 cm. Find its surface area.

SIMILAR QUESTIONS

Exercise 12C Questions 3(a)–(c), 4, 12, 14

Worked Example 11

(Finding the Radius of a Hemisphere, Given the Curved Surface Area)

A solid hemisphere has a curved surface area of 175 cm^2. Find its radius.

Solution:

Curved surface area of solid hemisphere = $\frac{1}{2} \times 4\pi r^2 = 2\pi r^2 = 175$

$$r^2 = \frac{175}{2\pi}$$

$$\therefore r = \sqrt{\frac{175}{2\pi}} \text{ (since } r > 0)$$

= 5.28 cm (to 3 s.f.)

PRACTISE NOW 11

A hemisphere has a curved surface area of 200 cm^2. Find its radius.

SIMILAR QUESTIONS

Exercise 12C Questions 5(a)–(f), 6, 13

Exercise 12C

BASIC LEVEL

1. Find the volume of each of the following spheres with the given radius.
   (a) 8 cm
   (b) 14 mm
   (c) 4 m

2. Find the radius of each of the following spheres with the given volume.
   (a) $1416\ \text{cm}^3$ (b) $12\ 345\ \text{mm}^3$
   (c) $780\ \text{m}^3$ (d) $972\pi\ \text{cm}^3$
   (e) $498\pi\ \text{mm}^3$ (f) $15\frac{3}{16}\pi\ \text{m}^3$

3. Find the surface area of each of the following spheres with the given radius.
   (a) 12 cm
   (b) 9 mm
   (c) 3 m

4. Find the total surface area of a hemisphere of radius 7 cm. (Take π to be 3.142.)

5. Find the radius of each of the following spheres with the given surface area.
   (a) $210\ \text{cm}^2$ (b) $7230\ \text{mm}^2$
   (c) $3163\ \text{m}^2$ (d) $64\pi\ \text{cm}^2$
   (e) $911\pi\ \text{mm}^2$ (f) $49\pi\ \text{m}^2$

6. A hemisphere has a curved surface area of $364.5\pi\ \text{cm}^2$. Find its radius.

INTERMEDIATE LEVEL

7. Find the number of steel ball bearings, each of diameter 0.7 cm, which can be made from 1 kg of steel, given that $1\ \text{cm}^3$ of steel has a mass of 7.85 g.

8. A hollow aluminium sphere has an internal radius of 20 cm and an external radius of 30 cm. Given that the density of aluminium is $2.7\ \text{g/cm}^3$, find the mass of the sphere in kg.

9. Fifty-four solid hemispheres, each of diameter 2 cm, are melted to form a single sphere. Find the radius of the sphere.

10. A sphere of diameter 26.4 cm is half-filled with acid. The acid is drained into a cylindrical beaker of diameter 16 cm. Find the depth of the acid in the beaker.

11. A cylindrical tin has an internal diameter of 18 cm. It contains water to a depth of 13.2 cm. A heavy spherical ball bearing of diameter 9.3 cm is dropped into the tin. Find the new height of water in the tin, leaving your answer correct to 2 decimal places.

12. A sphere has a volume of $850\ \text{m}^3$. Find its surface area.

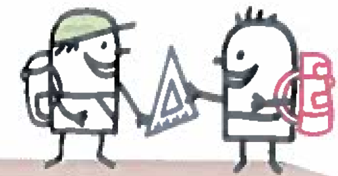
13. A basketball has a surface area of $1810\ \text{cm}^2$. Find its volume.

ADVANCED LEVEL

14. There are 20 identical solid hemispheres. The curved surface and the flat surface of each hemisphere are to be painted red and yellow respectively. Find the ratio of the amount of red paint needed to the amount of yellow paint needed.

15. A cylindrical can has a base radius of 3.4 cm. It contains a certain amount of water such that when a sphere is placed inside the can, the water just covers the sphere. If the sphere fits exactly inside the can, find
    (i) the surface area of the can that is in contact with the water when the sphere is inside the can,
    (ii) the depth of water in the can before the sphere was placed inside the can.

12.4 Volume and Surface Area of Composite Solids

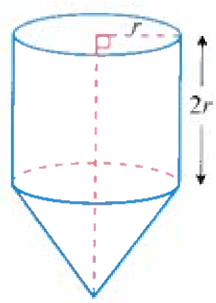


In this section, we will learn how to solve problems involving the volume and the surface area of composite solids.

Worked Example 12

(Problem involving a Cylinder and a Cone)

A container is made up of a hollow cone with an internal base radius of r cm and a hollow cylinder with the same base radius and an internal height of $2r$ cm. Given that the height of the cone is two-thirds of the height of the cylinder and 5 litres of water is needed to fill the conical part of the container completely, find the amount of water needed to fill the container completely, giving your answer in litres.



Solution:

Height of cone $= \frac{2}{3} \times$ height of cylinder

$$= \frac{2}{3} \times 2r$$

$$= \frac{4}{3}r$$

Volume of cone $= \frac{1}{3}\pi r^2\left(\frac{4}{3}r\right)$

$$= \frac{4}{9}\pi r^3$$

RECALL

$1\ l = 1000\ \text{cm}^3$

Since volume of cone $= 5\ l = 5000\ \text{cm}^3$,

then $\frac{4}{9}\pi r^3 = 5000$

$$r^3 = \frac{5000 \times 9}{4\pi}$$

$$= \frac{11\ 250}{\pi}$$

Volume of cylinder $= \pi r^2(2r)$

$$= 2\pi r^3$$

$$= 2\pi \times \frac{11\ 250}{\pi}$$

$$= 22\ 500\ \text{cm}^3$$

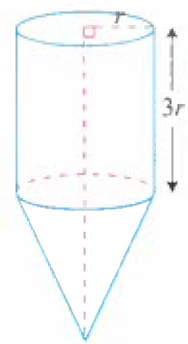
$$= 22.5\ l$$

$\therefore$ Amount of water needed to fill container completely $= 22.5 + 5$

$$= 27.5\ l$$

PRACTISE NOW 12

A container is made up of a hollow cone with an internal base radius of r cm and a hollow cylinder with the same base radius and an internal height of $3r$ cm. Given that the height of the cone is three-quarters of the height of the cylinder and 10 litres of water is needed to fill the conical part of the container completely, find the amount of water needed to fill the container completely, giving your answer in litres.



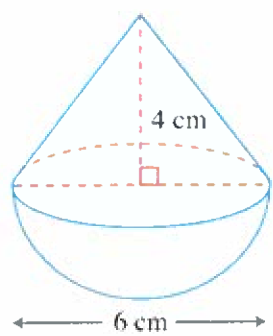
SIMILAR QUESTIONS

Exercise 12D Questions 1–2, 5–6

Worked Example 13

(Problem involving a Cone and a Hemisphere)

A solid consists of a cone and a hemisphere which share a common base. The cone has a height of 4 cm and a base diameter of 6 cm.



(a) Find
- **(i)** the volume,
- **(ii)** the total surface area,

of the solid.

(b) The solid is melted and recast to form a solid cylinder with a height of 4 cm. Find the radius of the cylinder.

(c) If 1000 identical cylinders are to be painted and each tin of paint is enough to paint an area of 5 m^2, find the number of tins of paint needed.

Solution:

(a) Radius of cone = radius of hemisphere
$= 6 \div 2$
$= 3$ cm

(i) Volume of solid = volume of cone + volume of hemisphere

$$= \frac{1}{3} \times \pi \times 3^2 \times 4 + \frac{1}{2} \times \frac{4}{3} \times \pi \times 3^3$$

$$= 12\pi + 18\pi$$

$$= 30\pi$$

$$= 94.2 \text{ cm}^3 \text{ (to 3 s.f.)}$$

(ii) Using Pythagoras' Theorem,

Slant height of cone $= \sqrt{3^2 + 4^2}$

$= 5$ cm

Total surface area of solid = curved surface area of cone + curved surface area of hemisphere

$= \pi \times 3 \times 5 + 2 \times \pi \times 3^2$

$= 15\pi + 18\pi$

$= 33\pi$

$= 104$ cm² (to 3 s.f.)

(b) Volume of cylinder $= \pi r^2(4) = 30\pi$

$4\pi r^2 = 30\pi$

$r^2 = \frac{30}{4}$

$= \frac{15}{2}$

$\therefore r = \sqrt{7.5}$

$= 2.739$ cm (to 4 s.f.)

$= 2.74$ cm (to 3 s.f.)

(c) Surface area of one cylinder $= 2 \times \pi \times \left(\sqrt{7.5}\right)^2 + 2 \times \pi \times 2.739 \times 4$

$= 15\pi + 21.912\pi$

$= 36.912\pi$

$= 116.0$ cm² (to 4 s.f.)

Surface area of 1000 cylinders $= 116\,000$ cm²

$= 11.60$ m² (to 4 s.f.)

Since $\frac{11.60}{5} = 2.32$, then 3 tins of paint are needed to paint 1000 cylinders.

ATTENTION

We cannot round 2.32 off to the nearest whole number because 2 tins of paint are not sufficient to paint 1000 cylinders.

PRACTISE NOW 13

SIMILAR QUESTIONS

Exercise 12D Questions 3–4, 7–10

A solid consists of a cone and a hemisphere which share a common base. The solid has a height of 50 cm and the hemisphere has a diameter of 30 cm.

(a) Find

(i) the volume,

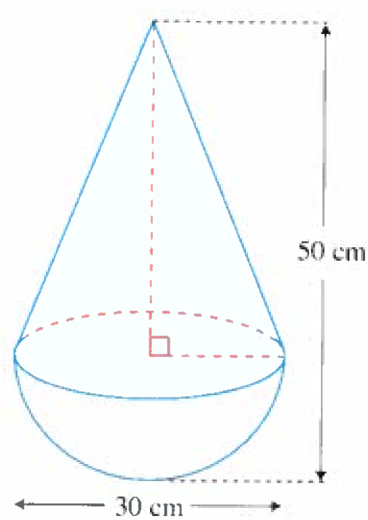
(ii) the total surface area,

of the solid.

(b) The solid is melted and recast to form a solid cylinder with a radius of 12.5 cm. Find

(i) the height of the cylinder,

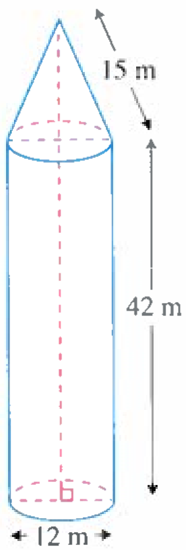
(ii) the surface area of the cylinder, leaving your answer in terms of π.



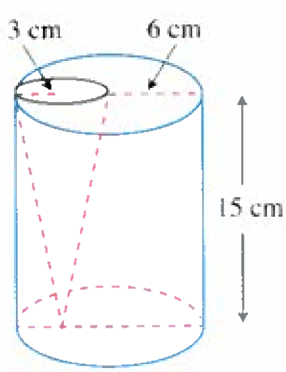
Exercise 12D

BASIC LEVEL

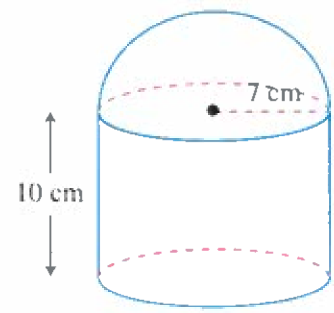
1. A rocket in the shape of a cone is attached to a cylinder with the same base radius. The cone has a slant height of 15 m. The cylinder has a base diameter of 12 m and a height of 42 m. Find the total surface area of the rocket.



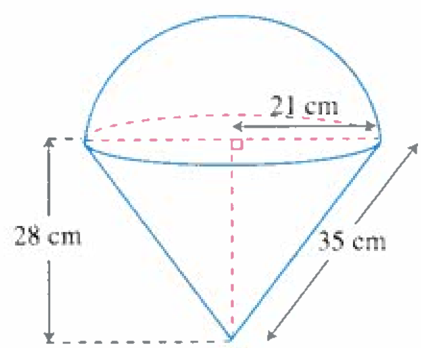
2. A cylinder has a radius of 6 cm and a height of 15 cm. A hole in the shape of a cone is bored into one of its ends. If the cone has a radius of 3 cm, find the volume of the remaining solid.



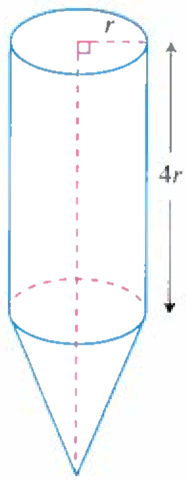
3. A solid consists of a hemisphere and a cylinder which share a common base. The cylinder has a base radius of 7 cm and a height of 10 cm. Find
(i) the volume,
(ii) the total surface area,
of the solid.



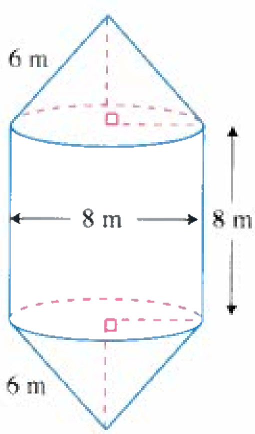
4. A solid consists of a hemisphere and a cone which share a common base. The cone has a base radius of 21 cm, a height of 28 cm and a slant height of 35 cm. Find
(i) the volume,
(ii) the total surface area,
of the solid.



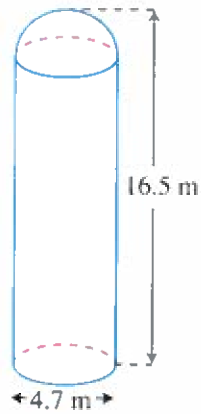
5. A container is made up of a hollow cone with an internal base radius of r cm and a hollow cylinder with the same base radius and an internal height of $4r$ cm. Given that the height of the cone is three-fifths of the height of the cylinder and 7 litres of water is needed to fill the conical part of the container completely, find the amount of water needed to fill the container completely, giving your answer in litres.



6. Find
(i) the total surface area,
(ii) the volume,
of the solid cylinder with conical ends as shown.



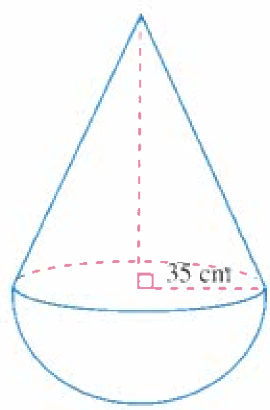
7. A storage tank consists of a hemisphere and a cylinder which share a common base. The tank has a height of 16.5 m and the cylinder has a base diameter of 4.7 m. Find the capacity of the tank.



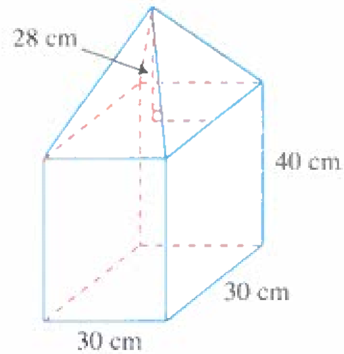
8. A solid metal ball of radius 3 cm is melted and recast to form a solid circular cone of radius 4 cm. Find the height of the cone.

ADVANCED LEVEL

9. A solid consists of a cone and a hemisphere which share a common base. The cone has a base radius of 35 cm. Given that the volume of the cone is equal to $1\frac{1}{5}$ of the volume of the hemisphere, find
(i) the height of the cone,
(ii) the total surface area of the solid, leaving your answer in terms of π.



10. A solid consists of a pyramid of height 28 cm attached to a cuboid with a square base of sides 30 cm and a height of 40 cm. Find
(i) the volume,
(ii) the total surface area,
of the solid.



1. For a pyramid and its *corresponding* prism with the *same base* and the *same height*,

Volume of pyramid $= \frac{1}{3} \times$ volume of *corresponding* prism

$= \frac{1}{3} \times$ base area $\times$ height

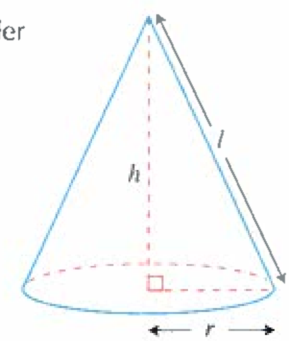
2. Total surface area of pyramid = total area of all faces

3. For a cone and its corresponding cylinder with the *same base radius r* and the *same height h*,

Volume of cone $= \frac{1}{3} \times$ volume of corresponding cylinder

$= \frac{1}{3} \times$ base area $\times$ height

$= \frac{1}{3}\pi r^2 h$



4. For a cone with base radius r and slant height l,

Curved surface area of cone $= \pi rl$

Total surface area of cone $= \pi rl + \pi r^2$

$= \pi r(l + r)$

5. For a sphere with radius r,

Volume of sphere $= \frac{4}{3}\pi r^3$

Surface area of sphere $= 4\pi r^2$

ATTENTION

Just remember 'volume of pyramid $= \frac{1}{3} \times$ volume of prism' and 'volume of prism = base area × height'. These formulae are similar to the formulae for finding the volume of a cone and the volume of a cylinder.

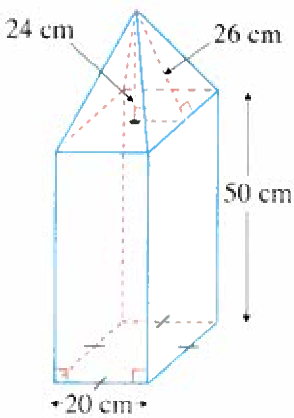
ATTENTION

Just remember the units of volume and area are, e.g. cm^3 and cm^2 respectively, so the volume of a sphere involves r^3 and the surface area of a sphere involves r^2.

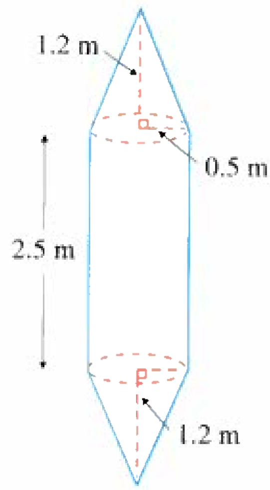
Review Exercise 12

1. For each of the following solids, find
   (i) the volume,
   (ii) the total surface area,
   of the solid.

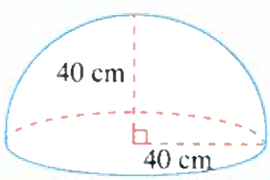
(a)



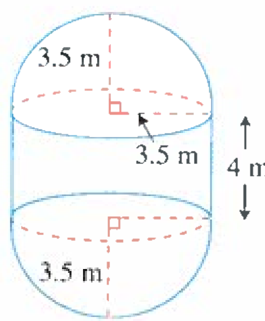
(b)



(c)



(d)

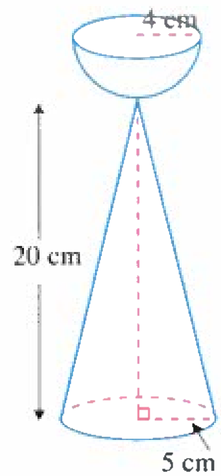


2. The figure shows a decorative structure made up of a solid cone and a solid hemisphere. The hemisphere has a radius of 4 cm. The cone has a base radius of 5 cm and a height of 20 cm. Find

(i) the volume,

(ii) the total surface area,

of the structure.



3. A solid metal model of a rocket is made up of a cone attached to a cylinder with the same base radius. The cone has a height of 49 cm. The cylinder has a base radius of 18 cm and a height of 192 cm. Given that the mass of the model rocket is 2145 kg, find the density, in kg/m^3, of the metal which the model rocket is made of, giving your answer correct to the nearest whole number. (Density $= \frac{\text{Mass}}{\text{Volume}}$)

4. Two solid spheres have surface areas of 144π cm^2 and 256π cm^2 respectively. They are melted and recast to form a larger sphere. Find the surface area of the larger sphere in square centimetres.

5. A hollow metal sphere has an external diameter of 12 cm and a thickness of 2 cm.

(i) Given that the mass of 1 cm^3 of the metal is 5.4 g, find the mass of the hollow sphere in kg.

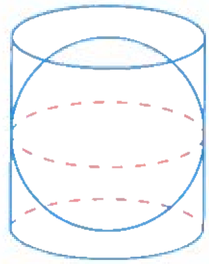
(ii) The hollow sphere is melted and recast to form a solid sphere. Find the radius of the solid sphere.

6. In an experiment, a small spherical drop of oil is allowed to fall onto the surface of water so that it produces a thin film of oil covering a large area. The volume of one drop of oil is 12.5 mm^3. Find

(i) the number of drops which can be produced by 5000 mm^3 of oil,

(ii) the radius of one drop of oil.

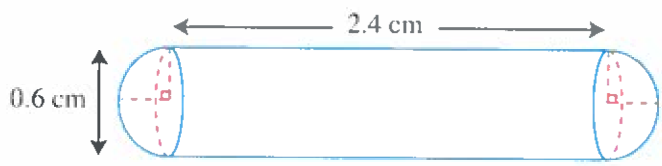
7. A sphere fits exactly into a cylinder, i.e. the sphere touches the cylinder at the top, the bottom and the curved surface. Show that the surface area of the sphere is equal to the area of the curved surface of the cylinder.



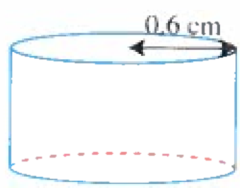
8. A house has a hemispherical roof 10 m in diameter. Find the cost of painting the curved surface of the roof at \$1.50 per square metre.

9. A metal hemispherical bowl has an external diameter of 50.8 cm and a thickness of 2.54 cm.
   (i) Given that the empty bowl weighs 97.9 kg, find the density, in kg/m^3, of the metal which the bowl is made of.
   (ii) If the bowl is completely filled with a liquid of density 31.75 kg/m^3, find the mass of the liquid in grams.

10. Capsule *A* is 2.4 cm long and is in the shape of a cylinder with hemispheres of diameter 0.6 cm attached to both ends. Capsule *B* is in the shape of a cylinder of radius 0.6 cm.
   (i) Given that the surface area of Capsule *B* is equal to that of Capsule *A*, find the height of Capsule *B*.
   (ii) Find the volume of each capsule.

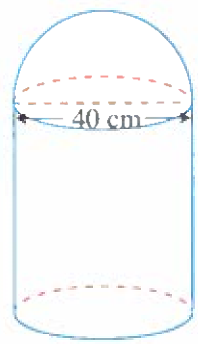


Capsule *A*



Capsule *B*

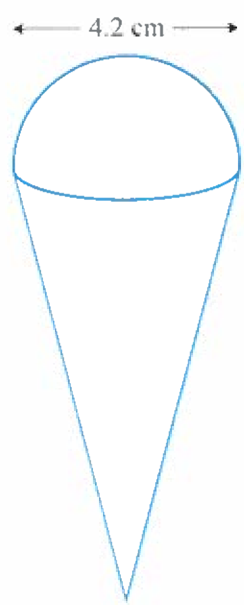
11. The figure shows a solid cylindrical stone pillar with a hemispherical top. The pillar has a diameter of 40 cm. If the pillar has the same mass as a solid stone sphere of the same material of radius 40 cm, find the height of the pillar.



12. A cylinder and a cone are of the same height $2r$ and of the same base diameter $2r$. A sphere has a diameter of $2r$. Find the ratio of the volume of the cylinder to that of the cone and to that of the sphere.

13. During a cooking contest, Huixian melts 40 solid chocolate hemispheres, each of diameter 2 cm, to form a solid chocolate cone with a base diameter of 6 cm. Find the height of the chocolate cone.

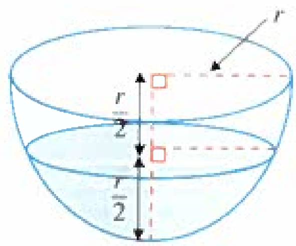
14. A wafer cone is completely filled with 56 cm^3 of ice cream. The top of the ice cream cone is in the shape of a hemisphere. Given that the diameter of the cone is 4.2 cm, find its height.



Challenge Yourself

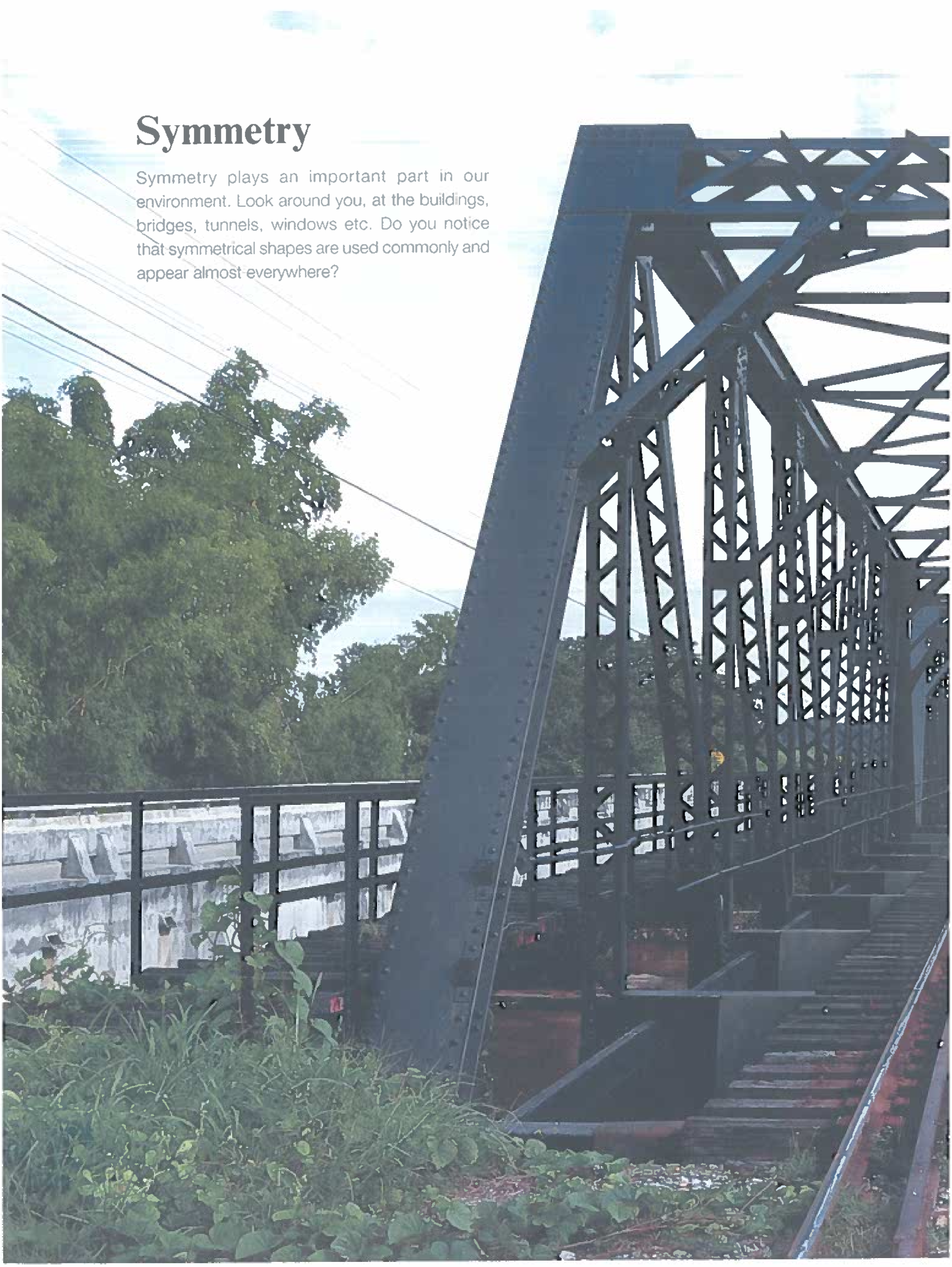
1. A regular tetrahedron has a volume of 500 cm^3. Find its total surface area.

2. The figure shows a bowl in the shape of a hemisphere with a radius of r cm. It is filled with water to a depth of $\frac{r}{2}$ cm. Show that the volume of the water is *more* than $\frac{1}{8}$ of the volume of the bowl.

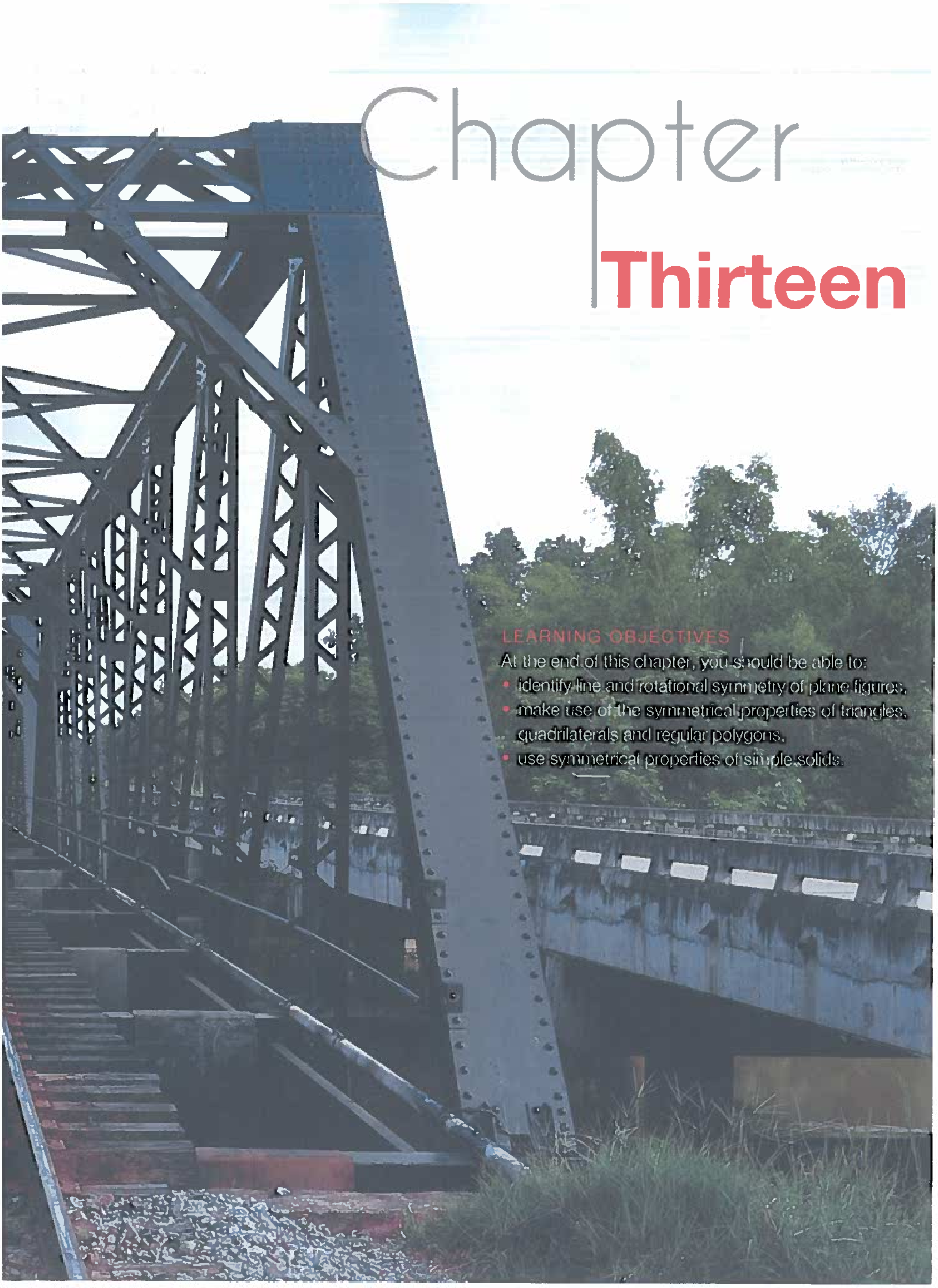


Symmetry

Symmetry plays an important part in our environment. Look around you, at the buildings, bridges, tunnels, windows etc. Do you notice that symmetrical shapes are used commonly and appear almost everywhere?



Chapter Thirteen

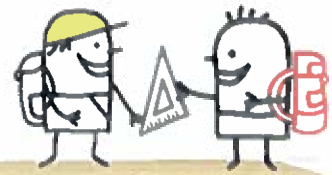


LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- identify line and rotational symmetry of plane figures,
- make use of the symmetrical properties of triangles, quadrilaterals and regular polygons,
- use symmetrical properties of simple solids.

13.1 Line Symmetry



Investigation

Line Symmetry in Two Dimensions

1. Fold a sheet of paper into two equal parts and mark two points, *X* and *Y*, on the folded line (see Fig. 13.1**(a)**).

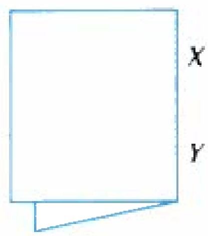


Fig. 13.1(a)

2. Use a pair of scissors to cut out a shape, starting at *X* and ending at *Y* (see Fig.13.1**(b)**).

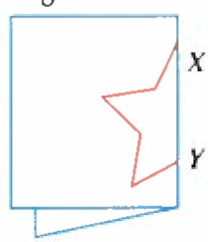


Fig. 13.1(b)

3. Unfold the paper. Do you get a shape similar to the one shown in Fig. 13.1**(c)** when you cut along the lines shown?

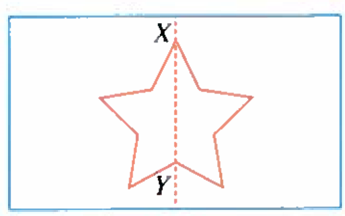


Fig. 13.1(c)

4. The "half-shapes" on both sides of the line *XY* in Fig. 13.1**(c)** are the same. The shape cut out is said to be symmetrical about the line *XY*. The line *XY* is known as the **line of symmetry** or the **axis of symmetry**.

5. Place a small thin mirror along *XY* in Fig. 13.1**(c)**. What do you notice about the reflected image and the actual figure?

From the investigation, we conclude that the line of symmetry or axis of symmetry is the same as the *line of reflection* or *axis of reflection* as we have learnt earlier in Chapter 9 on geometrical transformation. When an object is symmetrical, we can see that its two halves are **mirror images**, also referred to as *reflections*, of each other. We can also see that the symmetry of a figure about an axis may be tested by folding it along that axis.

If the figure is symmetrical, the part to the left of the axis will fit exactly onto the part to the right of the axis and vice versa (see Fig. 13.1 (**c**))

Symmetry Around Us

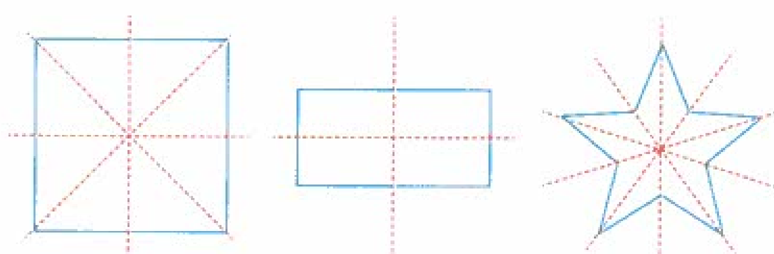
Many things around us are symmetrical. Fig. 13.2 shows some common things that often have symmetrical shapes.

Fig. 13.2

Thinking Time

Are human beings symmetrical? Do you know of any animal that is not symmetrical?

A figure may have more than one line of symmetry. For example, the square shown in Fig. 13.3(**a**) has 4 lines of symmetry, the rectangle in Fig. 13.3(**b**) has 2 lines of symmetry and the star shape shown in Fig. 13.3(**c**) has 5 lines of symmetry.



(**a**) (**b**) (**c**)

Fig. 13.3

Thinking Time

Can you draw a triangle that has exactly two lines of symmetry?

Recap (Mirror Image of a Point and of a Line)

In Chapter 9, we have learnt how to find the mirror image of a point and a line.

Consider a line XY on a plane. To find the mirror image of a point P with respect to XY, we draw a line PM perpendicular to XY and produce PM to P' so that $PM = MP'$ as shown in Fig. 13.4. P' is the mirror image of P with respect to the line XY.

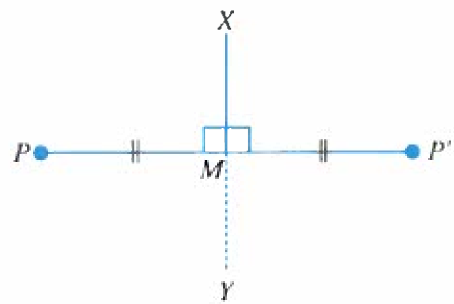


Fig. 13.4

We have also learnt that if P lies on XY, the mirror image of P with respect to XY is P itself and P will be known as the invariant point.

To find the mirror image of line PQ with respect to XY, we locate the images of P and Q. They are denoted by P' and Q' respectively as shown in Fig. 13.5. The line $P'Q'$ is the image of PQ with respect to XY where $PQ = P'Q'$ and $Q\hat{R}S = Q'\hat{R}S$.

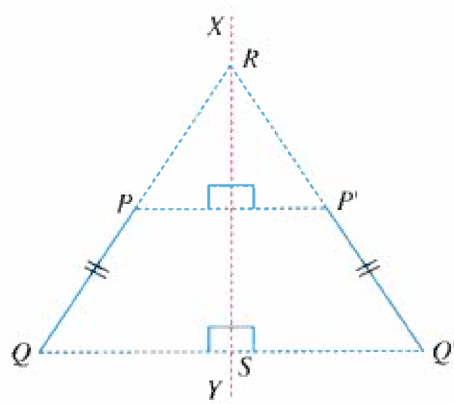
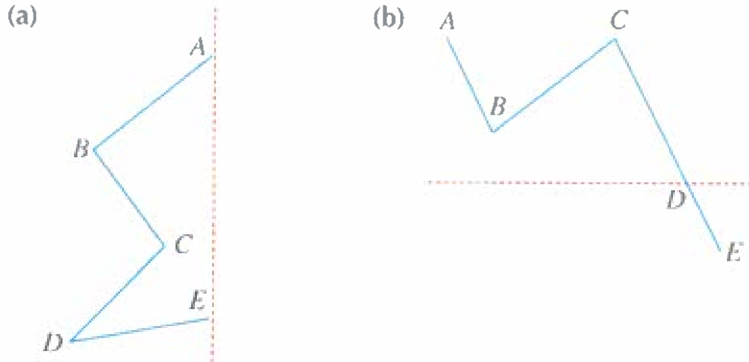


Fig. 13.5

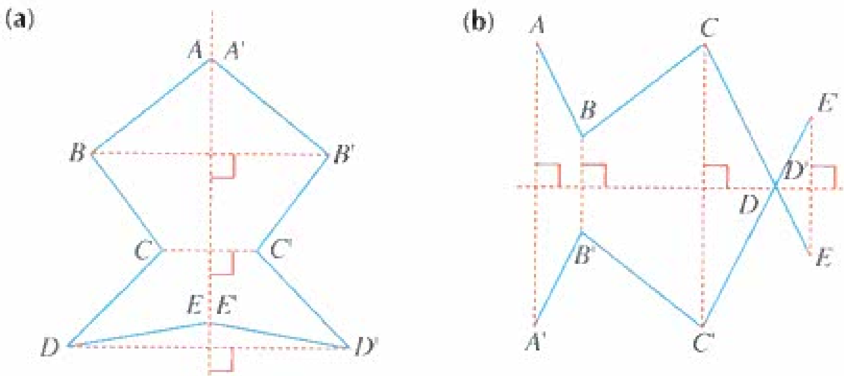
Worked Example 1

(Problem involving Line Symmetry)

Copy the following diagrams and make each of them symmetrical about the dotted line.



Solution:

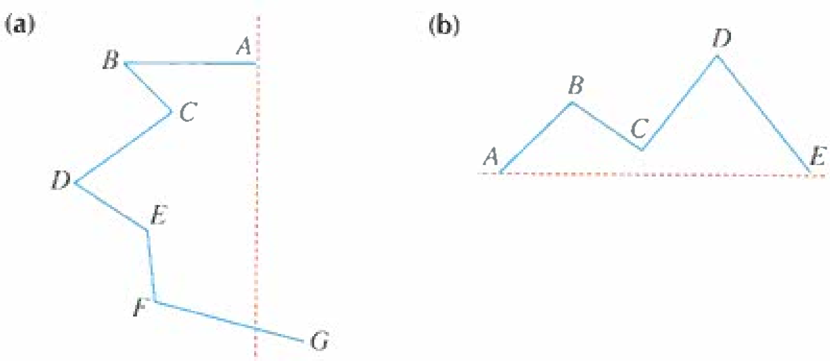


PRACTISE NOW 1

SIMILAR QUESTIONS

Exercise 13A Questions 1–5, 7, 9–11

Copy the following diagrams and make each of them symmetrical about the dotted line.

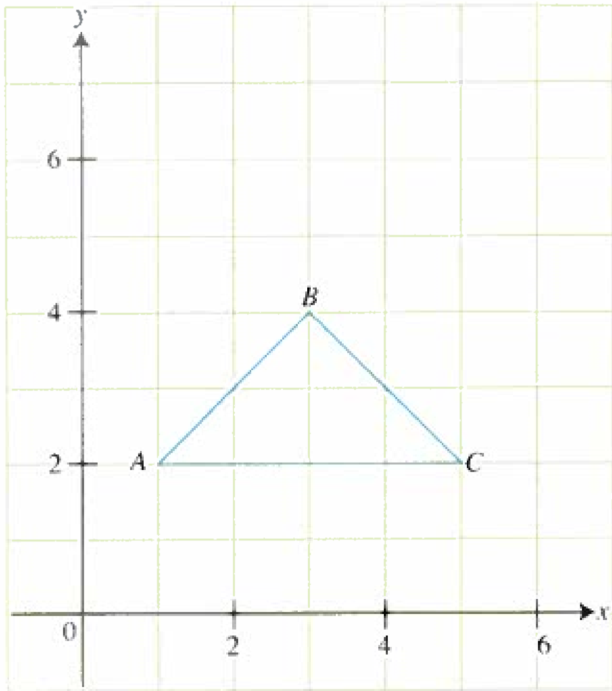


Worked Example 2

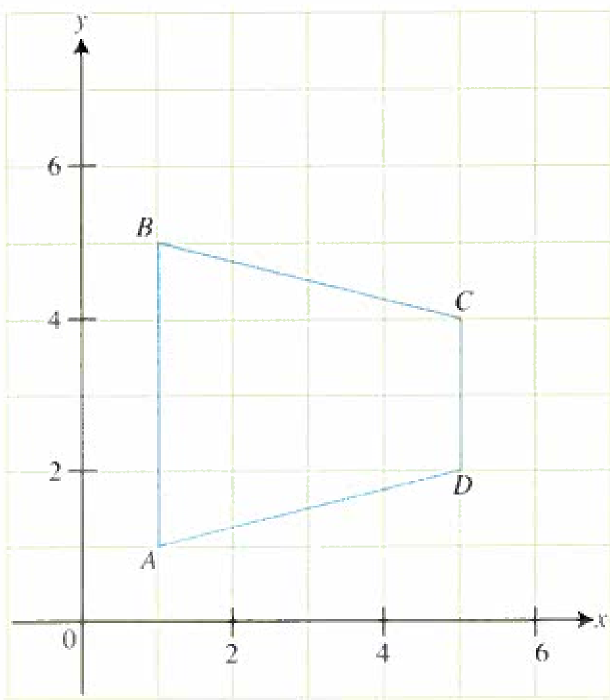
(Problem involving Line Symmetry)

Each of the following figures has only one line of symmetry. Draw the line of symmetry and write down the equation of the line of symmetry.

(a)

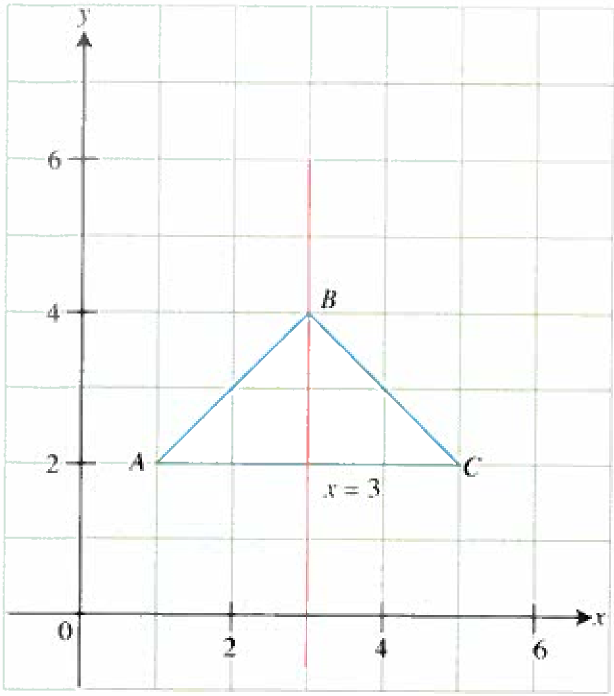


(b)



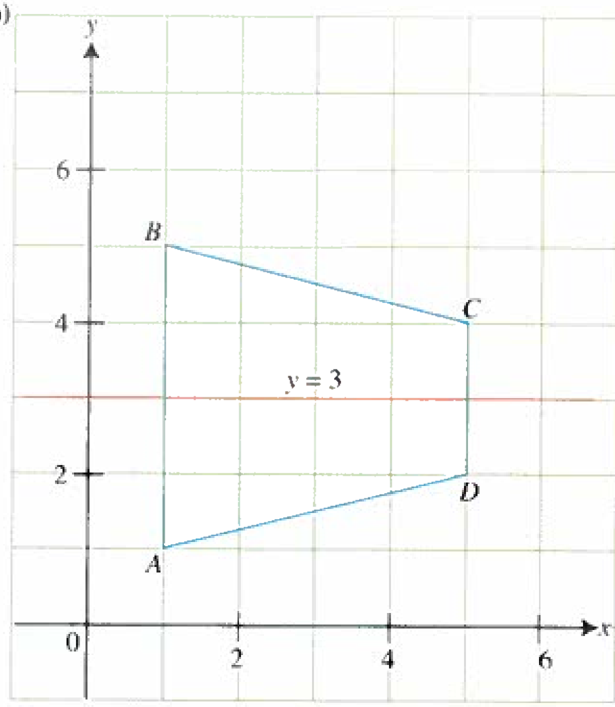
Solution:

(a)



The equation of the line of symmetry is $x = 3$.

(b)



The equation of the line of symmetry is $y = 3$.

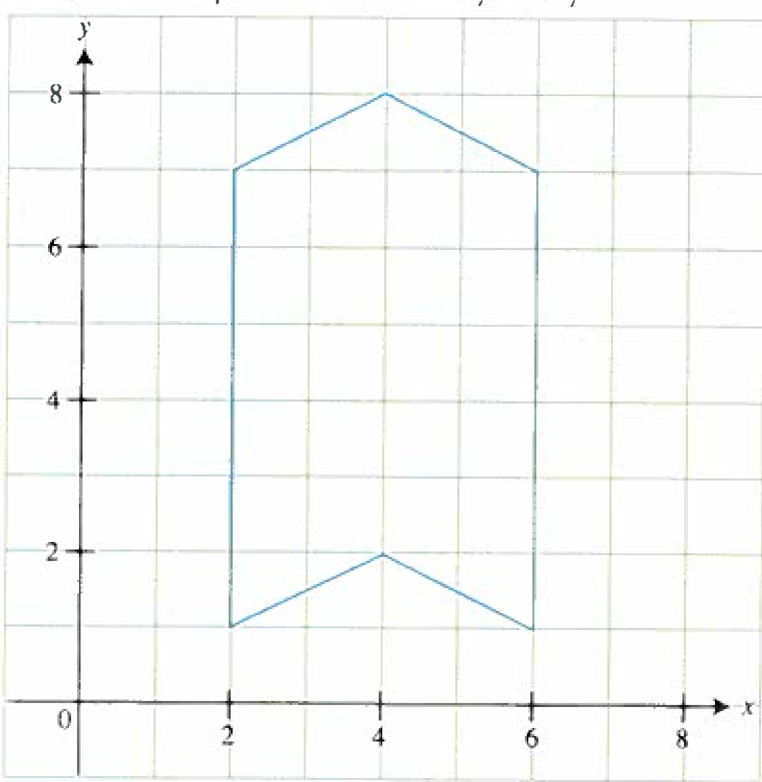
PRACTISE NOW 2

SIMILAR QUESTIONS

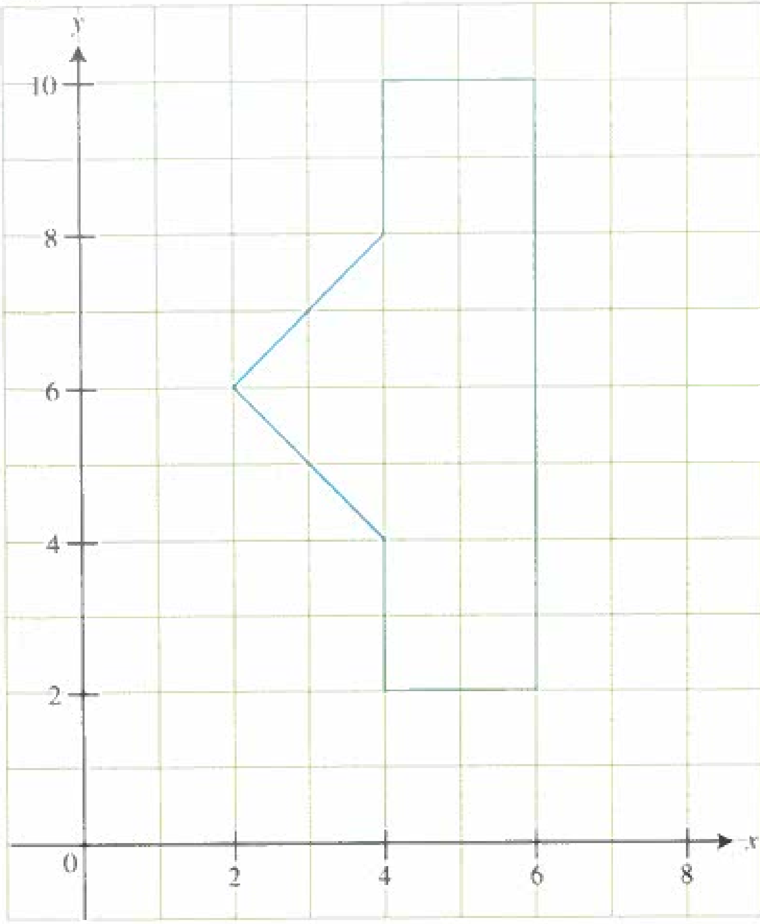
Exercise 13A Questions 6, 8

Each of the following figures has only one line of symmetry. Draw the line of symmetry and write down the equation of the line of symmetry.

(a)



(b)



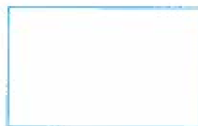
Exercise 13A

BASIC LEVEL

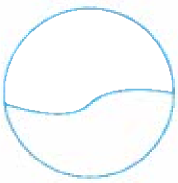
1. Which of the following shapes have no lines of symmetry at all?

(a)

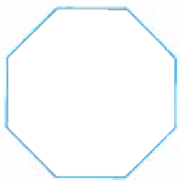
(b)



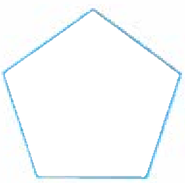
(c)



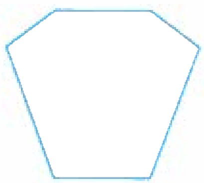
(d)



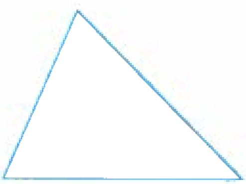
(e)



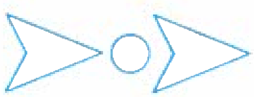
(f)



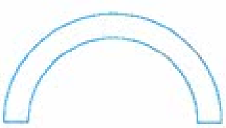
(g)



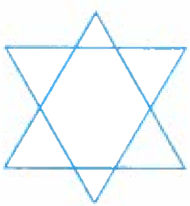
(h)



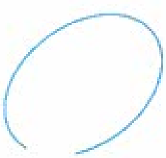
(i)



(j)



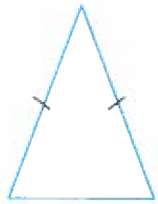
(k)



(l)

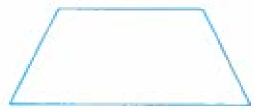
2. How many lines of symmetry are there in each of the following shapes?

(a)

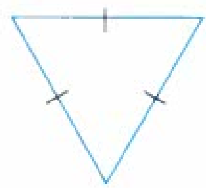


(b)

(c)

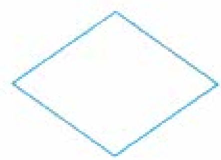


(d)

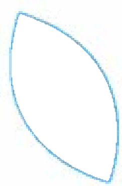


(e)

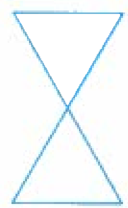
(f)



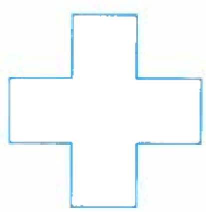
(g)



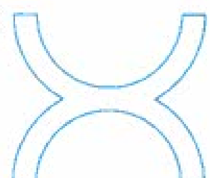
(h)



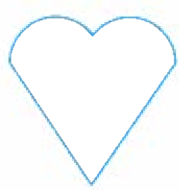
(i)



(j)



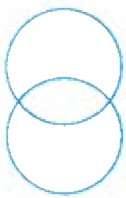
(k)



3. Show with dotted lines the line of symmetry of each of the following capital letters, where it exist.

B	**I**	**P**	**X**
C	**J**	**Q**	**Y**
D	**K**	**R**	**Z**
F	**L**	**U**	**G**
O	**V**		

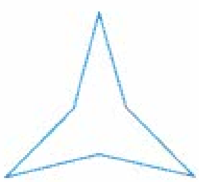
4. Copy the following diagrams and draw the axis of symmetry for each diagram. How many lines of symmetry does each of them have?

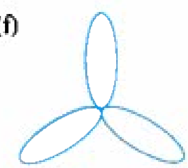
(a) 

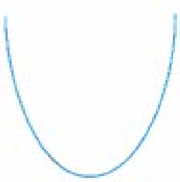
(b) 

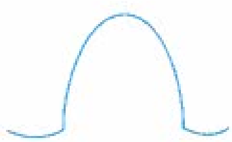
(c) 

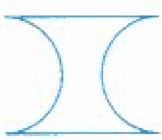
(d) 

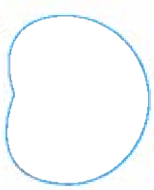
(e) 

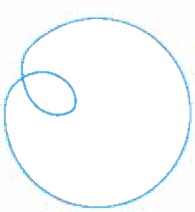
(f) 

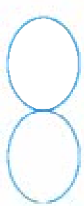
(g) 

(h) 

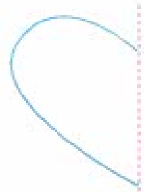
(i) 

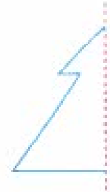
(j) 

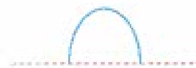
(k) 

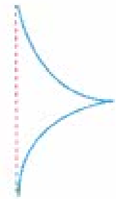
(l) 

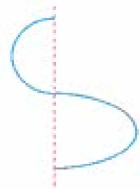
5. Copy the following diagrams and make each of them symmetrical about the dotted line.

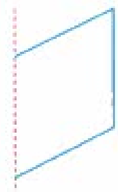
(a) 

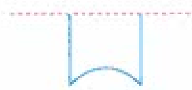
(b) 

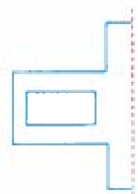
(c) 

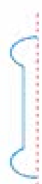
(d) 

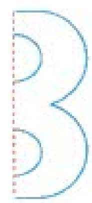
(e) 

(f) 

(g) 

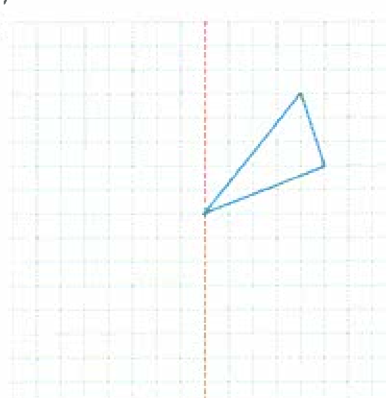
(h) 

(i) 

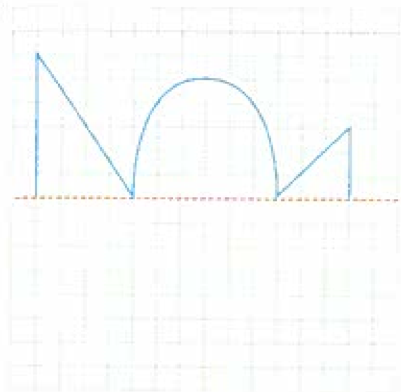
(j) 

INTERMEDIATE LEVEL

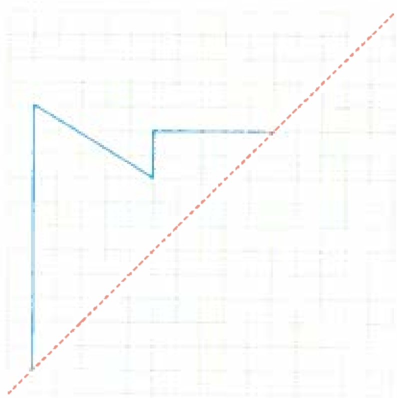
6. Complete each of the following diagrams to make it symmetrical about the dotted line.

(a) 

(b)

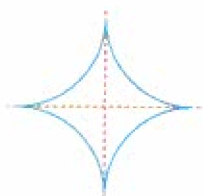


(c)

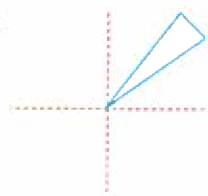


7. Copy the following diagrams and make each of them symmetrical about the two dotted lines. ((a) has been done for you.)

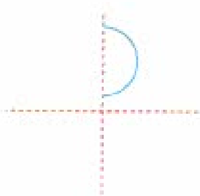
(a)



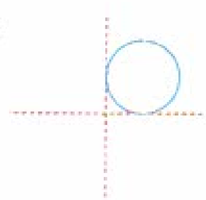
(b)



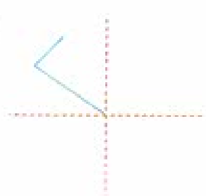
(c)



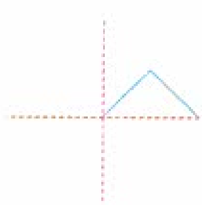
(d)



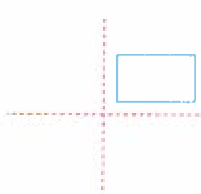
(e)



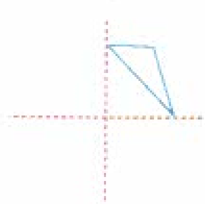
(f)



(g)

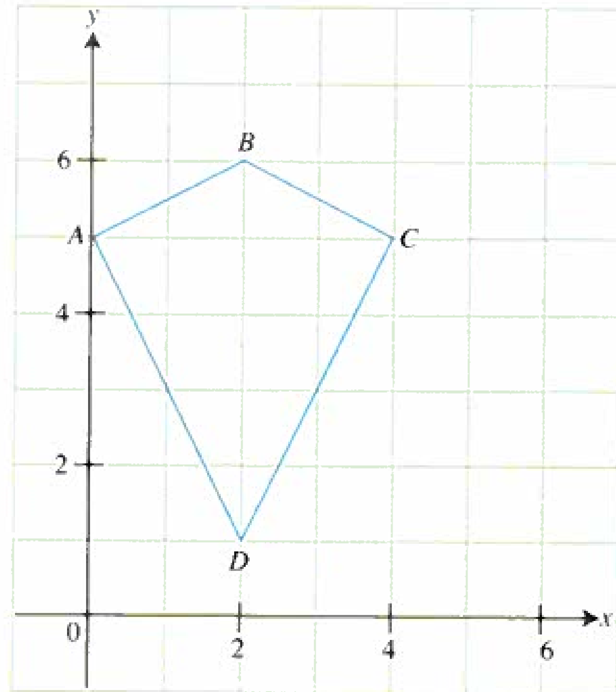


(h)

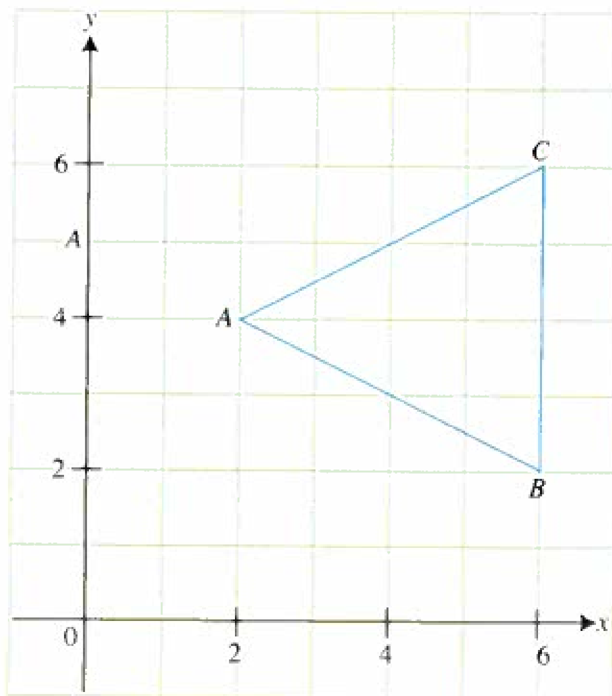


8. Each of the following figures has one line of symmetry. Draw the line of symmetry and write down the equation of the line of symmetry.

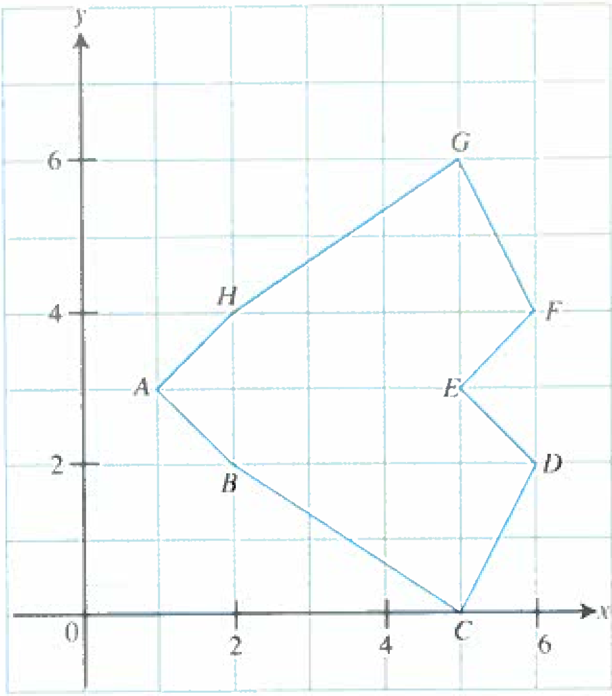
(a)



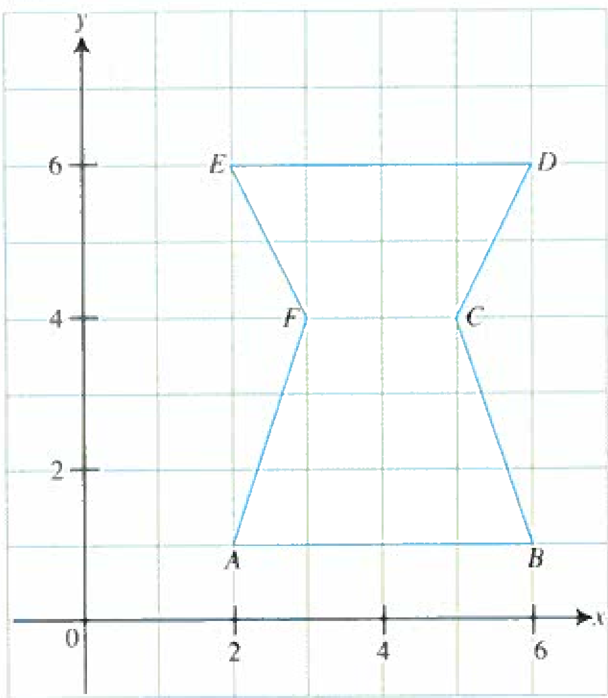
(b)



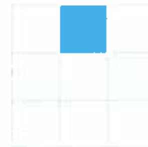
(c)



(d)



9. Shade in one small square on the diagram so that the resulting diagram has 2 lines of symmetry.



10. Add one square to the figure so that the resulting diagram has one line of symmetry.

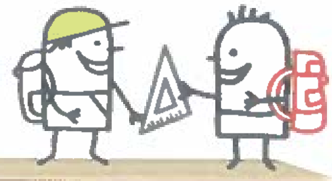
ADVANCED LEVEL

11. Study the eight capital letters below.

NEW MATHS

(a) Which letter(s) have a vertical line of symmetry?
(b) Which letter(s) have a horizontal line of symmetry?
(c) Which letter(s) have two lines of symmetry?
(d) Which letter(s) are not symmetrical?
(e) Copy the letter(s) above which are symmetrical and draw lines of symmetry on each one.

13.2 Rotational Symmetry in Plane Figures



Investigation

Rotational Symmetry in Two Dimensions

1. Trace Fig. 13.6 on two separate pieces of tracing paper. Place one piece of the paper exactly on top of the other such that the two figures coincide, and then place a thumbtack at the centre of the figure.

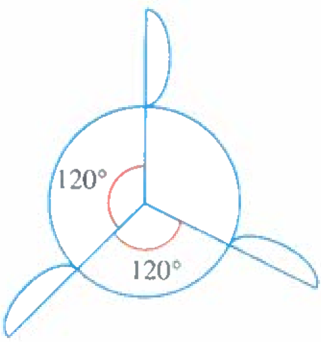


Fig. 13.6

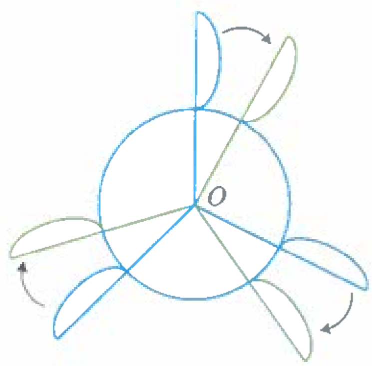
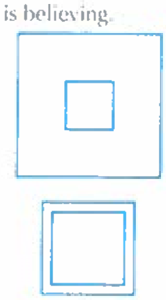


Fig. 13.7

2. Rotate the top tracing paper clockwise (see Fig. 13.7) until the two figures totally coincide again. What fraction of a complete turn has been made?
3. Rotate the top figure again until it covers the lower figure. What fraction of a complete turn has been made?
4. How many thirds of a turn have to be made before the top figure is to be back in its original position?

When a figure can be rotated to fit the outline of its original position, we say that the figure possesses **rotational symmetry**. The point *O* where we place the thumbtack is called the centre of rotation. Since the figure can be rotated 3 times to fit the original figure, the figure is said to have rotational symmetry of order 3, or the order of rotational symmetry is 3.

Seeing is believing.



Which inner square is larger?

From the investigation, we note that a point is the **centre of rotation** of a figure if the figure maps onto itself under rotation about the point and that the **order of rotational symmetry** is the number of distinct ways in which a figure can map onto itself by rotation.

The following shows figures with different orders of rotational symmetry:

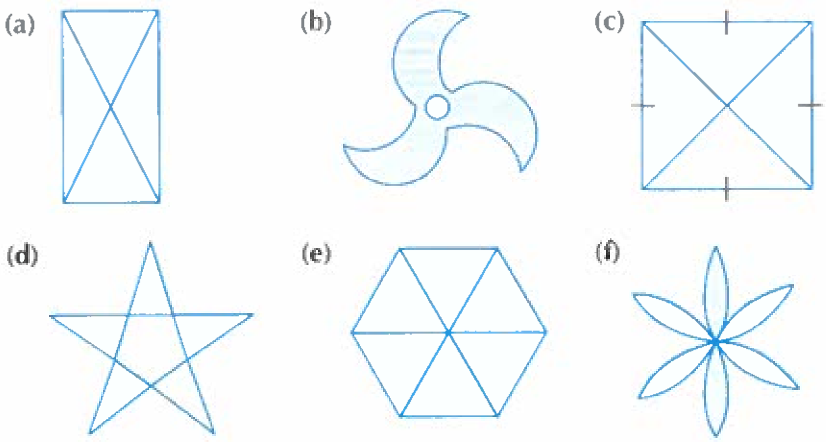
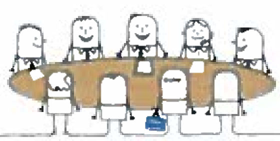


Fig. 13.8

INFORMATION

In the case of order 1, we say that there is no rotational symmetry.

If an identical figure can be obtained only by turning the original figure through 360°, we say that the order of rotational symmetry is 1. Hence, every figure has at least an order of rotational symmetry 1.



Class Discussion

Line and Rotational Symmetry in Circles

Work in pairs. Using paper cut-outs of a circle, fold or rotate to explore its symmetry.

1. How many ways can you fold a circle into half? Deduce the number of lines of symmetry in a circle.
2. Does the circle have rotational symmetry? If so, what is the order of rotational symmetry?

From the class discussion, we see that both sides of a circle will match up when we fold the circle over any line through its centre and that it maps onto itself at every angle. We can conclude that the circle is a special figure with infinite lines of symmetry and that its order of rotational symmetry is also infinite.

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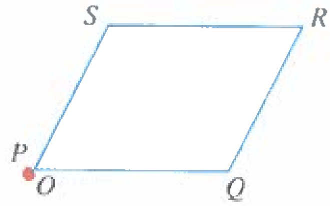
Worked Example 3

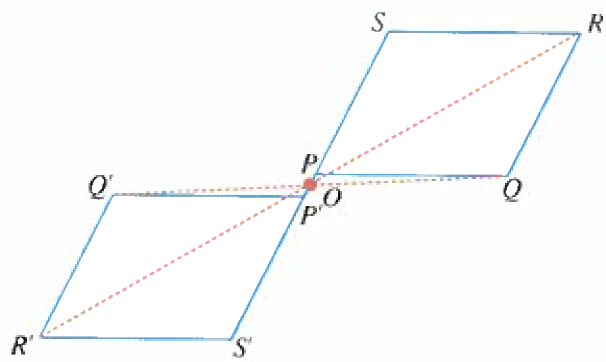
(Problem involving Rotational Symmetry)

The incomplete figure has rotational symmetry of order 2, with O as the centre of rotation. Complete the figure by drawing the other half.



Solution:

1. We start by locating the image of S with respect to O.
2. Produce SO to S' such that $SO = S'O$. S' is the image of S with respect to O. Mark the point P' such that $PO = P'O$. P' is the image of P with respect to O. We continue in the same way to obtain the images of R and Q which are denoted by R' and Q' respectively.
3. Join $P'Q'$, $Q'R'$, $R'S'$ and $S'P'$ to complete the figure which has rotational symmetry of order 2 as shown in the figure below.

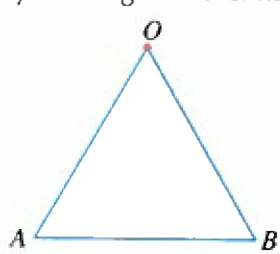


PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 13B Questions 1–5

The incomplete figure has rotational symmetry of order 2, with O as the centre of rotation. Complete the figure by drawing the other half.



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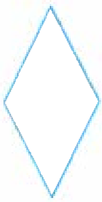
Exercise 13B

BASIC LEVEL

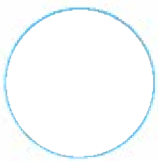
1. Copy the following diagrams and mark the centre of rotation for each diagram.

(a)



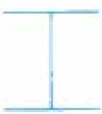
(b)

(c)



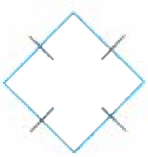
(d)

(e)

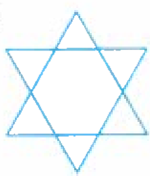


(f)

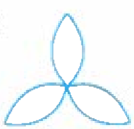
(g)



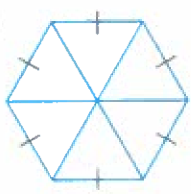
(h)



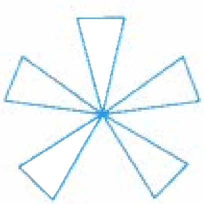
(i)



(j)



2. The object shown has rotational symmetry of order r. Find the value of r.

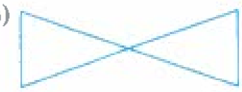


INTERMEDIATE LEVEL

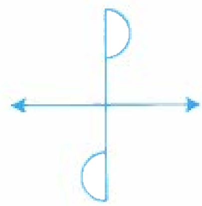
3. For each of the following figures, state (i) the number of lines of symmetry, (ii) the order of rotational symmetry.

(a)

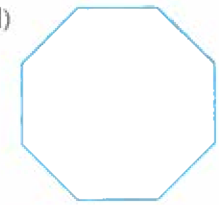
(b)



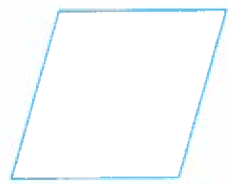
(c)



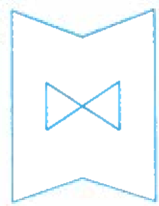
(d)



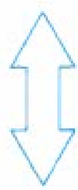
(e)



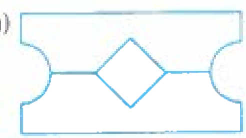
(f)



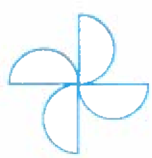
(g)



(h)

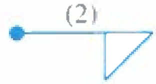


(i)

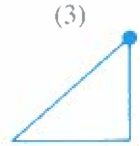


4. Copy the following diagrams and complete each of them so that the point marked becomes the centre of rotation with the order of rotational symmetry as given in the brackets.

(a) (2)

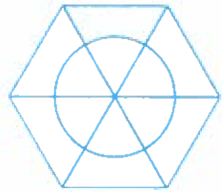


(b) (3)



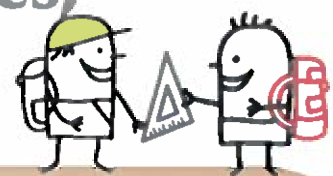
(c) (4)

5. Copy the figure and shade certain parts of the figure so that the resulting figure has rotational symmetry of



(a) order 2,   (b) order 3,   (c) order 6.

13.3 Symmetry in Triangles, Quadrilaterals and Polygons



In Book 1, we have learnt about triangles, quadrilaterals and polygons and some of their properties. In this section, we will look into their symmetric properties.

Symmetry in Triangles and Quadrilaterals

Investigation

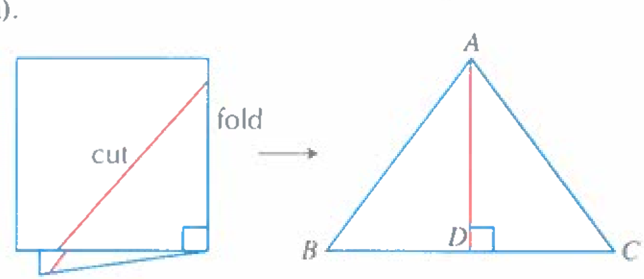
Symmetry in Triangles

In this investigation, we shall take a look at the symmetric properties of triangles.

Using paper cut-outs of the three types of triangles learnt, i.e. equilateral triangle, isosceles triangle and a scalene triangle, fold or rotate them to explore their symmetry.

We shall first take a look at the symmetric properties of an isosceles triangle.

1. Fold a sheet of paper in half and cut off a right-angled triangle as shown in Fig. 13.9(**a**).



(**a**)   (**b**)

Fig. 13.9

RECALL

An isosceles triangle is a triangle with at least 2 equal sides, an equilateral triangle is a triangle with 3 equal sides and a scalene triangle is a triangle with no equal sides.

2. Open up the triangle and we obtain the shape as shown in Fig. 13.9(**b**).
   How many lines of symmetry does the triangle have?
3. From the symmetries, are you able to observe any geometrical relationships? For example, can you say, without measuring, that the two sides *AB* and *AC* are equal and that the two angles *B* and *C* are equal? If so, why?
4. Does the triangle have rotational symmetry? If so, what is its order of rotational symmetry?
5. Repeat this with an equilateral triangle and a scalene triangle.

Table 13.1 gives a summary of the symmetric properties of the triangles which have been covered in the investigation.

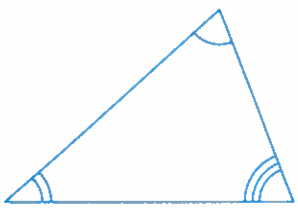
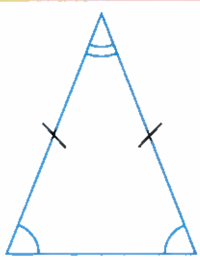
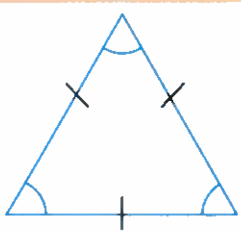
Triangle	Number of lines of symmetry	Order of rotational symmetry
 Scalene triangle	0	1
 Isosceles triangle	1	1
 Equilateral triangle	3	3

Table 13.1

Fig. 13.10 shows an equilateral triangle ABC with AP, BQ and CR as lines of symmetry. If a triangle has three lines of symmetry, the triangle is an equilateral triangle. A proof of the above statement is given as follows:

Consider the triangle in Fig. 13.10

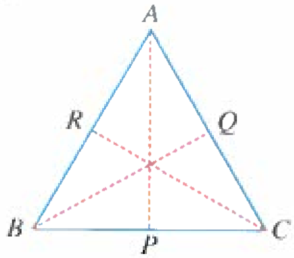


Fig. 13.10

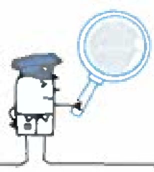
Since RC is a line of symmetry, $\hat{A} = \hat{B}$ and $CA = BC$.

Similarly, $\hat{A} = \hat{C}$ and $AB = BC$ (BQ is a line of symmetry)

$\hat{B} = \hat{C}$ and $AB = CA$ (AP is a line of symmetry)

$AB = BC = CA$ and $\hat{A} = \hat{B} = \hat{C} = \frac{180°}{3} = 60°$

By proving with the same method, we can show that if a triangle has only one line of symmetry, then the triangle is an isosceles triangle.



Investigation

Symmetry in Special Quadrilaterals

In this investigation, we shall take a look at the symmetric properties of quadrilaterals.

Let us now consider a rectangle $ABCD$.

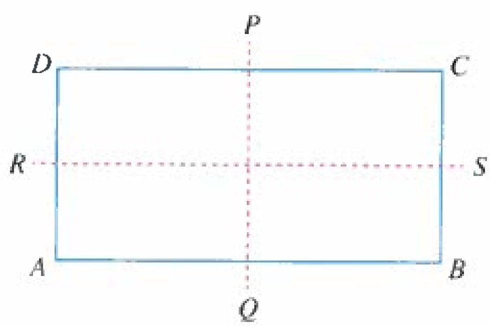


Fig. 13.11

Since each of the lines, PQ and RS, divides the rectangle into two identical halves, the rectangle $ABCD$ has 2 lines of symmetry.

Now, let us consider the same rectangle $ABCD$ where the diagonals AC and BD intersect at E.

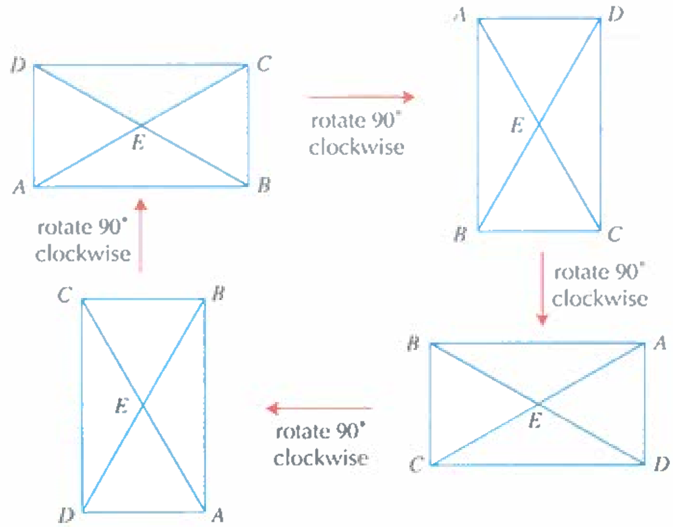


Fig. 13.12

Since the rectangle maps onto itself twice in a 360° rotation about E, the order of rotational symmetry of the rectangle is 2.

In Book 1, we learnt about trapeziums. Now we shall look into a specific type of trapezium.

Fig. 13.13 shows an isosceles trapezium.

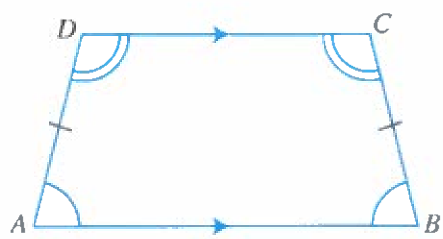


Fig. 13.13

An isosceles trapezium is a quadrilateral which has one pair of parallel sides ($AB // CD$), one pair of equal sides ($AD = BC$) and two pairs of equal angles ($A\hat{B}C = B\hat{A}D$ and $A\hat{D}C = B\hat{C}D$).

Consider a square, parallelogram, rhombus, isosceles trapezium and a kite.
Using paper cut-outs of these quadrilaterals, fold or rotate them to explore their symmetry. Discuss with your classmates the following questions:

(a) How many lines of symmetry are there in the quadrilateral?

(b) Does the quadrilateral have rotational symmetry? If so, what is its order of rotational symmetry?

(c) From the symmetries, are you able to observe any geometrical relationships? For example, from the line of symmetry of a kite, we can see which angles are equal.

Table 13.2 gives a summary of the symmetric properties of the special quadrilaterals which have been covered in the investigation.

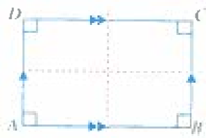
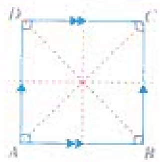
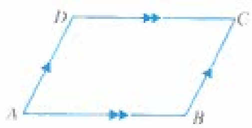
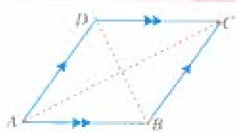
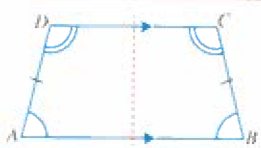
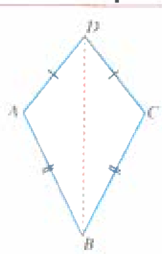
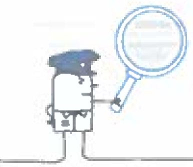
Quadrilateral	Number of lines of symmetry	Order of rotational symmetry
 Rectangle	2	2
 Square	4	4
 Parallelogram	0	2
 Rhombus	2	2
 Isosceles trapezium	1	1
 Kite	1	1

Table 13.2

Symmetry in Regular Polygons



Investigation

Symmetry in Regular Polygons

In this investigation, we shall deduce a general expression for the number of lines of symmetry and order of rotational symmetry of an n-sided polygon.

Fig. 13.14 shows some regular polygons.

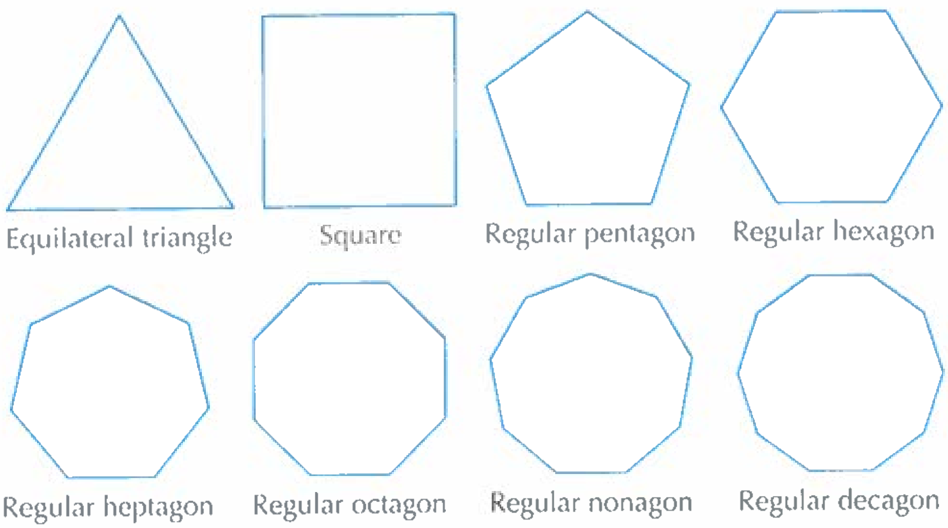
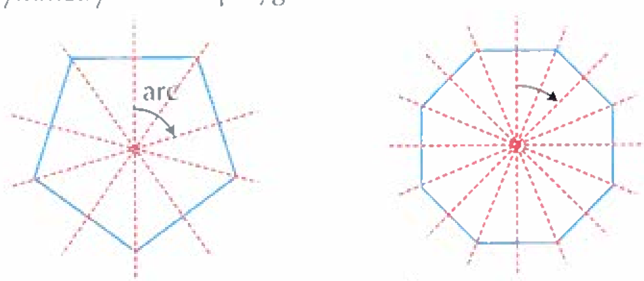


Fig 13.14

RECALL

A regular polygon is a polygon with all sides equal and all angles equal.

1. Copy the polygons in Fig. 13.14 and use a colour pencil to mark their lines of symmetry.
2. Use a different colour pencil to mark the centres of rotational symmetry. Draw arcs showing the smallest angle of rotational symmetry. Fig. 13.15**(a)** and Fig. 13.15**(b)** are examples of what is to be done. Measure the smallest angle of rotational symmetry for each polygon.



(a) Regular pentagon (b) Regular octagon

Fig. 13.15

3. Complete Table 13.3.

Types of quadrilateral	Number of lines of symmetry	Order of rotational symmetry
Equilateral triangle		
Square		
Regular pentagon	5	$\frac{360°}{72°} = 5$
Regular hexagon		
Regular heptagon		
Regular octagon	8	$\frac{360°}{45°} = 8$
Regular nonagon		
Regular decagon		

Table 13.3

4. Deduce

(a) the number of lines of symmetry, and

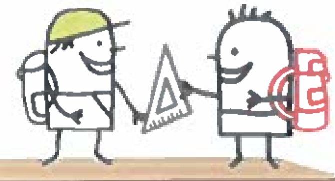
(b) the order of rotational symmetry of a regular polygon with n sides.

In the case of a regular pentagon, the smallest angle of rotation for which the pentagon can map onto itself is $72° = \frac{360°}{5}$. For a regular octagon, the smallest angle of rotational symmetry is $45° = \frac{360°}{8}$.

5. What is the smallest angle of rotational symmetry for a general regular polygon with n sides?

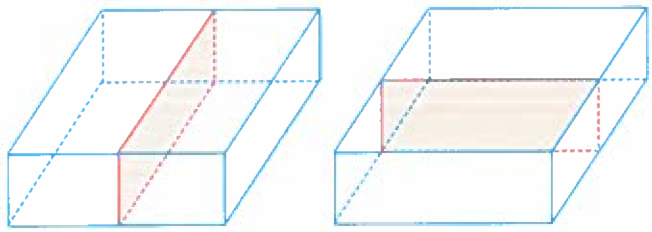
From the investigation, we can conclude that a regular polygon with n sides will have n lines of symmetry and a rotational symmetry of order n.

13.4 Symmetry in Three Dimensions



Plane Symmetry

Fig. 13.16 shows a cuboid being cut by a plane symmetrically in two different ways.



(a) (b)

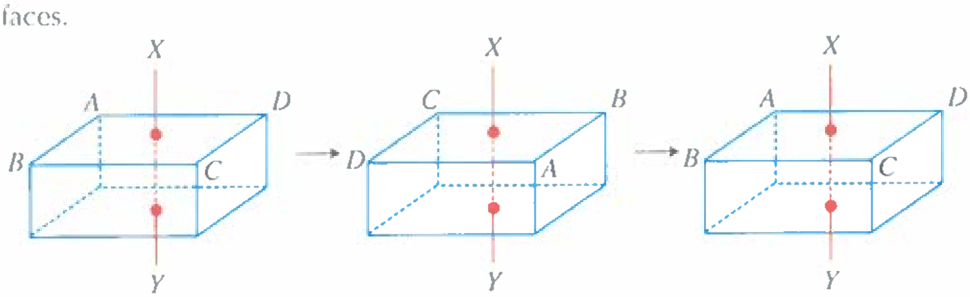
Fig. 13.16

The cuboid is said to be *symmetrical to a plane* since that part of the cuboid on one side of the plane is a mirror image of the part on the other side of the plane. We call the plane a **plane of symmetry**.

Can you find a few more planes of symmetry for the cuboid other than those shown?

Rotational Symmetry

Fig. 13.17 shows a cuboid, where $AB \neq BC$, rotating about a line XY which passes through the centre of two parallel faces.



(a) (b) (c)

Fig 13.17

XY is called the **axis of rotational symmetry**. The cuboid is rotated 180° clockwise about XY. The cuboid looks the same although we notice that the position of the vertex A of the upper face of the cuboid has changed (see Fig. 13.17(**b**)). If the cuboid is rotated another 180° clockwise about XY, the vertex A returns to the original position (see Fig. 13.17(**c**)). As in the case of rotational symmetry in two dimensions, we say that the cuboid has a rotational symmetry of order 2 about the axis XY.

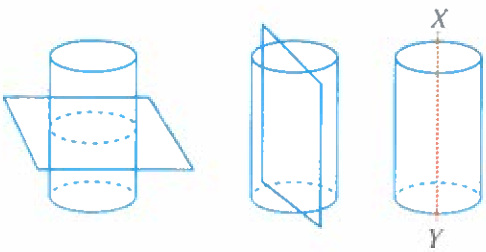
Thinking Time

Can you find a few more axes of rotational symmetry for the cuboid other than XY?

Investigation

Symmetry in Cylinders and Cones

In this investigation, we shall explore the symmetric properties of a cylinder and a cone.

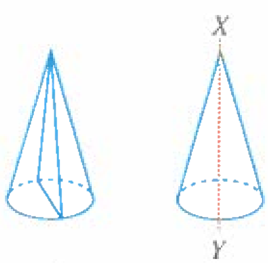


(a) (b) (c)

Fig. 13.18

Fig. 13.18(**a**) and Fig. 13.20(**b**)shows a cylinder being cut by a plane symmetrically in two different ways. Fig. 13.20(**c**) shows the cylinder rotating about the axis XY.

1. Since the cylinder has a circular cross section, find
   (i) the total number of planes of symmetry,
   (ii) the order of rotational symmetry when rotated about the axis XY.



(a) (b)

Fig. 13.19

Fig. 13.21(**a**) shows a cone being cut by a plane symmetrically in one way and Fig. 13.19(**b**) shows the cone rotating about the axis XY.

2. Since the cone has a circular base, find
   (i) the total number of planes of symmetry,
   (ii) the order of rotational symmetry when rotated about the axis XY.

From the investigation, we conclude that:

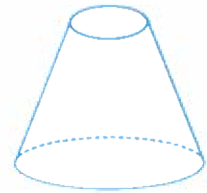
(a) Cylinders have
- infinite number of planes of symmetry (1 that cuts across the cylinder horizontally and infinite ones that cut across the cylinder vertically),
- infinite order of rotational symmetry when rotated about the axis that passes through the centre of the circular top and base.

(b) Cones have
- infinite planes of symmetry that pass through the vertex and the centre of the circular base,
- infinite order of rotational symmetry when rotated about the axis that passes through the vertex of the cone and the centre of the circular base.

Worked Example 4

(Symmetry in Three Dimensions)

The figure shows a frustum. Find
(i) the number of planes of symmetry,
(ii) the number of axes of rotational symmetry,
(iii) the order of rotational symmetry
of the frustum.



Solution:

(i) Since the frustum has a circular base, it has infinite number of planes of symmetry.

(ii) It has 1 axis of rotational symmetry passing the centres of the circles at the top and the base.

(iii) Since the frustum has a circular base, the order of rotational symmetry is infinite.

PRACTISE NOW 4

The figure shows a hemispherical bowl. Find
(i) the number of planes of symmetry,
(ii) the number of axes of rotational symmetry,
(iii) the order of rotational symmetry
of the hemispherical bowl.

SIMILAR QUESTIONS

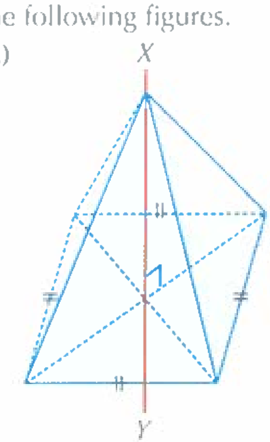
Exercise 13C Questions 1–7

Exercise 13C

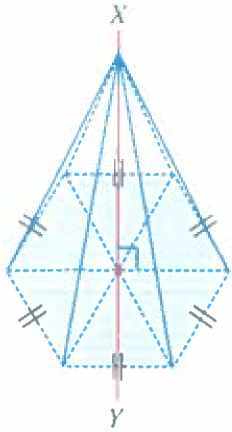
BASIC LEVEL

1. State whether each of the following statements is true or false.
   (a) An equilateral triangle is an isosceles triangle.
   (b) An isosceles triangle is an equilateral triangle.
   (c) An equilateral triangle has rotational symmetry of order 3.
   (d) If a triangle has two equal sides, then it is an isosceles triangle.
   (e) If a triangle has two equal sides, then it has two equal angles.
   (f) If a triangle has two equal angles, then it has two equal sides.
   (g) A circle has no rotational symmetry.
   (h) An n sided polygon has a rotational symmetry of order n.
   (i) A scalene triangle has 2 lines of symmetry.
   (j) An isosceles trapezium has two pairs of equal sides.

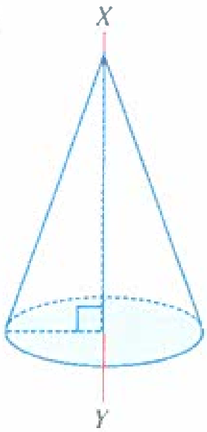
2. Find the order of rotational symmetry for each of the following figures.
   (a)



(b)



(c)

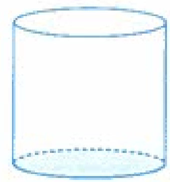


(d)

INTERMEDIATE LEVEL

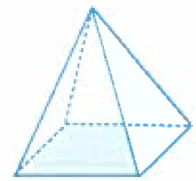
3. Copy and draw the following figures. For each of the following figures, draw two planes of symmetry and one axis of rotational symmetry.

(a)



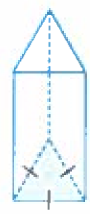
A right cylinder

(b)



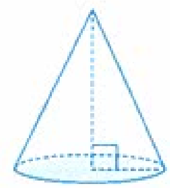
A right pyramid with a rectangular base

(c)



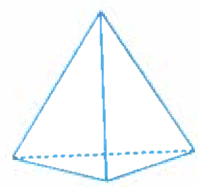
A right prism with an equilateral triangle as the base

(d)



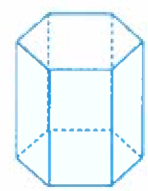
A right circular cone

(e)



A regular tetrahedron

(f)

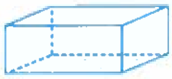


A regular right hexagonal prism

4. How many planes of symmetry does a right pyramid with a square base have altogether?

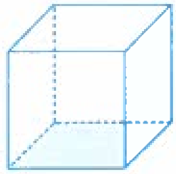
5. State (i) the number of planes of symmetry, (ii) the number of axes of rotational symmetry for the following solids

(a)



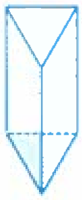
A cuboid

(b)



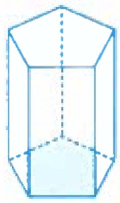
A cube

(c)



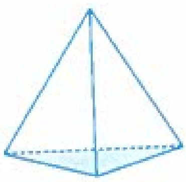
A right prism with an equilateral triangle as the base

(d)



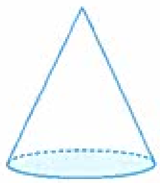
A right pentagonal prism

(e)



A regular tetrahedron

(f)

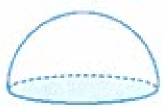


A right circular cone

(g)

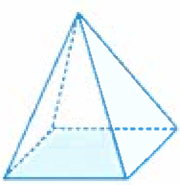
A right cylinder

(h)



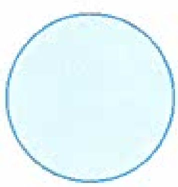
A hemisphere

(i)



A square pyramid

(j)

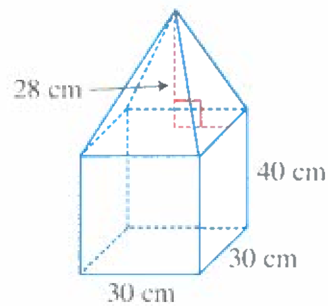


A sphere

ADVANCED LEVEL

6. How many planes of symmetry are there in a cube? Does a cube have more planes of symmetry than a cuboid?

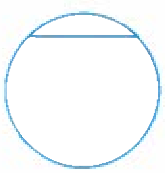
7. A structure is made up of a pyramid attached to a cuboid with a square base of sides 30 cm and a height of 40 cm. Find
   (i) the number of planes of symmetry,
   (ii) the number of axes of rotational symmetry ,
   (iii) the order of rotational symmetry,
   of the structure.

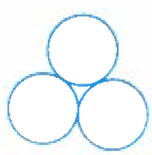


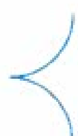
1. A line is a **line of symmetry** of a figure if the "half-shapes" on either side of the line are congruent. This figure is symmetrical. A symmetrical figure may have more than 1 line of symmetry.
2. A point is a **centre of rotational symmetry** of a figure if the figure maps onto itself under rotation about the point.
3. The **order of rotational symmetry** is the number of different ways in which a figure can map onto itself by rotation. In the case of order 1, we say that there is no rotational symmetry.
4. A plane is a **plane of symmetry** of a solid if the "half-shapes" on either side of the plane are congruent.
5. A line is an **axis of rotational symmetry** of a solid if the solid is invariant under rotation about that line.

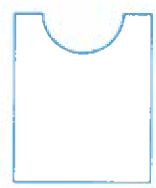
Review Exercise 13

1. Copy each of the following diagrams and draw the axis/axes of symmetry for each of them.

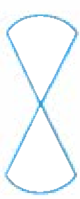
(a)


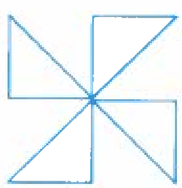
(b)


(c)


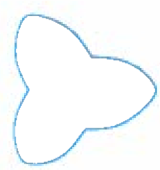
(d)


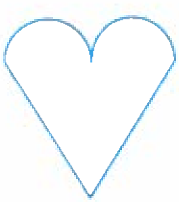
2. For each of the following figures, state **(i)** the number of lines of symmetry and **(ii)** the order of rotational symmetry.

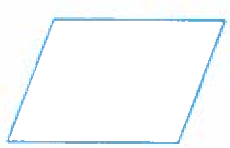
(a)


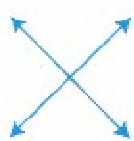
(b)


(c)


(d)


(e)


(f)


(g)


3. State the order of rotational symmetry of each of the following letters.

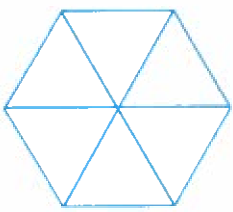
(a) A
(b) I
(c) M
(d) N
(e) X
(f) W
(g) Z
(h) H

4.

The diagram has 16 small squares. On separate diagrams, shade exactly two small squares so that the resulting figure will have

(a) 2 lines of symmetry,
(b) 1 line of symmetry,
(c) rotational symmetry of order 2,
(d) no rotational symmetry,
(e) no line of symmetry

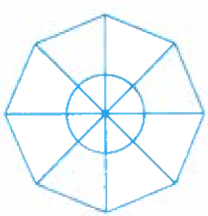
5.



The diagram shows a regular hexagon with 6 equilateral triangles. On separate diagrams, shade exactly two of the triangles so that the resulting figure will have

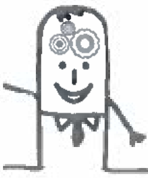
(a) 2 lines of symmetry,
(b) 1 line of symmetry,
(c) rotational symmetry of order 2,
(d) no rotational symmetry.

6.



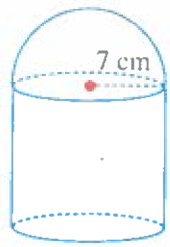
The diagram shows a regular octagon with an inner circle. On separate figures, shade certain parts of the figure so that the resulting figure will have

(a) 1 line of symmetry,
(b) 2 lines of symmetry,
(c) 4 lines of symmetry,
(d) 8 lines of symmetry,
(e) no line of symmetry,
(f) no rotational symmetry.

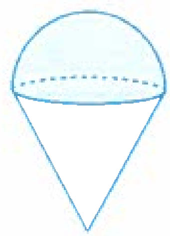


Challenge Yourself

1. The figure below shows a solid made up of a hemisphere and a cylinder which share a common base.
   (i) How many planes of symmetry does the solid have?
   (ii) Draw the axis of rotational symmetry.

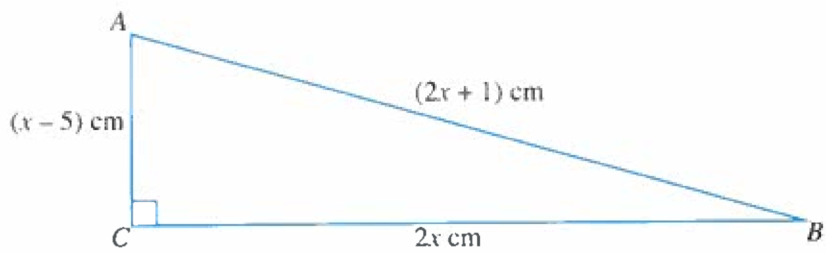


2. The figure below shows a cone of ice cream made up of a hemisphere and a cone.
   (i) How many planes of symmetry does the solid have?
   (ii) Draw the axis of rotational symmetry.

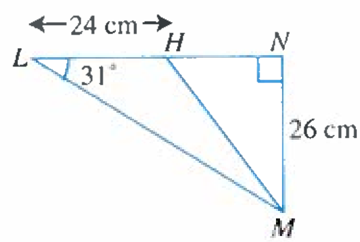


C1 Revision Exercise

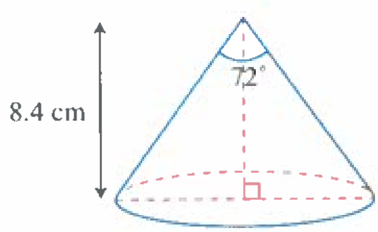
1. The figure shows a right-angled triangle where $\angle ACB = 90°$. Given that $AB = (2x + 1)$ cm, $BC = 2x$ cm and $AC = (x - 5)$ cm,
   (i) write down an equation in x,
   (ii) find the value of x,
   (iii) find the area of $\triangle ABC$.



2. In $\triangle LMN$, $MN = 26$ cm, $\angle MNL = 90°$ and $\angle MLN = 31°$. H lies on LN such that $LH = 24$ cm. Find $\angle MHN$.



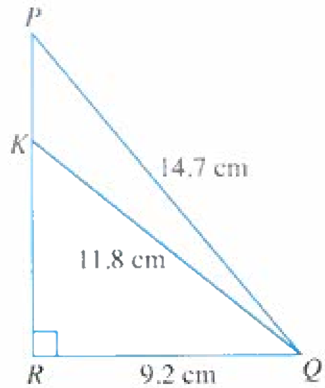
3. A cone has a height of 8.4 cm. Given that the cone has a vertical angle of 72°, find the diameter of its circular base.



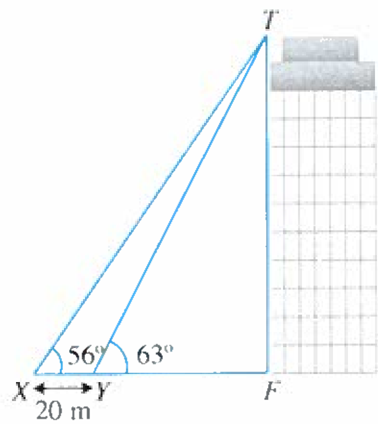
4. An open rectangular fish tank is 0.6 m long and 0.2 m wide. A solid glass decorative structure in the shape of a pyramid with a square base of sides 10 cm and height 27 cm is placed inside the tank. The tank is then filled with water until the water just covers the pyramid. If the pyramid is removed, what will be the fall in the water level?
   (*Hint:* Decrease in volume of water = volume of pyramid)

C2 Revision Exercise

1. In ΔPQR, $PQ = 14.7$ cm, $QR = 9.2$ cm and $\angle PRQ = 90°$. K lies on PR such that $KQ = 11.8$ cm. Find
   (i) the length of PK,
   (ii) $\angle PQK$.



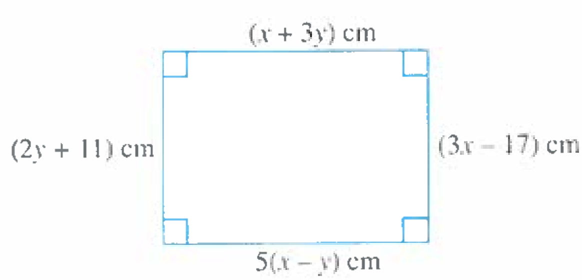
2. Two points X and Y, 20 m apart on level ground, are due West of the foot F of a tower TF. Given that $\angle TXF = 56°$ and $\angle TYF = 63°$, find the height of the tower.



3. A solid square pyramid has a mass of 750 g. It is made of a material with a density of 8.05 g/cm^3. Given that the height of the pyramid is 13.5 cm, find the length of its square base.

4. A solid cone has a circular base of diameter 7 cm and a height of 4 cm.
   (a) Find
   (i) the volume,
   (ii) the surface area,
   of the cone.
   (b) The cone is melted to form identical spheres each of radius 0.9 cm. Find the maximum number of spheres that can be obtained.

5. The figure shows a rectangle with its length and breadth as indicated. Find the length of the longest line segment that can be drawn on the rectangle.



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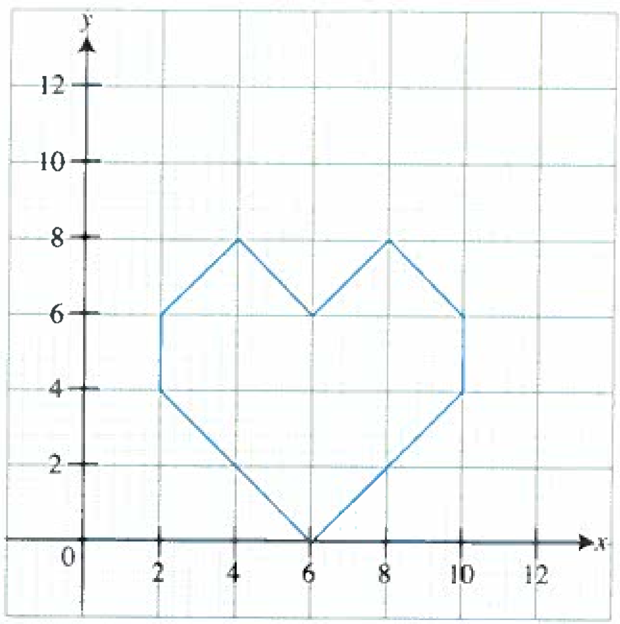
5. Lip balm is sold in hemispherical plastic containers. Given that the volume of lip balm in each container is 30 cm^3, find

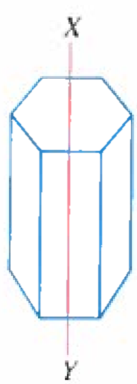
(i) the radius of the container,

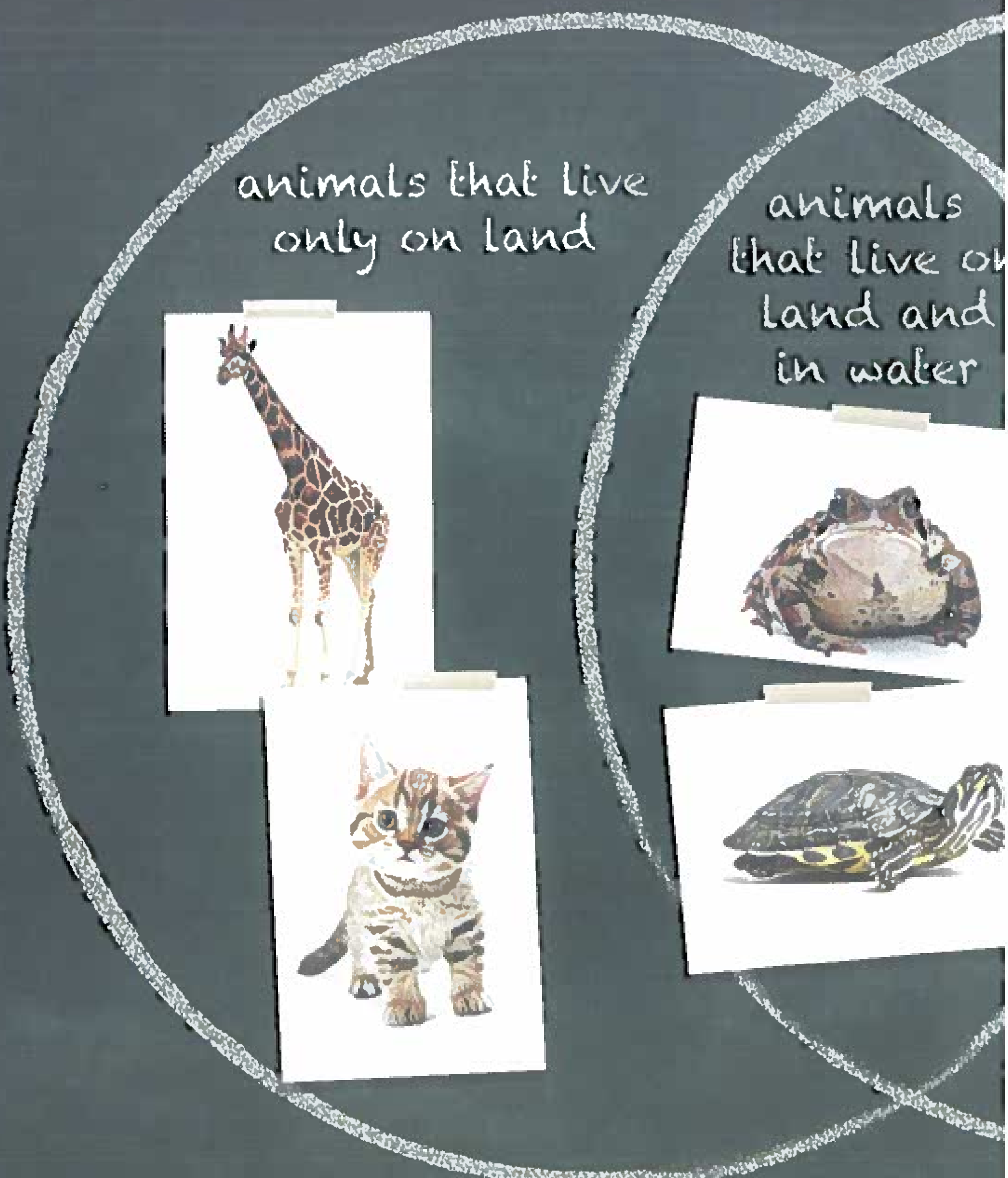
(ii) the surface area of the container that is in contact with the lip balm. (You may ignore the lid of the container.)

6. Draw the line of symmetry and state the equation of the line of symmetry.



7. Given that XY is the axis of rotational symmetry, state the order of rotational symmetry for the solid below.

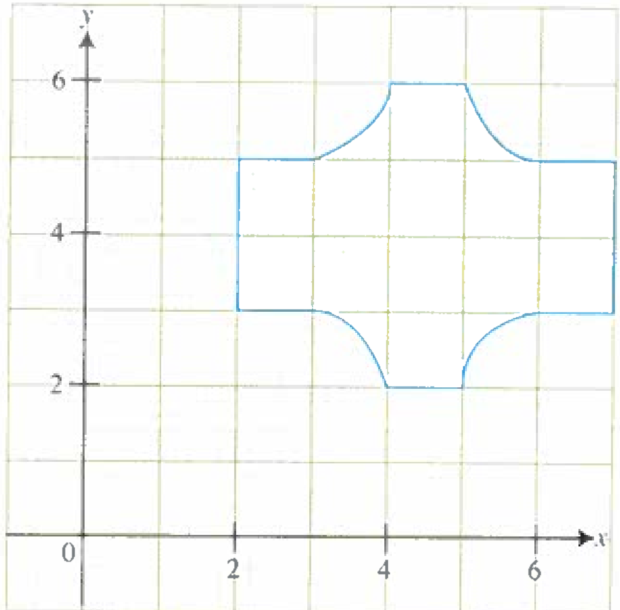




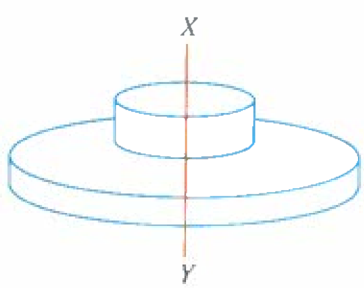
Sets

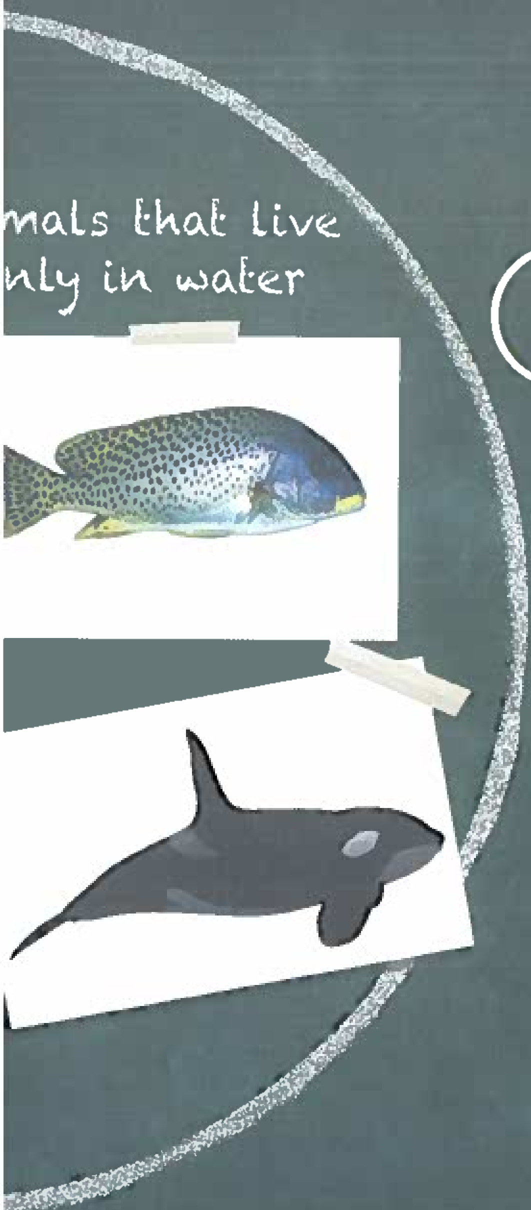
There are many ways to classify animals. One way is to classify them according to whether they live only on land, only in water, or both. The diagram seen here, known as a Venn diagram, can be used to show this manner of classification by grouping animals according to where they live.

6. Draw the line(s) of symmetry and state the equation(s) of the line(s) of symmetry.



7. Given that XY is the axis of rotational symmetry, state the order of rotational symmetry for the solid below.





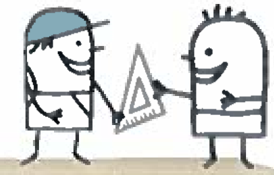
Chapter Fourteen

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- describe a set in words, list all the elements in a set, and describe the elements in a set,
- state and use the terms 'set', 'element', 'equal sets', 'empty set', 'universal set', 'complement of a set', 'subset', 'proper subset', 'intersection of two sets', and 'union of two sets',
- use Venn diagrams to represent sets, including universal sets, complement of a set, proper subset, intersection and union of two sets,
- solving problems using set notations and Venn diagrams.

14.1 Introduction to Set Notations



In everyday life, we often encounter a collection of objects such as a pile of books, a bunch of keys, a team of players and a school of dolphins. In English, we use different terms such as 'pile', 'bunch' and 'team' to describe different collections of objects.

In Mathematics, we use the term 'set' to describe any collection of **well-defined** and **distinct** objects.

For example, if we let S be the set of all the vowels in the English alphabet, we can list all the members or the elements of S in set notation $S = \{a, e, i, o, u\}$.

Since the letter 'a' is an element of S, we write: $a \in S$
Since the letter 'b' is not an element of S, we write: $b \notin S$
Since there are 5 elements in S, we write $n(S) = 5$

> $\in$ denotes 'is an element of',
> $\notin$ denotes 'is not an element of', and
> $n(S)$ denotes 'the number of elements in the set S'.

ATTENTION

'Alphabet' refers to the whole set of 'letters'. By convention, we use capital letters to denote a set, e.g. S, and small letters to denote the elements in a set, e.g. a. Sometimes, the elements can be in capital letters.

PRACTISE NOW

1. A is the set of even positive integers less than 10.

(a) List all the elements of A in set notation.

(b) State whether each of the following statements is true or false.

(i) $8 \in A$ **(ii)** $7 \notin A$
(iii) $10 \in A$ **(iv)** $0 \notin A$

(c) Using the notation $\in$ or $\notin$, describe whether each of the following numbers is an element of, or is not an element of, A.

(i) 2 **(ii)** 5 **(iii)** 9 **(iv)** 6

2. Given that $B = \{3, 6, 9, 12, 15, \ldots, 30\}$, find the value of $n(B)$.

SIMILAR QUESTIONS

Exercise 14A Questions 1–2, 4, 8–9, 12, 14–15

INFORMATION

The set of all positive multiples of 3, i.e. $\{3, 6, 9, 12, 15, \ldots\}$, contains an infinite number of elements, and it is called an **infinite** set. Sets A and B are **finite** sets.

Class Discussion

Well-defined and Distinct Objects in a Set

Work in pairs.

1. Let H be a collection of all the handsome boys in the class. Is H a set?
   Hint: A set is a collection of **well-defined** objects. Is H well-defined?

2. Let T be a collection of 2 identical pens. How should we list the elements of T? {P, P}, {P} or $\{P_1, P_2\}$?
   Hint: How many elements does T have? A set is a collection of **distinct** objects. Are the elements in {P, P} distinct?

3. Let S be the set of letters in the word 'CLEVER'. How should we list the elements?
   Hint: Is the letter 'E' distinct?

In general, a set is *not* any collection of objects. The objects in a set must be **well-defined** and **distinct**.

In the above class discussion, we cannot write T = {P, P} because the elements in {P, P} are not distinct. As there are 2 distinct elements in the set T (the 2 identical pens are distinct), we have to write $T = \{P_1, P_2\}$.

However, in Question 3 of the class discussion, the letter 'E' is not distinct in the word 'CLEVER' because it is the same letter 'E' that is used to form the word. Therefore, S = {C, L, E, V, R}.

ATTENTION

Although the 2 pens are identical, they are still distinct.

To understand this, consider a pair of identical twins, Ethan and Michael. Ethan and Michael are identical. However, Ethan is not Michael, and Michael is not Ethan, i.e. each of them is distinct.

Describing a Set

There are a few ways to describe a set.

1. *Describing a set in words*, e.g. S is the set of all positive even integers less than 10.
2. *Listing all the elements in a set in set notation*, e.g. S = {2, 4, 6, 8}.
3. *Describing the elements in a set in set notation*, e.g.

   $S = \{x : x$ is a positive even integer less than 10}.

   We read this as 'x is such that x is a positive even integer less than 10.'

Worked Example 1

(Listing the Elements in a Set in Set Notation)

It is given that $A = \{x : x$ is a positive integer such that $2 \leqslant x < 11\}$ and $B = \{x : x$ is a positive integer between 3 and 11 inclusive$\}$.

(i) List all the elements in A and in B in set notation.

(ii) Do A and B contain the same elements? If not, explain why.

Solution:

(i) $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$B = \{3, 4, 5, 6, 7, 8, 9, 10, 11\}$

(ii) No. $2 \in A$ but $2 \notin B$.

Problem Solving Tip

If an integer is **between** 3 and 11, the integer cannot be equal to 3 or 11.

If an integer is between 3 and 11 **inclusive**, the integer can be 3 or 11.

PRACTISE NOW 1

It is given that $C = \{x : x$ is a positive integer between 10 and 18$\}$ and $D = \{x : x$ is a positive integer such that $10 \leqslant x < 18\}$.

(i) List all the elements in C and in D in set notation.

(ii) Do C and D contain the same elements? If not, explain why.

SIMILAR QUESTIONS

Exercise 14A Questions 3, 10–11, 16

Worked Example 2

(Well-defined Sets)

State whether each of the following collections is a well-defined set. Give a reason for your answer.

(a) {A book well-liked by my classmates}

(b) {A naughty pupils in my class}

(c) {A Mathematics teacher in my class}

Solution:

(a) No, because a book may be well-liked by some, but not others.

(b) No, because some teachers and pupils may consider a certain pupil naughty while others may not.

(c) Yes, because it is clear whether someone is a Mathematics teacher in the class.

PRACTISE NOW 2

State whether each of the following collections is a well-defined set. Give a reason for your answer.

(a) {A movie well-liked by my classmates}

(b) {A fourteen year old pupil in my class}

(c) {An English teacher in my school}

SIMILAR QUESTIONS

Exercise 14A Question 5

Equal Sets

Two sets A and B are **equal** if they contain exactly the same elements, and we write $A = B$. For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 1, 3\}$, then all the elements of A and of B are the same, i.e. $A = B$, although the order of the elements are different in A and in B.

We also notice that if $A = B$, then $n(A) = n(B)$.

INFORMATION

By convention, we should list the elements
(i) in ascending order for numbers,
(ii) in alphabetical order for letters, or
(iii) according to the given order.
E.g. if S is the set of letters in the word 'MATH', listing the elements according to the given order is often clearer and more convenient: $S = \{M, A, T, H\}$.

Thinking Time

If A and B are two sets such that $n(A) = n(B)$, is $A = B$? If not, give a counter-example.

Empty Sets

Consider the sets $A = \{0, 1, 2\}$, $B = \{0\}$ and $C = \{\ \}$.

The set A contains 3 elements: 0, 1 and 2.
The set B contains 1 element: 0.
The set C does not contain any elements.

In other words, C is called an **empty set** (or a **null set**).
We use the symbol $\emptyset$ (pronounced as 'phi') to describe an empty set, i.e. $C = \emptyset$.

INFORMATION

The set $\{\emptyset\}$ is not an empty set. It is a set containing one element: the symbol $\emptyset$.

Worked Example 3

(Equal Sets and Empty Sets)

It is given that
  A is the set of vowels in the word 'RHYTHM'
and $B = \{H\}$.
(i) List all the elements of A in set notation.
(ii) Are A and B equal sets? Why?

Solution:

(i) $A = \{\ \}$

(ii) A and B are not equal sets, as A is an empty set, i.e. it has no elements while B consists of one element, H.

PRACTISE NOW 3

It is given that $P = \{x : x \text{ is a positive integer less than } 1\}$ and $Q = \{0\}$.
(i) List all the elements of P in set notation.
(ii) Are P and Q equal sets? Why?

SIMILAR QUESTIONS

Exercise 14A Questions 6–7, 13, 17

Exercise 14A

BASIC LEVEL

1. B is the set of odd positive integers less than 11.
   (a) List all the elements of B in set notation.
   (b) State whether each of the following statements is true or false.
   (i) $1 \in B$ (ii) $4 \notin B$
   (iii) $0 \in B$ (iv) $11 \notin B$

2. State the number of elements in each of the following sets.
   (a) {Months in the year}
   (b) {Odd numbers between 10 and 26}
   (c) {Pupils in your class who are taller than you}
   (d) {Colours in your school flag}
   (e) {Colours of a rainbow}
   (f) {Horoscope signs}

3. List all the elements in each of the following sets in set notation.
   (a) $A = \{x : x$ is a positive integer between 1 and 10$\}$
   (b) $B = \{x : x$ is a negative integer between -10 and -1 inclusive$\}$
   (c) $C = \{x : x$ is a positive even integer such that $-2 < x \leq 12\}$
   (d) D is the set of vowels in the word 'HAPPY'.

4. List the elements of the following sets.
   (a) {Vowels in the word "MATHEMATICS"}
   (b) {The seven colours of the rainbow}
   (c) {Multiples of 9 which are less than 50}
   (d) {Colours in your school flag}
   (e) {Even numbers which are between 10 and 23}

5. State whether each of the following collections is a well-defined set. Give a reason for each answer.
   (a) {A pupils in my class who has two brothers}
   (b) {A pupil in my class who is shy}
   (c) {A TV actor who is well-liked by my classmates}
   (d) {A dish well-liked by my family members}
   (e) {A textbook used in my school}
   (f) {The most attractive actor in Hollywood}

6. State whether each of the following sets is an empty set.
   (a) {Polar bears living in the Sahara Desert}
   (b) {Bald men with a crew cut}
   (c) {Buses with 50 seats}
   (d) {Boys who wear skirts to school}
   (e) {Orators who cannot talk well}
   (f) {Odd numbers that are divisible by 4}
   (g) {Multiples of 3 that are divisible by 9}
   (h) {Triangles having three equal sides}
   (i) {Parallelograms having five vertices}
   (j) {Quadrilaterals having three obtuse angles}

7. List all the elements in each of the following sets in set notation, and state whether it is an empty set.
   (a) E is the set of odd numbers that are divisible by 2.
   (b) $F = \{x : x$ is a month of the year with more than 31 days$\}$
   (c) G is the set of quadrilaterals with 5 vertices each.
   (d) $H = \{x : x$ is an even prime number$\}$

INTERMEDIATE LEVEL

8. D is the set of days in a week.
   (a) List all the elements of D in set notation.
   (b) Using the notation $\in$ or $\notin$, describe whether each of the following is an element of, or is not an element of D.
   (i) Tuesday (ii) Sunday
   (iii) March (iv) Holiday

9. P is the set of all perfect squares bigger than 1 and less than 50.
   (i) Is $10 \in P$?
   (ii) List all the elements of P in set notation.

10. List all the elements in each of the following sets in set notation.
   (a) $J = \{x : x$ is a colour of the rainbow$\}$
   (b) K is the set of consonants in the word 'SYMMETRY'.
   (c) $L = \{x : x$ is a teacher teaching my current class$\}$
   (d) $M = \{x : x$ is the month of the year beginning with the letter J$\}$
   (e) $N = \{x : x$ is an odd number between 10 and 18$\}$
   (f) $O = \{x : x$ is the first five consonants of the English alphabet$\}$
   (g) $P = \{x : x$ is the day of the week beginning with the letter T$\}$
   (h) Q is the set of even numbers on a clock face
   (i) $R = \{x : x$ is the month of the year with fewer than 30 days$\}$

11. Describe each of the following sets in words.
   (a) $M = \{0, 2, 4, 6, 8, \ldots\}$
   (b) $N = \{0, 2, 4, 6, 8\}$
   (c) $O = \{1, 8, 27, 64, 125, \ldots\}$
   (d) $P = \{\ldots, -15, -10, -5, 0, 5, 10, 15, \ldots\}$
   (e) $Q = \{a, b, c, d, e\}$

12. Pick out the element that does not belong in each of the following sets and describe the remaining elements in the set.
   (a) {Malaysia, Singapore, Cambodia, the Philippines, Vietnam, Laos, Myanmar, Brunei, Thailand, Indonesia, China}
   (b) {rubber, coconut, apple, mango, rambutan, orange}
   (c) $\{4, 9, 1, 25, 16, 49, 20, 36\}$
   (d) $\{8, 1, 64, 75, 27\}$
   (e) {mode, mean, pie chart, median}

13. It is given that
   $Q = \{x : x$ is a perfect square between 10 and 15$\}$
   and $R = \{x : x$ is a positive integer less than 5 that is both a perfect square and a perfect cube$\}$.
   (i) List all the elements of Q and of R in set notation.
   (ii) Are Q and R empty sets? Use the notation $\varnothing$ to describe Q and R.

ADVANCED LEVEL

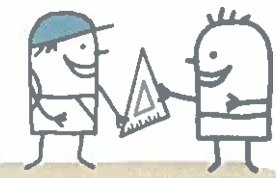
14. State whether each of the following statements is true or false. If it is false, explain why.
   (i) $c \notin \{c, a, r\}$
   (ii) $car \in \{c, a, r\}$
   (iii) $\{c\} \in \{c, a, r\}$
   (iv) $\{c, a, r\} = 3$
   (v) $5 \in \{1, 3, 5, 7\}$
   (vi) $4 \in \{1, 3, 5, 7\}$
   (vii) $bus \in \{b, u, s\}$
   (viii) $b \in \{b, u, s\}$

15. State whether each of the following statements is true or false.
   (a) $5 \notin \{3, 7, 11, 14\}$
   (b) $i \in \{a, i, j, k\}$
   (c) $4 \notin \{x : x$ is an even number$\}$
   (d) $\{S, C, O, H, L\} \in \{x : x$ is a letter of the word "SCHOOL"$\}$
   (e) $\left(3\frac{3}{4} + 1\frac{1}{4}\right) \in \{x : x$ is an even number$\}$
   (f) $\{3\} \in \{1, 2, 3, 4, 5\}$

16. Describe the elements of each of the following sets in set notation.
   (a) S is the set of girls in my current class who wear spectacles.
   (b) $T = \{2, 3, 5, 7, 11, 13, \ldots\}$
   (c) $U = \{\ldots, -8, -4, 0, 4, 8, 12, \ldots\}$
   (d) $V = \{-8, -4, 0, 4, 8, 12\}$

17. State whether each of the following statements is true or false. If it is false, explain why.
   (i) $\{0\} = \varnothing$
   (ii) $\varnothing = \{\ \}$
   (iii) $\{\varnothing\}$ is an empty set.
   (iv) $n(\varnothing) = 0$

14.2 Venn Diagrams, Universal Set and Complement of a Set



We can represent a set using a **Venn diagram** as shown in Fig. 14.1.

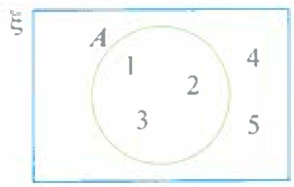


Fig. 14.1

In Fig. 14.1, the rectangle represents the set of all the elements that are under consideration for this particular situation, i.e. $\{1, 2, 3, 4, 5\}$. This is called the **universal set** and is denoted by the symbol ξ, i.e. $\xi = \{1, 2, 3, 4, 5\}$.

The circle represents the set $A = \{1, 2, 3\}$.

We observe that the elements 4 and 5 are outside the circle but inside the rectangle, i.e. $4 \notin A$ and $5 \notin A$. The set of all the elements in ξ but *not* in A is called the **complement** of the set A, and is denoted by A' (pronounced as 'A prime'), i.e. $A' = \{4, 5\}$.

ATTENTION

When drawing a Venn diagram,

- do not put commas between the elements,
- do not write the elements too close together,
- write the elements inside the set, but label the set ξ or the set A outside the set (i.e. outside the rectangle or the circle respectively).

Worked Example 4

(Universal Set and Complement of a Set)

It is given that $\xi = \{x : x$ is an integer between 1 and 10 inclusive$\}$ and $A = \{x : x$ is a perfect square$\}$.

(i) List all the elements of ξ and of A in set notation.

(ii) Draw a Venn diagram to represent the sets ξ and A.

(iii) From the Venn diagram, list all the elements of A' in set notation.

(iv) Describe the set A' in words.

(v) State the values of $n(\xi)$, $n(A)$ and $n(A')$.

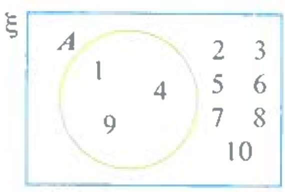
(vi) Is $n(A) + n(A') = n(\xi)$? Why or why not?

Solution:

(i) $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 4, 9\}$

(ii)



(iii) $A' = \{2, 3, 5, 6, 7, 8, 10\}$

(iv) A' is the set of all integers between 1 and 10 inclusive which are not perfect squares.

(v) $n(\xi) = 10$, $n(A) = 3$ and $n(A') = 7$

(vi) Yes. Since A and A' contain all the elements of ξ, and A and A' do not contain the same elements, $n(A) + n(A') = n(\xi)$.

Problem Solving Tip

The set A does *not* contain all the perfect squares because all the elements under consideration for this particular situation are only the integers between 1 and 10 inclusive.

PRACTISE NOW 4

It is given that $\xi = \{x : x$ is an integer between 1 and 13 inclusive$\}$, and $B = \{x : x$ is a prime number$\}$.

(i) List all the elements of ξ and of B in set notation.

(ii) Draw a Venn diagram to represent the sets ξ and B.

(iii) From the Venn diagram, list all the elements of B' in set notation.

(iv) Describe the set B' in words.

SIMILAR QUESTIONS

Exercise 14B Questions 1–6, 10–11, 17

Thinking Time

Given that $A = \{1, 2, 3\}$, can we find A' if we do not define what the universal set ξ is?

Subsets and Proper Subsets

Given two sets A and B, we say that B is a **subset** of A if every element of B is an element of A and we write $B \subseteq A$. $\subseteq$ expresses the idea "includes" or "contains". Following the definition of a subset, we see that B can be equal to A and any set itself is also a subset of the same set.

Consider the sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3\}$.

How can we draw a Venn diagram to represent the sets A and B?

Since all the elements in a set are distinct (i.e. we *cannot* write the same element twice or more), we can draw the Venn diagram as shown in Fig. 14.2.

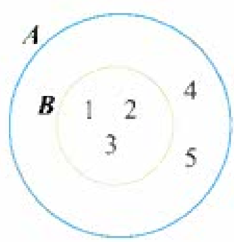


Fig. 14.2

We observe that B is completely inside A, i.e. every element of B is an element of A, and $B \neq A$. We say that B is a **proper subset** of A and we write $B \subset A$.

In other words, $B \subset A$ if every element of B is an element of A and there exists at least one element of A that is not in B.

INFORMATION

If B is a subset of A, then B can be a proper subset of A, or $B = A$.

Class Discussion

Understanding Subsets

Work in pairs.

1. Is a subset also a set?

2. In the Venn diagram shown in Fig 14.3, is P a subset of Q or vice versa?

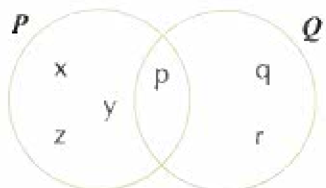


Fig. 14.3

Worked Example 5

(Subsets)

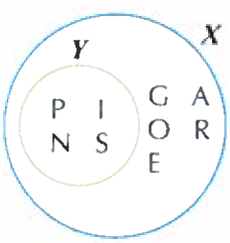
It is given that X is the set of letters in the word 'SINGAPORE' and Y is the set of letters in the word 'PINS'}.

(i) Draw a Venn diagram to represent the sets X and Y.

(ii) Is Y a subset of X? Explain.

Solution:

(i)



(ii) Yes, Y is a proper subset of X because every element of Y is an element of X, and $Y \neq X$.

PRACTISE NOW 5

SIMILAR QUESTIONS

Exercise 14B Questions 7–9, 12–14, 18–19

1. It is given that $C = \{1, 2, 3, 4, 5, 6, 7\}$ and $D = \{1, 3, 5, 7\}$.
   (i) Draw a Venn diagram to represent the sets C and D.
   (ii) Is D a proper subset of C? Explain.

2. It is given that $P = \{x : x$ is an integer such that $0 < x \leqslant 13\}$,
   $Q = \{x : x$ is a prime number less than $13\}$
   and $R = \{x : x$ is a positive integer not more than $13\}$.
   (i) List all the elements of P and of Q.
   (ii) Is $Q \subset P$ or $P \subset Q$? Explain.
   (iii) List all the elements of R.
   (iv) Hence, state the relationship between sets R and P. Explain.

Listing all the Subsets and Proper Subsets of a Set

Consider the set $S = \{1, 2\}$.

What are all the subsets and proper subsets of S?

Firstly, it is obvious that $\{1\}$ and $\{2\}$ are proper subsets of S as shown in Fig. 14.4(**a**) and (**b**).

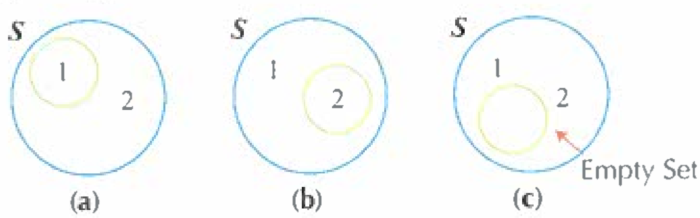


Fig. 14.4

There is no need to draw any Venn diagram when listing all the subsets.

In Fig. 14.4(**c**), we observe that we can draw the empty set completely inside S.

Thus, the empty set $\varnothing$ is also a proper subset of S.

Therefore, all the proper subsets of $S = \{1, 2\}$ are $\varnothing$, $\{1\}$ and $\{2\}$.

Since the definition of a subset includes the set itself, all the subsets of $S = \{1, 2\}$ are $\varnothing$, $\{1\}$, $\{2\}$ and $\{1, 2\}$.

PRACTISE NOW

SIMILAR QUESTIONS

Exercise 14B Questions 15–16

List all the (**a**) subsets and (**b**) proper subsets of
(i) $S = \{7, 8\}$,
(ii) $T = \{a, b, c\}$.

Exercise 14B

BASIC LEVEL

1. If $\xi = \{1, 2, 3, \ldots, 20\}$, A = {multiples of 2}, B = {multiples of 4} and C = {multiples of 3}, list the elements of the following.

(a) A **(b)** B
(c) C **(d)** A'
(e) B' **(f)** C'

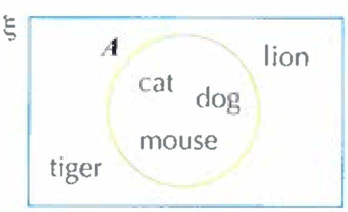
2. If $\xi = \{30, 31, 32, \ldots, 45\}$, list the complements of the following sets.

(a) $A = \{33, 37, 39, 42, 44\}$
(b) $B = \{30, 31, 32, 33, 34, 36, 37, 38, 39, 40, 41, 42, 45\}$
(c) C = {non-prime numbers}
(d) D = {multiples of 4}
(e) E = {multiples of 3}

3. If the universal set, $\xi = \{1, 2, 3, \ldots, 10\}$ and A = {prime numbers}, B = {even numbers} and C = {multiples of 5}, list the elements of

(a) A, **(b)** B,
(c) C, **(d)** A',
(e) B', **(f)** C'.

4. The Venn diagram shows the elements in ξ and A.



(i) List all the elements in A and in ξ.
(ii) List all the elements in A'.

5. It is given that $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2\}$, and $B = \{1, 2, 3, 4\}$.
Draw a Venn diagram to illustrate the given information.

6. It is given that

$\xi = \{x : x$ is an integer between 1 and 10 inclusive$\}$

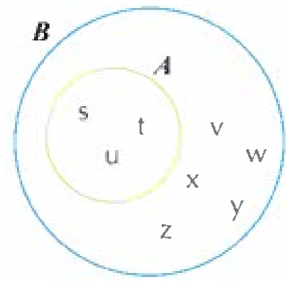
and $B = \{x : x$ is an even number$\}$.

(i) List all the elements of ξ and of B in set notation.
(ii) Draw a Venn diagram to represent the sets ξ and B.
(iii) From the Venn diagram, list all the elements of B' in set notation.
(iv) Describe the set B' in words.

7. If $A = \{10, 20, 40, 60, 80\}$, list the elements of the following subsets of A.

(a) {numbers divisible by 4}
(b) {numbers divisible by 6}
(c) {numbers divisible by 8}
(d) {numbers divisible by 7}

8. The Venn diagram shows the elements in A and in B.



(i) List all the elements in A and in B.
(ii) Is A a proper subset of B? Explain.

9. It is given that $A = \{-2, -1, 0, 1, 2, 3\}$ and $B = \{0, 1, 2, 3\}$.

(i) Draw a Venn diagram to represent the sets A and B.
(ii) Is B a proper subset of A? Explain.

INTERMEDIATE LEVEL

10. It is given that $\xi = \{x : x$ is an integer between 0 and 10$\}$ and $C = \{x : x$ is not a prime number$\}$.

(i) List all the elements of ξ, of C and of C' in set notation.

(ii) Describe the set C' in words.

11. It is given that ξ is the universal set containing the first 10 letters of the English alphabet and D is the set of consonants.

(i) List all the elements of ξ, of D and of D' in set notation.

(ii) Describe the set D' in words.

12. It is given that

$E = \{x : x$ is an integer such that $0 < x \leqslant 20\}$,

$F = \{x : x$ is a positive multiple of 4 that is less than 20$\}$

and $G = \{x : x$ is a positive integer less than 21$\}$.

(i) List all the elements of E and of F.

(ii) Is $F \subset E$ or $E \subset F$? Explain.

(iii) List all the elements of G.

(iv) Hence state the relationship between sets E and G.

13. It is given that

$H = \{x : x$ is a rational number$\}$,

and $I = \{x : x$ is an integer$\}$.

Is $I \subset H$? Explain.

14. State whether each of the following statements below is true or false.

(a) If $\xi = \{a, b, c, d, e\}$ and $A = \{a, b, c\}$, then $A' = \{d, e\}$.

(b) If $a \in A$ and $A \subset B$, then $a \in B$.

(c) If $A \subset B$ and $B \subset C$, then $A \subset C$.

(d) If $a \in A$ and $a \in B$, then $A = B$.

(e) If $\varnothing = A'$, then $A = \xi$.

15. List all the subsets of the following sets.

(a) $P = \{1, 2\}$

(b) $Q = \{$pen, ink, ruler$\}$

(c) $R = \{$Thailand, Vietnam$\}$

(d) $S = \{a, e, i, o\}$

16. List all the proper subsets of each of the following sets.

(a) $K = \{x, y\}$

(b) $L = \{$Singapore, Malaysia$\}$

(c) $M = \{3, 4, 5\}$

(d) $N = \{a, b, c, d\}$

ADVANCED LEVEL

17. It is given that

$\xi = \{x : x$ is a positive integer less than 21$\}$

and $O = \{x : x$ is a number divisible by 3$\}$.

(i) List all the elements of O'.

(ii) Describe the elements in O' in set notation.

18. Given the following sets:

$A = \{1, 2, 3, 4\}$,
$B = \{a, b, c, d\}$,
$C = \{a, o, u, z\}$,
$D = \{a, e, i, o, u\}$,
$E = \{2, 3, 4\}$,
$F = \{4, 3, 2\}$,
$G = \{$the vowels in the English alphabet$\}$,
$H = \{$the first four letters in the English alphabet$\}$,
$I = \{$the first four counting numbers$\}$,

fill in the blanks with one of the following symbols: $=, \neq, \subset$ or $\supset$.

There may be more than one answer in certain cases.

(a) A _____ B **(b)** A _____ F

(c) B _____ H **(d)** H _____ C

(e) G _____ I **(f)** F _____ E

(g) F _____ I **(h)** D _____ G

(i) C _____ I **(j)** I _____ A

(k) E _____ A **(l)** G _____ A

19. If $A = \{a, \{a\}, b, \{c\}, d\}$, state whether each of the following is true or false.

(a) $a \in A$ **(b)** $\{a\} \in A$

(c) $c \in A$ **(d)** $\{d\} \in A$

(e) $\{a, \{a\}\} \in A$ **(f)** $\{a\} \subset A$

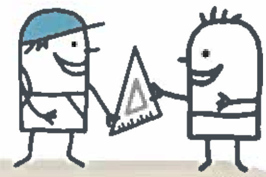
(g) $\{\{a\}\} \subset A$ **(h)** $\{a, b\} \in A$

(i) $\{a, b\} \subset A$ **(j)** $\{b, \{c\}, d\} \subset A$

(k) $\{a, \{a, c\}, d\} \subset A$ **(l)** $\{a, \{a\}, b, \{c\}, d\} \subseteq A$

(m) $\{a, \{a\}, b, \{c\}, d\} \subset A$

14.3 Intersection of Two Sets



In the chapter opener, we see a Venn diagram with two sets. One set represents animals that live only on land and the other set represents animals that live only in water. At the middle of the diagram where the two sets intersect, there are animals that live on land and in water. How could we relate this diagram using set notation?

Consider the sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 5, 6, 7\}$.

How can we draw a Venn diagram to represent the sets A and B?

Since all the elements in a set are distinct (i.e. we *cannot* write the same element twice or more), we draw the Venn diagram as shown in Fig. 14.5.

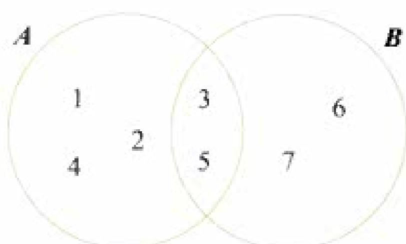


Fig. 14.5

We notice that the elements 3 and 5 are *common* to both sets A and B, and they lie in the **intersection** of A and B.

We write $A \cap B = \{3, 5\}$. In other words, $A \cap B$ is a *set*.

> We read this as 'A intersect B'.

In general,

> the **intersection** of sets A and B, denoted by $A \cap B$, is the set of all the elements which are common to both A and B.

Worked Example 6

(Intersection of Two Sets)

It is given that $A = \{r, s, t, u, v, w, x\}$ and $B = \{s, t, w, y, z\}$.

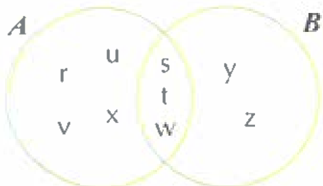
(i) List all the elements in $A \cap B$ in set notation.

(ii) Draw a Venn diagram to represent the sets A and B.

Solution:

(i) $A \cap B = \{s, t, w\}$

(ii)



PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 14C Questions 1–3, 11–13

1. It is given that $C = \{x : x \text{ is a multiple of 6 such that } 0 < x \leqslant 18\}$ and $D = \{x : x \text{ is a multiple of 3 such that } 0 < x \leqslant 18\}$.

(i) List all the elements in C and in D in set notation.

(ii) Find $C \cap D$.

(iii) Draw a Venn diagram to represent the sets C and D.

(iv) Is $C \cap D = C$? Explain.

2. It is given that $E = \{x : x \text{ is a positive integer and a factor of } 12\}$ and $F = \{x : x \text{ is a prime number between 5 and 13 inclusive}\}$.

(i) List all the elements in E and in F in set notation.

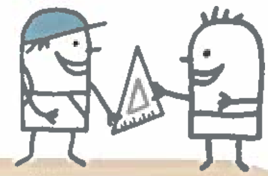
(ii) Find $E \cap F$. Explain.

(iii) Draw a Venn diagram to represent the sets E and F.

From Practise Now 6 Questions 1 and 2, we observe that

- If all the elements of A are also in B, A is a subset of B, then $A \cap B = A$.
- If A and B do not share any common elements, A and B are disjoint sets, then $A \cap B = \varnothing$.

14.4 Union of Two Sets



Consider the same sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 5, 6, 7\}$ as in Fig. 14.5, which is shown again as Fig. 14.6.

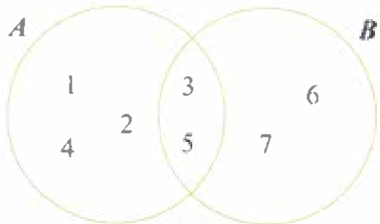


Fig. 14.6

If we list all the elements in A or in B together, we will get $\{1, 2, 3, 4, 5, 6, 7\}$.

This is called the **union** of A and B, and is denoted by $A \cup B$,
i.e. $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$.

In other words, $A \cup B$ is also a *set*.

We read this as 'A union B'.

In general,

> the **union** of sets A and B, denoted by $A \cup B$ is the set of all the elements which are in A or in B.

ATTENTION

In Mathematics, all the elements which are in A or in B include the elements which are in both A and B.

Worked Example 7

(Union of Two Sets)

It is given that $A = \{f, g, h, i, j, k\}$ and $B = \{h, i, p, q\}$.

(i) Draw a Venn diagram to represent the sets A and B.

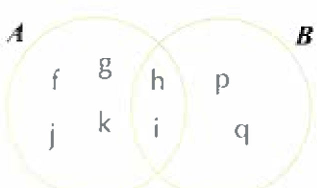
(ii) From the Venn diagram, list all the elements in $A \cup B$ in set notation.

Problem Solving Tip

Identify A □ B before drawing the Venn diagram.

Solution:

(i)



(ii) $A \cup B = \{f, g, h, i, j, k, p, q\}$

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 14C Questions 4–6, 14–15

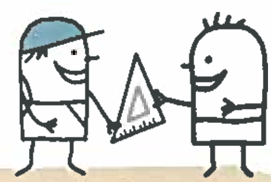
1. It is given that $C = \{x : x \text{ is a positive integer and a factor of } 8\}$ and
$D = \{x : x \text{ is a positive integer and a factor of } 16\}$.
   (i) List all the elements in C and in D in set notation.
   (ii) Draw a Venn diagram to represent the sets C and D.
   (iii) From the Venn diagram, find $C \cup D$.
   (iv) Is $C \cup D = D$? Explain.

2. It is given that $E = \{x : x \text{ is a multiple of 7 such that } 0 < x < 63\}$
and $F = \{x : x \text{ is a multiple of 9 such that } 0 < x < 63\}$.
   (i) List all the elements in E and F in set notation.
   (ii) Draw a Venn diagram to represent the sets E and F.
   (iii) From the Venn diagram, find $E \cup F$.

From Practise Now 7 Question 1, we observe that

if $C \subset D$, then $C \cup D = D$.

14.5 Combining Universal Set, Complement of a Set, Subset, Intersection and Union of Sets



Worked Example 8

Problem involving Universal Set, Intersection and Union of Sets

It is given that $\xi = \{x : x \text{ is a positive integer less than } 11\}$,
$A = \{x : x \text{ is an even number}\}$
and $B = \{x : x \text{ is a factor of } 12\}$.

(a) List all the elements in ξ, A and B.

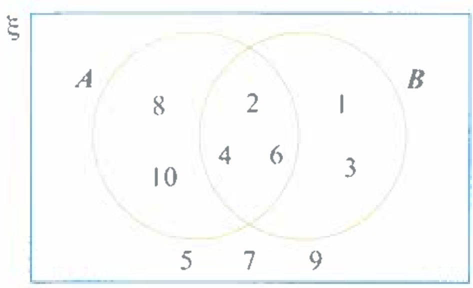
(b) Draw a Venn diagram to represent the sets ξ, A and B.

(c) Find
   (i) $(A \cup B)'$, (ii) $A \cap B'$.

Solution:

(a) $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$A = \{2, 4, 6, 8, 10\}$
$B = \{1, 2, 3, 4, 6\}$

(b)



(c) (i) $(A \cup B)' = \{5, 7, 9\}$

(ii) $A = \{2, 4, 6, 8, 10\}$
$B' = \{5, 7, 8, 9, 10\}$
$\therefore A \cap B' = \{8, 10\}$

Problem Solving Tip

Since ξ is the set of positive integers less than 11, then A cannot contain all the even numbers, but only those that are positive and less than 11.

Problem Solving Tip

For **(b)**, to draw a Venn diagram, always fill in the elements for $A \cap B$ first.

For **(c)(iii)**, it may be easier to list the elements in A and in B' first, if you cannot obtain the answer from the Venn diagram directly.

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 14C Questions 7–9, 16–17, 21–23

1. It is given that $\xi = \{x : x$ is a positive integer not more than $9\}$,
$A = \{x : x$ is an odd number$\}$
and $B = \{x : x$ is a multiple of $3\}$.

(a) List all the elements in ξ, in A and in B in set notation.

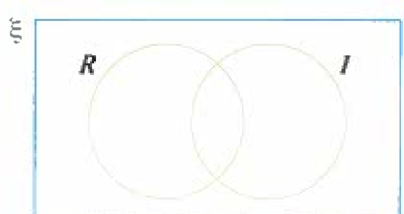
(b) Draw a Venn diagram to represent the sets ξ, A and B.

(c) Find
(i) $(A \cup B)'$,
(ii) $A \cap B'$.

2. It is given that $\xi = \{x : x$ is a triangle$\}$,
$R = \{x : x$ is a right-angled triangle$\}$
and $I = \{x : x$ is a triangle with exactly two equal sides$\}$.

T is a triangle with sides 3 cm, 4 cm and 5 cm.
U is a triangle with angles 90°, 45° and 45°.
V is a triangle with sides 5 cm, 8 cm and 8 cm.
W is a triangle with two sides, 5 cm and 5 cm, and an included angle of 60°.

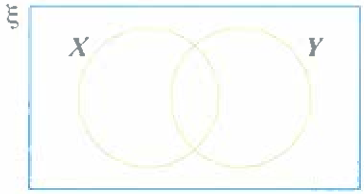
On the Venn diagram below, write T, U, V and W in the appropriate subsets.



Worked Example 9

(Problem involving Shading of Sets)

Identify and shade the following regions on separate Venn diagrams.



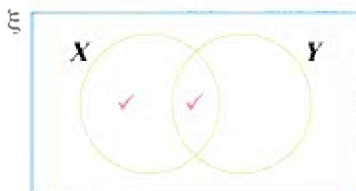
(i) $X \cup Y'$

(ii) $X \cap Y'$

(iii) $(X \cap Y)'$

Solution:

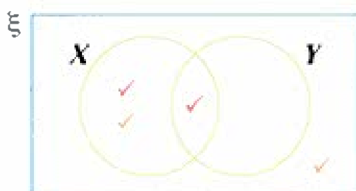
(i) **Step 1:** Put a tick in each of the two regions for X.



Problem Solving Tip

The diagram is divided into 4 regions.

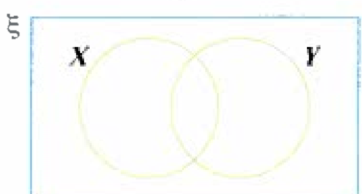
Step 2: Put a tick in each of the two regions for Y'.



Problem Solving Tip

$X \cup Y'$ is the set of all the elements which are in X **or** in Y', so shade all the regions with at least one tick.

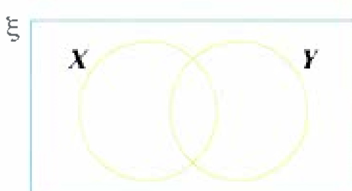
Step 3: As it is a union, shade all the regions with at least one tick.



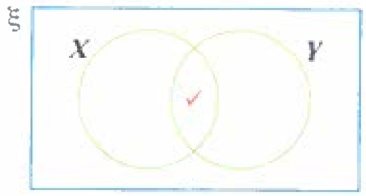
Problem Solving Tip

$X \cap Y'$ is the set of all the elements which are common to both X **and** Y'.

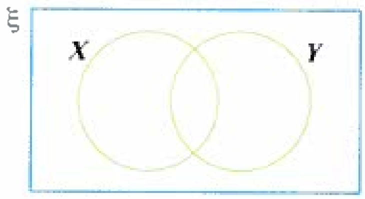
(ii) As it is an intersection, shade all the regions with exactly two ticks.



(iii) **Step 1:** Put a tick in the region for $X \cap Y$.



Step 2: As it is a complement of $X \cap Y$, shade all the regions without any tick.

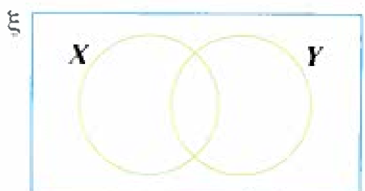


PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 14C Questions 10, 18–20

Identify and shade the following regions on separate Venn diagrams.



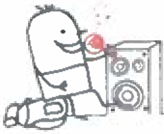
(i) $X' \cup Y$ **(ii)** $X' \cap Y$ **(iii)** $(X \cup Y)'$

(iv) $X' \cup Y'$ **(v)** $X' \cap Y'$ **(vi)** $(X \cap Y')'$

From your answers in Practise Now 9, answer each of the following.

1. Is $(X \cup Y)'$ equal to $X' \cup Y'$ or $X' \cap Y'$?

2. Is $(X \cap Y')'$ equal to $X' \cap Y$ or $X' \cup Y$?

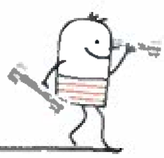


Performance Task

Find out from your classmates how they usually travel to school every morning. Present your findings on a vanguard sheet by drawing a Venn diagram to display the following sets where appropriate.

A = {x : x is a student in your class who travels to school by public bus}
B = {x : x is a student in your class who travels to school by MRT}
C = {x : x is a student in your class who travels to school by chartered bus}
D = {x : x is a student in your class who travels to school by taxi}
E = {x : x is a student in your class who travels to school by car}
F = {x : x is a student in your class who travels to school by foot only}
G = {x : x is a student in your class who travels to school by helicopter}
H = {x : x is a student in your class who travels to school by other modes of transport not stated above}

1. What would you take as the universal set ξ?
2. Is there any empty set? Do you want to include an empty set in your Venn diagram?
3. Every student who travels to school will have to do some walking. Do you want to include walking for every student when doing such a survey? Explain.
4. Do some sets intersect each other? How would you represent these sets in the Venn diagram?
5. Do you want to include yourself in the above survey? Explain.



Exercise 14C

BASIC LEVEL

1. It is given that ξ = {letters of the English alphabet}. Draw Venn diagrams to illustrate the following sets. In each case, find $A \cap B$.

(a) A = {a, b, c , e, f}, B = {b, c, o, p, q}
(b) A = {a, c, e, g}, B = {p, q, r}
(c) A = {x, y, z, m, n}, B = {consonants in the word 'money'}
(d) A = {consonants in the word 'mathematics'}, B = {consonants in the word 'statistics'}
(e) A = {letters of the word 'universal'}, B = {letters of the word 'probability'}
(f) A = {vowels in the word 'transformations'}, B = {vowels in the word 'combinations'}

2. It is given that

A = {1, 2, 3, 4, 7}

and B = {2, 4, 8, 10}.

(i) List all the elements in $A \cap B$ in set notation.
(ii) Draw a Venn diagram to represent the sets A and B.

3. It is given that

D = {blue, green, yellow, orange, red, pink}

and C = {blue, yellow, pink, purple, black}.

(i) List all the elements in $C \cap D$ in set notation.
(ii) Draw a Venn diagram to represent the sets C and D.

4. It is given that $\xi = \{1, 2, 3, \ldots, 9\}$. Draw Venn diagrams to illustrate the following sets. In each case, find $A \cup B$.
   (a) $A = \{1, 2, 3, 4\}$, $B = \{3, 5, 7, 9\}$
   (b) $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 4, 6, 8\}$
   (c) $A = \{4, 8\}$, $B = \{2, 4, 6, 8\}$
   (d) A = {multiples of 3}, B = {prime numbers}
   (e) A = {multiples of 4}, B = {multiples of 2}

5. It is given that $E = \{11, 13, 15, 17\}$
   and $F = \{12, 14, 15, 16, 17, 18, 20\}$.
   (i) Draw a Venn diagram to represent the sets E and F.
   (ii) From the Venn diagram, list all the elements in $E \cup F$ in set notation.

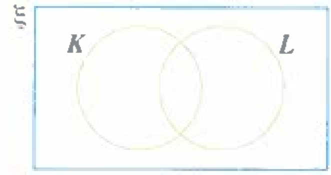
6. It is given that
   G = {apple, orange, banana, grape, durian, pear}
   and H = {apple, banana, grape, strawberry}.
   (i) Draw a Venn diagram to represent the sets G and H.
   (ii) From the Venn diagram, list all the elements in $G \cup H$ in set notation.

7. If A = {durian, mango, pineapple} and B = {durian, rambutan, mango, soursop}, find
   (i) $A \cup B$, (ii) $A \cap B$.

8. Find the union and intersection of each of the following pairs of sets.
   (a) $A = \{3, 6, 9, 12\}$, $B = \{6, 8, 9\}$
   (b) $C = \{a, b, x, y\}$, $D = \{m, n, o, p\}$
   (c) E = {monkey, goat, lion}, F = {tiger, goat}
   (d) $G = \{a, m, k, y\}$, $H = \varnothing$

9. It is given that
   $\xi = \{x : x$ is a positive integer less than 16},
   $I = \{x : x$ is a multiple of 4},
   and $J = \{x : x$ is a factor of 8}.
   (a) List all the elements in ξ, in I and in J in set notation.
   (b) Draw a Venn diagram to represent the sets ξ, I and J.
   (c) From the Venn diagram, find
      (i) $(I \cup J)'$, (ii) $I \cap J'$.

10. Identify and shade the following regions on separate Venn diagrams.



   (i) $K \cap L$, (ii) $K \cup L$, (iii) $(K \cup L)'$.

INTERMEDIATE LEVEL

11. It is given that
   $M = \{x : x$ is a perfect square such that $0 < x < 70\}$
   and $P = \{x : x$ is a perfect cube such that $0 < x < 70\}$.
   (i) List all the elements in M and P in set notation.
   (ii) Draw a Venn diagram to represent the sets M and P.
   (iii) From the Venn diagram, list all the elements in $M \cap P$ in set notation.

12. It is given that
   $N = \{x : x$ is a multiple of 8 such that $0 < x \leqslant 32\}$
   and $Q = \{x : x$ is a multiple of 4 such that $0 < x \leqslant 32\}$.
   (i) List all the elements in N and in Q in set notation.
   (ii) Find $N \cap Q$.
   (iii) Draw a Venn diagram to represent the sets N and Q.
   (iv) Is $N \cap Q = N$? Explain.

13. It is given that
   $R = \{x : x$ is a positive integer and a factor of 18}
   and $S = \{x : x$ is a composite number between 9 and 18}.
   (i) List all the elements in R and S in set notation.
   (ii) Find $R \cap S$. Explain.
   (iii) Draw a Venn diagram to represent the sets R and S.

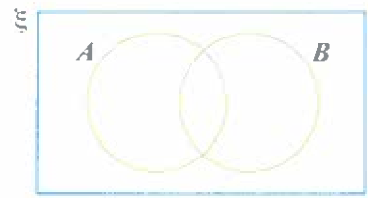
14. It is given that
$T = \{x : x$ is a multiple of 4 such that $0 < x < 16\}$
and $U = \{x : x$ is a positive integer and a factor of $24\}$
(i) List all the elements in T and in U in set notation.
(ii) Draw a Venn diagram to represent the sets T and U.
(iii) From the Venn diagram, find $T \cup U$.
(iv) Is $T \cup U = U$? Explain.

15. It is given that
$V = \{x : x$ is a positive integer and a factor of $25\}$
and $W = \{x : x$ is a multiple of 6 such that $0 < x < 25\}$.
(i) List all the elements in V and W in set notation.
(ii) Draw a Venn diagram to represent the sets V and W.
(iii) From the Venn diagram, find $V \cup W$.

16. It is given that
$\xi = \{x : x$ is a positive integer such that $3 < x \leqslant 18\}$,
$Y = \{x : x$ is a multiple of $3\}$
and $Z = \{x : x$ is a multiple of $9\}$.
(a) List all the elements in ξ, in Y and in Z in set notation.
(b) Draw a Venn diagram to represent the sets ξ, Y and Z.
(c) From the Venn diagram, find
(i) $(Y \cup Z)'$, (ii) $Y \cap Z'$.

17. It is given that
$\xi = \{x : x$ is a non-negative integer less than $12\}$,
$P = \{x : x$ is a prime number$\}$
and $Q = \{x : x$ is not a prime number$\}$.
(a) List all the elements in ξ, in P and in Q in set notation.
(b) Draw a Venn diagram to represent the sets ξ, P and Q.
(c) From the Venn diagram, find
(i) $P \cup Q$, (ii) $(P \cup Q)'$,
(iii) $P' \cap Q$.

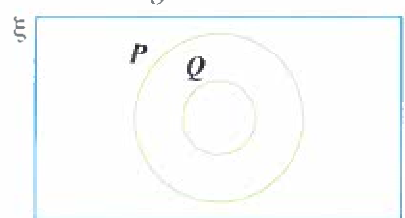
18. (a) Identify and shade the following regions on separate Venn diagrams.



(i) $A \cup B'$ (ii) $A \cap B'$
(iii) $(A \cup B)'$ (iv) $(A \cap B)'$
(v) $A' \cup B'$ (vi) $A' \cap B'$
(vii) $(A' \cup B)'$ (viii) $(A \cup B')'$

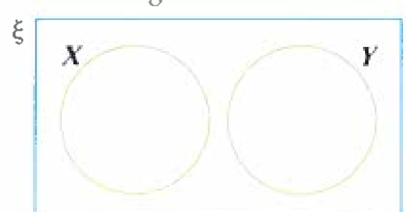
(b) What do you notice about the sets
(i) $(A \cup B)'$ and $A' \cap B'$
(ii) $(A \cap B)'$ and $A' \cup B'$

19. Identify and shade the following regions on separate Venn diagrams.



(i) $P \cup Q'$ (ii) $P \cap Q'$ (iii) $(P \cup Q)'$
(iv) $(P \cap Q)'$ (v) $P' \cup Q'$ (vi) $P' \cap Q'$
(vii) $(P' \cup Q)'$ (viii) $(P \cup Q')'$

20. Identify and shade the following regions on separate Venn diagrams.



(i) $X \cup Y'$ (ii) $X \cap Y'$ (iii) $(X \cup Y)'$
(iv) $(X \cap Y)'$ (v) $X' \cup Y'$ (vi) $X' \cap Y'$
(vii) $(X' \cup Y)'$ (viii) $(X \cup Y')'$

ADVANCED LEVEL

21. For any set A, simplify the following if possible.

(i) $A \cap \xi$ (ii) $A \cup \xi$

(iii) $A \cap \emptyset$ (iv) $A \cup \emptyset$

22. If $A \subset B$ and $A \cap C = \emptyset$, simplify the following if possible.

(i) $A \cap B$ (ii) $A \cup B$

(iii) $B \cap C$ (iv) $A \cup C$

(v) $(B \cup C) \cap A$ (vi) $(B \cap C) \cap A$

(vii) $(A \cup C) \cap B$ (viii) $(A \cap C) \cup B$

23. It is given that

$\xi = \{x : x \text{ is a quadrilateral}\}$,

$A = \{x : x \text{ is a quadrilateral with all sides equal}\}$

and $B = \{x : x \text{ is a quadrilateral with all angles equal}\}$.

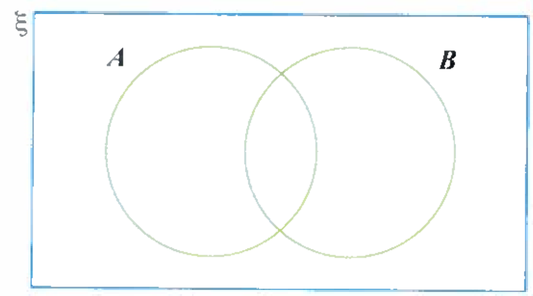
Q is a rectangle with length 12 cm and breadth 10 cm.

R is a rhombus with side 7 cm and angles 30°, 150°, 30° and 150°.

S is a square with side 8 cm.

T is a trapezium with sides 5 cm, 5 cm, 5 cm and 9 cm.

On the Venn diagram below, write Q, R, S and T in the appropriate subsets.



1. A set is a collection of **well-defined** and **distinct** objects. Each object in the set is called an **element**.
2. A set can be defined by:
   (a) *describing it in words*, e.g. S is the set of all positive even integers less than 10,
   (b) *listing all its elements in set notation*, e.g. $S = \{2, 4, 6, 8\}$,
   (c) *describing its elements in set notation*, e.g. $S = \{x : x \text{ is a positive even integer less than } 10\}$.
3. Two sets A and B are **equal** if they contain exactly the same elements, and we write $\boldsymbol{A = B}$.
4. The **empty** or **null** set is the set containing no element. It is described by $\emptyset$.
5. A **Venn diagram** can be used to represent the relationships among sets.
6. The **universal set** is the set of all elements that are under consideration for a particular situation. It is denoted by ξ.
7. The **complement** of a set A is the set of all the elements in ξ but *not* in A. It is denoted by A'.
8. B is a **subset** of A of if every element of B is an element of A. We write $\boldsymbol{B \subseteq A}$.
9. B is a **proper subset** of A if every element of B is an element of A, and $B \neq A$. We write $\boldsymbol{B \subset A}$.
10. The **intersection** of sets A and B is the set of all the elements which are *common* to both A and B. It is denoted by $\boldsymbol{A \cap B}$.
11. The **union** of sets A and B is the set of all the elements which are in A or B. It is denoted by $\boldsymbol{A \cup B}$. In Mathematics, all the elements which are in A or B include the elements which are in both A and B.

Review Exercise 14

1. A is the set of odd positive integers less than 11.
   (a) List all the elements of A in set notation.
   (b) State whether each of the following statements is true or false.
   (i) $7 \in A$ (ii) $8 \notin A$
   (iii) $11 \in A$ (iv) $1 \notin A$
   (c) Using set notation $\in$ or $\notin$, describe whether each of the following numbers is an element of, or is not an element of, A.
   (i) -3 (ii) 3
   (iii) 0 (iv) 9

2. List all the elements in each of the following sets in set notation, and then state whether it is an empty set. If yes, use the notation $\emptyset$ to describe the set.
   (a) B is the set of prime numbers that are divisible by 2.
   (b) $C = \{x : x$ is a day of the week that begins with the letter 'S'$\}$
   (c) $D = \{x : x$ is a positive integer, a factor of 24 and a multiple of 9$\}$
   (d) E is the set of quadrilaterals with three obtuse angles.

3. If $\xi = \{-5, -4, -3, -2, -1, 0, 1, 2, 3\}$, list the **complement** of each of the following sets.
   (a) $A = \{-5, -3, -1, 2\}$
   (b) $B = \{-5, -4, -3, -2, -1, 0, 1, 2, 3\}$
   (c) $C = \{x : x$ is a prime number$\}$
   (d) $D = \{x : x$ is a positive integer and a multiple of 3$\}$

4. State whether each of the following statements is true or false.
   (a) If $a \in X$ and $X \subset Y$, then $a \in Y$.
   (b) If $b \in Y$ and $X \subset Y$, then $b \in X$.
   (c) If $X \subset Y$ and $Y \subset Z$, then $X \subset Z$.
   (d) If $X' = \emptyset$, then $X = \xi$.

5. It is given that
   $\xi = \{x : x$ is an integer such that $1 \leqslant x < 24\}$,
   $A = \{x : x$ is a multiple of 4$\}$
   and $B = \{x : x$ is a factor of 36$\}$.
   (i) Draw a Venn diagram to illustrate this information.
   (ii) List the elements contained in the set $A \cup B'$.

6. It is given that
   $\xi = \{x : x$ is an integer such that $-7 \leqslant x \leqslant 7\}$,
   $A = \{x : x$ is an integer such that $-7 < x < 7\}$
   and $B = \{x : x$ is an integer such that $0 < x \leqslant 7\}$.
   List the elements in
   (i) A', (ii) $A \cap B$,
   (iii) $A \cup B'$.

7. It is given that
   $\xi = \{x : x$ is an integer such that $0 < x < 16\}$,
   $A = \{x : x$ is a perfect square$\}$
   and $B = \{x : x$ is a factor of 26$\}$.
   (i) Draw a Venn diagram to illustrate the given information.
   (ii) List the elements contained in the set $A' \cap B'$.

8. It is given that

$\xi = \{x : x \text{ is an integer between 0 and 21}\}$,

$A = \{x : x \text{ is divisible by 5}\}$

and $B = \{x : x \text{ is not a prime number}\}$.

(i) Draw a Venn diagram to illustrate the given information.

(ii) List the elements contained in the set $A \cap B'$.

9. It is given that $S = \{s, i, t\}$. List all the

(i) subsets,

(ii) proper subsets

of S.

10. It is given that

$\xi = \{x : x \text{ is a real number}\}$,

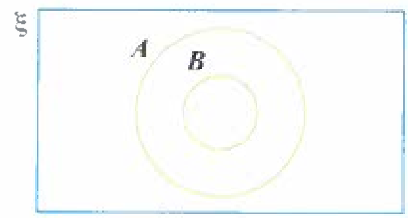
$A = \{x : x \text{ is a rational number}\}$

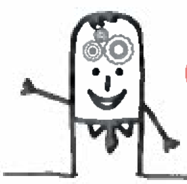
and $B = \{x : x \text{ is an integer}\}$.

On the Venn diagram given, write the following numbers in the appropriate subsets.

(i) -7 (ii) $\frac{2}{3}$

(iii) π (iv) 0





Challenge Yourself

1. It is given that $S = \{a, \{a\}\}$. State and explain whether the following statements can be true.

(i) $a \in S$, (ii) $\{a\} \in S$,

(iii) $\{a\} \subset S$, (iv) $\{\{a\}\} \subset S$.

2. If a set S has n elements, how many proper subsets does it have?

3. Draw a Venn diagram to illustrate the relationships among the following quadrilaterals.

(i) Square (ii) Rectangle

(iii) Rhombus (iv) Parallelogram

(v) Trapezium (vi) Kite

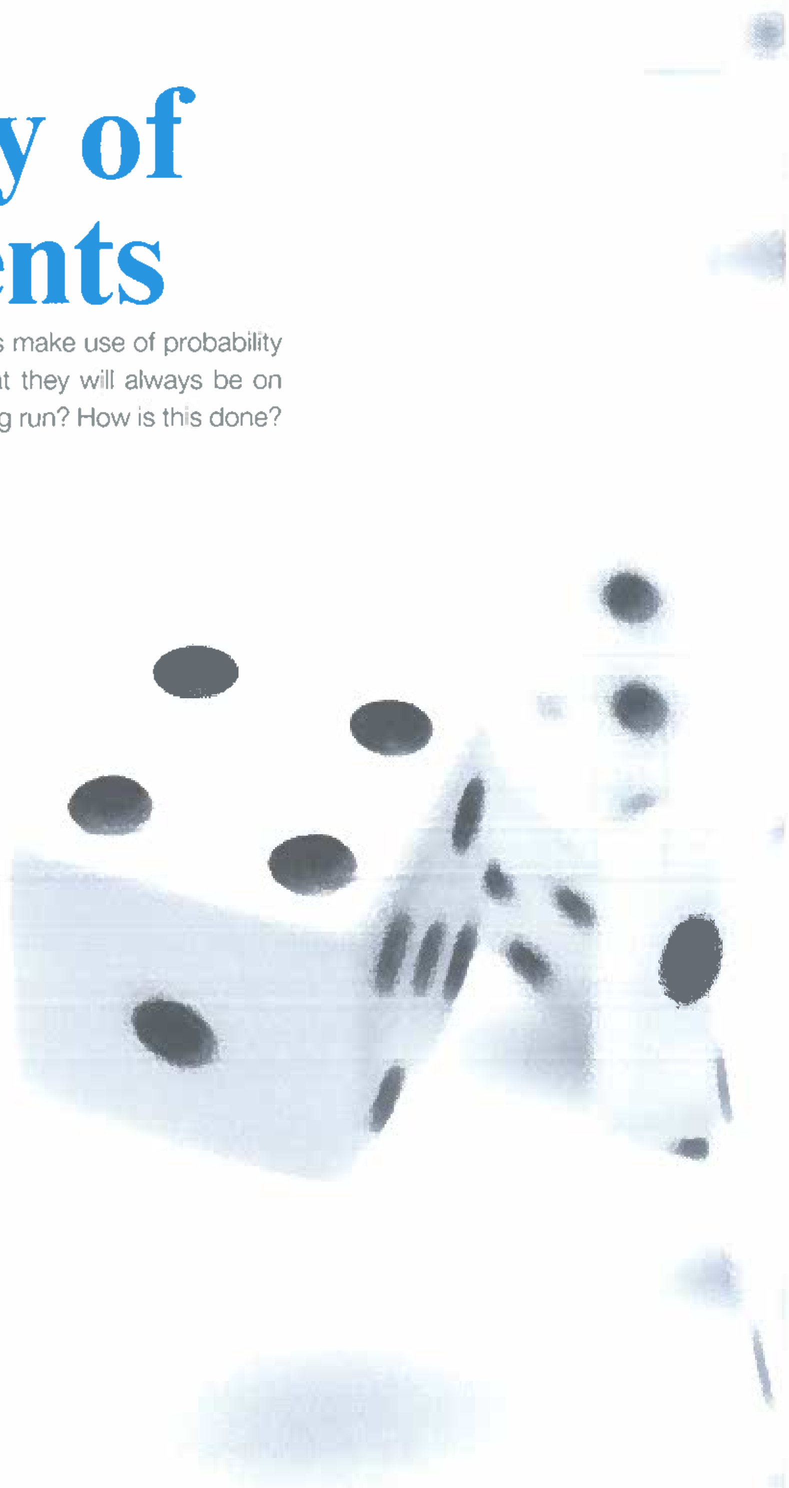
For the purpose of classification, adopt the following definitions (because there is more than one definition).

(a) A trapezium has at least two parallel sides.

(b) A kite has at least two equal adjacent sides.

Probability of Single Events

Do you know that casinos make use of probability to set rules to ensure that they will always be on the winning side in the long run? How is this done?



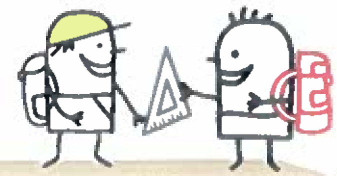
Chapter Fifteen

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- define probability as a measure of chance,
- list the sample space of a probability experiment,
- find the probability of a single event,
- solve problems involving the probability of single events.

15.1 Introduction to Probability



We often make statements such as:

- 'There is a 50 : 50 *chance* of our school winning the National Inter-School Basketball Championship.'
- 'I cannot *predict* whether I will obtain a 'six' in my next roll of a die.'
- 'It will *probably* rain today.'

We make such statements because we are uncertain whether an **event** will occur. For an uncertain event, we can discuss about its chance of occurrence.

Thinking Time

1. The following are some events which we may come across in our everyday life.

   Event A: The sun will rise from the west every day.
   Event B: Your friend will win the lottery this year.
   Event C: You obtain a 'tail' when you toss a coin.
   Event D: It will rain in Pakistan at least once a year.
   Event E: Babies drink milk every day.

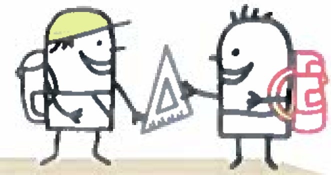
   Each of these events may or may not happen. Mark these events A to E on the line to show the likelihood they will occur.

2. We can use values between 0 and 1 inclusive to measure the chance of an event occurring, where an impossible event takes on the value 0 and a certain event takes on the value 1. If there is a 50 : 50 chance that an event will occur, what value does it take?

3. Write down an event that corresponds to each of the five categories above. Mark out each event on the number line based on the estimated chance of occurence.

In our everyday life, we use words such as 'unlikely', 'likely' or 'certain' to describe the chance of an event occurring. The measure of chance, which takes on values between 0 and 1 inclusive, is known as **probability**. We will learn about probability in this chapter.

15.2 Sample Space



When we perform a scientific experiment, we will obtain a certain result or outcome. However, in probability, the result or the outcome is not certain – it depends on chance. Table 15.1 shows some examples of probability experiments and their possible outcomes.

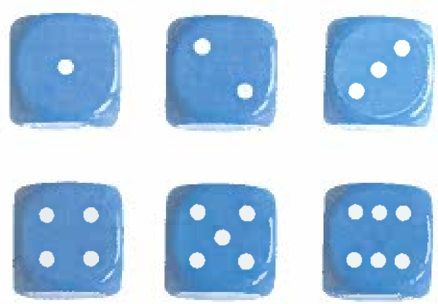
Probability experiment	Possible outcomes
 **Tossing a coin**	 **Head**  **Tail**
 **Rolling a die**	
 Ten identical cards numbered 11, 12, 13, …, 20 are placed in a box. One card is drawn at random from the box.	11 12 13 14 15 16 17 18 19 20
Two black balls and three white balls of the same size are placed in a bag. One ball is drawn at random from the bag. 	

Table 15.1

Consider the experiment where a coin is tossed. The results are either getting a 'head' or a 'tail'. These results are referred to as the **outcomes**.

The collection of all the possible outcomes of a probability experiment is called the **sample space**. In the case of tossing a coin, the sample space is a 'head' and a 'tail'. What is the sample space when a die is rolled?

Worked Example 1

(Sample Space)

A fair die is rolled. Write down the sample space and state the total number of possible outcomes.

Solution:

A die has the numbers 1, 2, 3, 4, 5 and 6 on its six faces,

i.e. the sample space consists of the numbers 1, 2, 3, 4, 5 and 6.

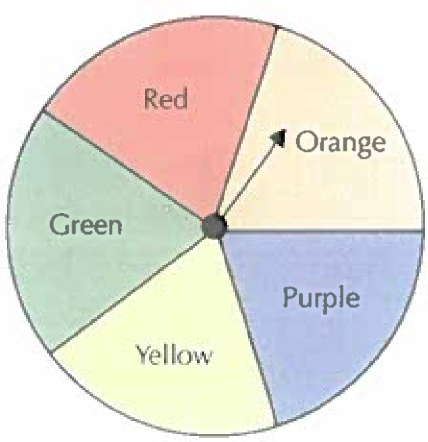
Total number of possible outcomes = 6

PRACTISE NOW 1

SIMILAR QUESTIONS

Exercise 15A Questions 1, 2(a)–(b)

A spinner is divided into 5 equal sectors of different colours. When the spinner is spun, the colour of the sector on which the pointer lands is noted. Write down the sample space and state the total number of possible outcomes.



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Worked Example 2

(Sample Space)

For each of the following experiments, write down the sample space and state the total number of possible outcomes.

(a) Drawing a ball at random from a bag containing 2 identical black balls and 3 identical white balls

(b) Choosing a two-digit number at random

Solution:

(a) Let B_1 and B_2 represent the 2 black balls; W_1, W_2 and W_3 represent the 3 white balls.

The sample space consists of B_1, B_2, W_1, W_2 and W_3.

Total number of possible outcomes = 5

(b) The sample space consists of the integers 10, 11, 12, …, 99.

Total number of possible outcomes = first 99 numbers – first 9 numbers
= 99 – 9
= 90

ATTENTION

For **(a)**, as there are two black balls in the bag, there is a difference between drawing the first or the second black ball. Thus we must differentiate between the two black balls by representing them using B_1 and B_2. Similarly, for the three white balls, we represent them using W_1, W_2 and W_3.

PRACTISE NOW 2

For each of the following experiments, write down the sample space and state the total number of possible outcomes.

(a) Drawing a marble at random from a bag containing 5 identical blue marbles and 4 identical red marbles

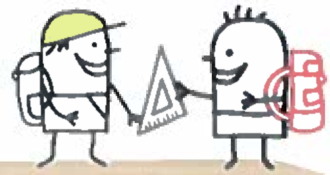
(b) Picking a letter at random from a box containing identical cards with letters that spell the word 'NATIONAL'

(c) Selecting a receipt at random from a receipt book with running serial numbers from 357 to 389

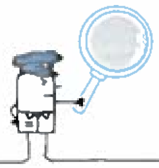
SIMILAR QUESTIONS

Exercise 15A Questions 2(c)–(e)

15.3 Probability of Single Events



In Section 15.1, we have learnt that probability is a measure of chance.



Investigation

Tossing a Coin

1. Are we able to state with certainty whether the outcome is a 'head' or a 'tail' *before* a coin is tossed?

2. Toss a coin 20 times.
   (i) Record the outcome of each toss in the following table.

Outcome	Tally	Number of 'heads' or 'tails' for 20 tosses	Fraction of obtaining a 'head' or a 'tail'
Head			
Tail			

   (ii) Write down the fraction of obtaining a 'head' or a 'tail' in the table above.
   (iii) Compare your results with those of your classmates. Are they the same? What can you deduce about the results of tossing a coin?

3. (a) In groups of 4 or 5, add and record the total number of 'heads' obtained by your group members. Repeat for the total number of 'tails'. Compute the fraction of obtaining a 'head' or a 'tail'.

Outcome for ____ tosses	Total number of 'heads' or 'tails'	Fraction of obtaining a 'head' or a 'tail'
Head		
Tail		

   (b) As a class, add and record the total number of 'heads' obtained by all students. Repeat for the total number of 'tails'. Compute the fraction of obtaining a 'head' or a 'tail'.

Outcome for ____ tosses	Total number of 'heads' or 'tails'	Fraction of obtaining a 'head' or a 'tail'
Head		
Tail		

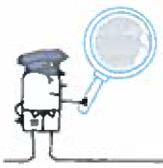
4. Look at the last column in the three tables. Do you notice that the probabilities of obtaining a 'head' or a 'tail' approach $\frac{1}{2}$ when there are more tosses?

5. If we toss a coin 1000 times, would we expect to obtain *exactly* 500 'heads' and *exactly* 500 'tails'? Explain your answer.

When a coin is tossed, if the chance of obtaining a 'head' is the same as the chance of obtaining a 'tail', we say that the coin is **fair** or **unbiased**. This means that for a fair coin, there are two equally likely outcomes, i.e. obtaining a 'head' and obtaining a 'tail'. Thus the chance of obtaining a 'head' is 1 out of 2. We say that the probability of obtaining a 'head' is $\frac{1}{2}$. What is the probability of obtaining a 'tail'?

Internet Resources

Search on the Internet for interactive applets on probability that involve tossing a coin.



Investigation

Rolling a Die

Go to http://www.shinglee.com.sg/StudentResources/ and open the spreadsheet 'Rolling a Die'.

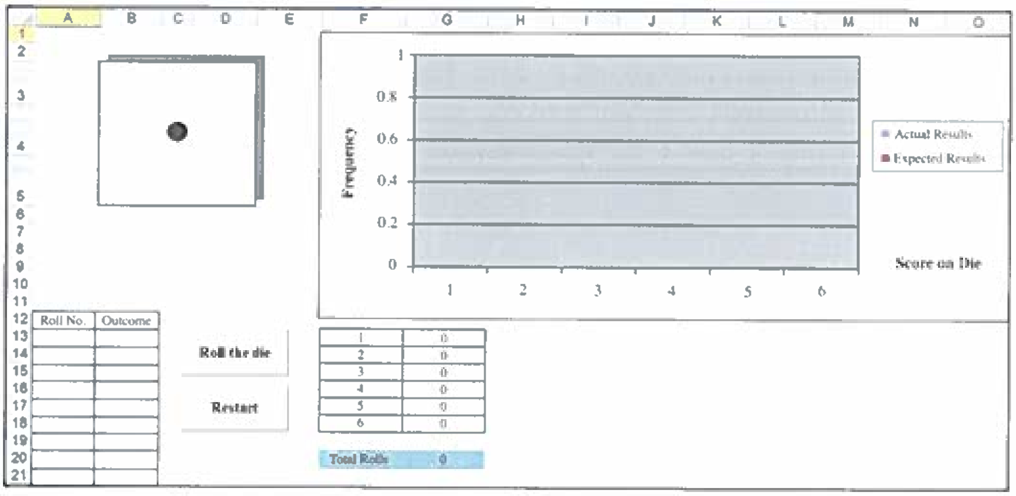


Fig. 15.1

1. Click on the button 'Roll the Die' and it will roll the die once. Repeat for a total of 20 rolls.
   Record the number of '1', '2', '3', '4', '5' and '6' obtained in the following table.
   Compute the fraction of obtaining each outcome.

Outcome	Number of corresponding outcomes for 20 rolls	Fraction of obtaining each corresponding outcome for 20 rolls
'1'		
'2'		
'3'		
'4'		
'5'		
'6'		

2. As a class, add and record the total number of '1', '2', '3', '4', '5' and '6' obtained by all students. Compute the fraction of obtaining each outcome.

Outcome for _____ rolls	Total number of corresponding outcomes	Fraction of obtaining each corresponding outcome
'1'		
'2'		
'3'		
'4'		
'5'		
'6'		

3. Look at the last column in the two tables. Do you notice that the probabilities of obtaining any one of the six outcomes approach $\frac{1}{6}$ when there are more rolls?

4. If we roll a die 600 times, would we expect to obtain *exactly* 100 '6'? Explain your answer.

When a die is rolled, there are six possible outcomes, i.e. 1, 2, 3, 4, 5 and 6. If the die is fair, then each of the six outcomes is equally likely to occur. Thus the chance of obtaining a 'six' is 1 out of 6. We say that the probability of obtaining a 'six' is $\frac{1}{6}$.

In general, in a probability experiment with m *equally likely* outcomes, if k of these outcomes favour the occurrence of an event E, then the probability, $P(E)$, of the event happening is given by:

$$P(E) = \frac{\text{Number of favourable outcomes for event } E}{\text{Total number of possible outcomes}} = \frac{k}{m}$$

This is known as **theoretical probability**, the probability that is obtained based on mathematical theory.

In the investigation, you conducted a probability experiment to determine the chance of obtaining a 'head' or a 'tail' when a coin is tossed. Based on theoretical probability, the probability of obtaining a 'head' is $\frac{1}{2}$, i.e. 10 'heads' in 20 tosses. However, it is unlikely that all your classmates obtained 10 'heads' in 20 tosses. The probability that you obtained in the experiment is known as **experimental probability**. Thus if you obtained 11 'heads' in 20 tosses, your experimental probability of getting a 'head' is $\frac{11}{20}$.

From the investigation, we can conclude that as the number of trials increases, the experimental probability of an outcome occurring tends towards the theoretical probability of the outcome happening.

Worked Example 3

(Probability involving Number Cards)

A card is drawn at random from a box containing 12 cards numbered 1, 2, 3, ..., 12. Find the probability of drawing

(i) a '7',

(ii) an even number,

(iii) a prime number,

(iv) a perfect square,

(v) a negative number,

(vi) a number less than 13.

Solution:

Total number of possible outcomes = 12

(i) P(drawing a '7') $= \frac{1}{12}$

(ii) There are 6 even numbers from 1 to 12, i.e. 2, 4, 6, 8, 10 and 12.

P(drawing an even number) $= \frac{6}{12}$

$= \frac{1}{2}$

(iii) There are 5 prime numbers from 1 to 12, i.e. 2, 3, 5, 7 and 11.

P(drawing a prime number) $= \frac{5}{12}$

(iv) There are 3 perfect squares from 1 to 12, i.e. 1, 4 and 9.

P(drawing a perfect square) $= \frac{3}{12}$

$= \frac{1}{4}$

(v) There are no negative numbers from 1 to 12.

P(drawing a negative number) $= \frac{0}{12}$

$= 0$

(vi) All the 12 numbers from 1 to 12 are less than 13.

P(drawing a number less than 13) $= \frac{12}{12}$

$= 1$

RECALL

A **prime number** is a positive integer that has *exactly 2 different factors*, 1 and itself. Thus 1 is not a prime number.

ATTENTION

- When the probability of an event occurring is 0, we say that it is an *impossible* event.
- When the probability of an event occurring is 1, we say that it is a *certain* event.

PRACTISE NOW 3

A ball is drawn at random from a bag containing some balls numbered 10, 11, 12, ..., 24. Find the probability of drawing

(i) a '21',

(ii) an odd number,

(iii) a composite number,

(iv) a perfect cube.

SIMILAR QUESTIONS

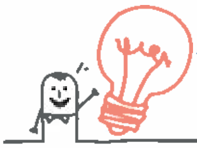
Exercise 15A Questions 3–4, 11, 18–20

RECALL

A **composite number** is a positive integer that has *more than 2 different factors*.

In Worked Example 3, we observe that the probability of drawing a negative number from the 12 cards numbered 1, 2, 3, ..., 12 is *0*. This means that we will *never* be able to draw a negative number from the 12 cards.

In the same worked example, we also notice that the probability of drawing a number less than 13 from the 12 cards is *1*. This means that we will *definitely* be able to draw a number less than 13 from the 12 cards.

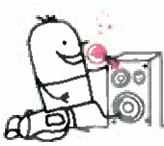


Thinking Time

1. In the thinking time on page 421, the event D 'It will rain in Pakistan at least once a year.' is a certain event, i.e. it will definitely occur. What can we say about the probability of D occurring?
2. In the thinking time on page 421, the event A 'The sun will rise from the west every day.' is an impossible event, i.e. it will never occur. What can we say about the probability of A occurring?
3. Is it possible that the probability of an event occurring is less than 0 or greater than 1?

From the above explanation and thinking time, we can conclude that:

> For any event E, $0 \leqslant P(E) \leqslant 1$.
>
> - $P(E) = 0$ if and only if E is an *impossible* event, i.e. it will *never* occur.
> - $P(E) = 1$ if and only if E is a *certain* event, i.e. it will *definitely* occur.



Performance Task

Probability theory was first used primarily in gambling problems. Girolamo Cardano (1501 – 1576), an Italian, wrote a gambler's manual which made use of probability theory. In 1654, Chevalier de Mere (1607 – 1684), a Frenchman, posed a gambling problem to his fellow countryman, Blaise Pascal (1623 – 1662). In response to this problem, Pascal and another French mathematician, Pierre Fermat (1601 – 1665), laid the foundations for the theory of probability.

This theory has widespread applications in business and in the sciences. Its applications range from the determination of life insurance premiums to the description of the behaviour of molecules in a gas. In fact, it can also be used to predict the outcome of an election.

Search on the Internet for other real-life applications of probability theory. Present your findings to the class.

Worked Example 4

(Probability involving Playing Cards)

A card is drawn at random from a standard pack of 52 playing cards. Find the probability of drawing

(i) a black card,

(ii) a red ace,

(iii) a diamond,

(iv) a card which is not a diamond.

ATTENTION

There are 4 suits in a standard pack of 52 playing cards, i.e. club ♣, diamond ♦, heart ♥ and spade ♠.

Each suit has 13 cards, i.e. Ace, 2, 3, ..., 10, Jack, Queen and King.

All the clubs and spades are black in colour.

All the diamonds and hearts are red in colour.

All the Jack, Queen and King cards are picture cards.

Solution:

Total number of possible outcomes = 52

(i) There are 26 black cards in the pack.

$$\text{P(drawing a black card)} = \frac{26}{52} = \frac{1}{2}$$

(ii) There are 2 red aces in the pack, i.e. the ace of hearts and the ace of diamonds.

$$\text{P(drawing a red ace)} = \frac{2}{52} = \frac{1}{26}$$

(iii) There are 13 diamonds in the pack.

$$\text{P(drawing a diamond)} = \frac{13}{52} = \frac{1}{4}$$

(iv) Since there are 13 diamonds in the pack, there are 52 – 13 = 39 cards which are not diamonds.

$$\text{P(drawing a card which is not a diamond)} = \frac{39}{52} = \frac{3}{4}$$

Notice that,

P(drawing a card which is not a diamond) = 1 – P(drawing a diamond).

In general, for any event E, we have:

$$\boxed{\text{P(not } E) = 1 - \text{P}(E)}$$

Internet Resources

Casinos make use of probability theory to set rules to ensure that they will always be on the winning side in the long run so that they will not go out of business. Search on the Internet for examples of how casinos use probability to their advantage.

PRACTISE NOW 4

A card is drawn at random from a standard pack of 52 playing cards. Find the probability of drawing

(i) a red card,

(ii) an ace,

(iii) the three of clubs,

(iv) a card which is not the three of clubs.

SIMILAR QUESTIONS

Exercise 15A Questions 5, 12–14

Worked Example 5

(Probability involving Letters of the English Alphabet)

A letter is chosen at random from the word 'MATHEMATICS'.
Find the probability that the letter is
(i) an 'A',
(ii) a vowel,
(iii) not a vowel.

Solution:

Total number of letters = 11

(i) There are 2 'A's.

P(an 'A' is chosen) = $\frac{2}{11}$

(ii) There are 4 vowels, i.e. 2 'A's, 1 'E' and 1 'I'.

P(letter chosen is a vowel) = $\frac{4}{11}$

(iii) **Method 1:**

There are 7 consonants, i.e. 1 'C', 1 'H', 2 'M's, 1 'S' and 2 'T's.

P(letter chosen is not a vowel) = $\frac{7}{11}$

Method 2:

P(letter chosen is not a vowel) = 1 – P(letter chosen is a vowel)

$$= 1 - \frac{4}{11}$$

$$= \frac{7}{11}$$

ATTENTION

There are 26 letters in the alphabets. 5 of them are vowels, namely 'a', 'e', 'i', 'o', and 'u'.

PRACTISE NOW 5

SIMILAR QUESTIONS

Exercise 15A Questions 6–10, 15–17

1. A letter is chosen at random from the word 'CHILDREN'. Find the probability that the letter is
   (i) a 'D',
   (ii) a consonant,
   (iii) not a consonant.

2. A marble is drawn at random from a bag containing 9 red marbles, 6 yellow marbles, 4 purple marbles and 5 blue marbles. Find the probability of drawing
   (i) a purple marble,
   (ii) a red or a blue marble,
   (iii) a white marble,
   (iv) a marble that is not white.

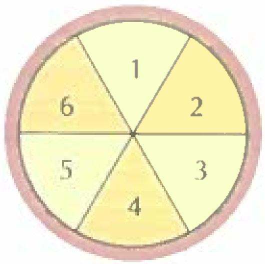
3. A box contains 24 balls, some of which are red, some of which are green and the rest are blue. The probabilities of drawing a red ball and a green ball at random from the box are $\frac{1}{3}$ and $\frac{1}{6}$ respectively. Find the number of blue balls in the box.

The first three children of a couple are boys. What is the probability that their next child will be a girl?

Exercise 15A

BASIC LEVEL

1. A dart board is divided into 6 equal sectors. When a dart lands on it, the number of the sector on which it lands is noted. Write down the sample space and state the total number of possible outcomes.



2. For each of the following experiments, write down the sample space and state the total number of possible outcomes.
   (a) Tossing a fair tetrahedral die with faces labelled 2, 3, 4 and 5 respectively
   (b) Drawing a card at random from a box containing ten identical cards labelled A, B, C, D, E, F, G, H, I, J
   (c) Drawing a disc at random from a bag containing 5 identical red discs, 3 identical blue discs and 2 identical green discs
   (d) Picking a letter at random from a box containing identical cards with letters that spell the word 'TEACHER'
   (e) Choosing a three-digit number at random

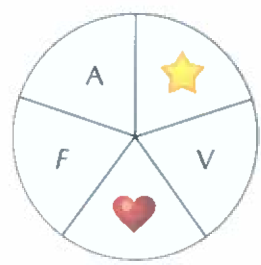
3. An 8-sided fair die with faces labelled 2, 3, 3, 4, 7, 7, 7 and 9 is rolled once. Find the probability of getting
   (i) a '7',
   (ii) a '3' or a '4',
   (iii) a number less than 10,
   (iv) a number which is not '2'.

4. A card is drawn at random from a box containing some cards numbered 10, 11, 12, ..., 22. Find the probability of drawing
   (i) an even number,
   (ii) a number between 13 and 19 inclusive,
   (iii) a prime number that is less than 18,
   (iv) a number greater than 22,
   (v) a number that is divisible by 4.

5. A card is drawn at random from a standard pack of 52 playing cards. Find the probability of drawing
   (i) the ace of spades,
   (ii) a heart or a club,
   (iii) a picture card,
   (iv) a non-picture card.

6. Each of the letters of the word 'PROBABILITY' is written on a card. All the cards are well-shuffled and placed face down on a table. A card is turned over. Find the probability that the card shows
   (i) the letter 'A',
   (ii) the letter 'B',
   (iii) a vowel,
   (iv) a consonant.

7. A spinner is divided into 5 equal sectors. When the spinner is spun, what is the probability that the pointer will stop at a sector whose label is
   (i)  ?
   (ii) a letter of the English alphabet?
   (iii) a vowel?
   (iv) a consonant?



8. A bag contains 4 pieces of candy – caramel, chocolate, gummies and licorice. A piece of candy is removed at random from the bag. Find the probability that the candy is
   (i) a caramel,
   (ii) either a chocolate or a gummy,
   (iii) not a licorice.

9. An envelope contains 40 shopping vouchers, of which 25 vouchers each have a value of \$50 and 15 vouchers each have a value of \$100. Amirah picks a voucher at random from the envelope. Find the probability that the voucher has a value of \$100.

10. A group of 30 people consisting of 9 men, 6 women, 12 boys and 3 girls, are waiting to get their passport photographs taken. A person is selected at random from the group. Find the probability that the person is

(i) a male,

(ii) either a woman, a boy or a girl.

INTERMEDIATE LEVEL

11. A two-digit number is chosen at random. Find the probability that the number is

(i) less than 20,

(ii) a perfect square.

12. Two Joker cards are added to a standard pack of 52 playing cards. A card is then drawn at random from the 54 cards. Find the probability of drawing

(i) a red card,

(ii) a two,

(iii) a joker,

(iv) a queen or a king.

Note: A Joker card is neither a black nor a red card.

13. All the clubs are removed from a standard pack of 52 playing cards. A card is drawn at random from the remaining cards. Find the probability of drawing

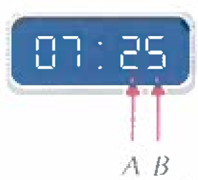
(i) a black card,

(ii) a diamond,

(iii) a picture card,

(iv) a card which is not an ace.

14. Raj wakes up in the morning and notices that his digital clock reads 07 25.



After noon, he looks at the clock again. What is the probability that

(i) the number in column A is a 4?

(ii) the number in column B is an 8?

(iii) the number in column A is less than 6?

(iv) the number in column B is greater than 5?

15. A box contains 2 dozen pairs of contact lenses, of which 8 pairs are tinted. A pair of contact lenses is drawn at random from the box. Find the probability that it is not tinted.

16. The table shows the number of each type of school personnel at a school.

Personnel	Management	Teaching	Laboratory	Administrative	Maintenance
Number of personnel	26	62	8	9	12

(a) If a school personnel is selected at random, find the probability that the school personnel is
  (i) a teacher,
  (ii) a management staff,
  (iii) an administrative or a maintenance staff.

(b) Two teachers and an administrative staff resign from the school. A school personnel is selected at random from the remaining staff. Find the probability that the school personnel is
  (i) an administrative staff,
  (ii) not a laboratory staff.

17. There are a total of 117 pairs of socks in a clothes bin. Each pair of socks is placed in a bag. The probabilities of selecting a yellow pair of socks and a grey pair of socks at random from the bin are $\frac{2}{9}$ and $\frac{3}{13}$ respectively. Find the number of pairs of socks in the bin which are
(i) yellow,
(ii) neither yellow nor grey.

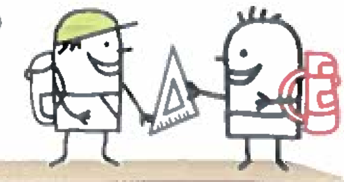
ADVANCED LEVEL

18. An IQ test consists of 80 multiple-choice questions. A question is selected at random. Find the probability that the question number
(i) contains only a single digit,
(ii) is greater than 67,
(iii) contains exactly one '7',
(iv) is divisible by both 2 and 5.

19. Each of the numbers 2, 3, 5 and 7 is written on a card. Two of the cards are drawn at random to form a two-digit number. Find the probability that the two-digit number is
(i) divisible by 4,
(ii) a prime number.

20. A biased tetrahedral die with faces labelled 1, 2, 3 and 4 is rolled once. The chance of getting a '3' is twice that of getting a '1'. The chance of getting a '2' is thrice that of getting a '3'. There is an equal chance of getting a '2' and a '4'. Find the probability of getting a prime number.

15.4 Further Examples on Probability of Single Events



In this section, we will take a look at more examples that involve probability.

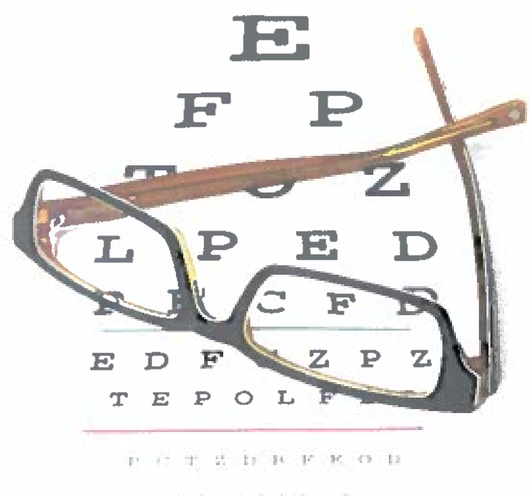
Worked Example 6

(Probability involving Groups of People)

In a class of 30 students, there are 12 girls and 2 of them are short-sighted. 6 of the boys are not short-sighted. If a student is chosen at random, find the probability that the student is

(i) a boy,

(ii) short-sighted.



Solution:

(i) Number of boys $= 30 - 12$

$= 18$

$$\text{P(student chosen is a boy)} = \frac{18}{30} = \frac{3}{5}$$

(ii) Number of girls who are short-sighted $= 2$

Number of boys who are short-sighted $= 18 - 6$

$= 12$

Number of students who are short-sighted $= 2 + 12$

$= 14$

$$\text{P(student chosen is short-sighted)} = \frac{14}{30} = \frac{7}{15}$$

PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 15B Questions 1–2, 6–7

In a class of 40 students, there are 24 boys and 16 of them are not short-sighted. 4 of the girls are short-sighted. If a student is chosen at random, find the probability that the student is

(i) a girl,

(ii) not short-sighted.

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Worked Example 7

(Probability involving Angles of Sectors)

A circle is divided into sectors of different colours. A point is selected at random in the circle. Find the probability that the point lies in the

(i) yellow sector,
(ii) green sector,
(iii) black sector.

Solution:

(i) $$\text{P(point selected lies in the yellow sector)} = \frac{\text{Area of the yellow sector}}{\text{Area of the circle}}$$

$$= \frac{\text{Angle of the yellow sector}}{360°}$$

$$= \frac{180°}{360°}$$

$$= \frac{1}{2}$$

(ii) Angle of the green sector $= 360° - 180° - 90° - 45°$

$= 45°$

$$\text{P(point selected lies in the green sector)} = \frac{\text{Area of the green sector}}{\text{Area of the circle}}$$

$$= \frac{\text{Angle of the green sector}}{360°}$$

$$= \frac{45°}{360°}$$

$$= \frac{1}{8}$$

(iii) $$\text{P(point selected lies in the black sector)} = \frac{\text{Area of the black sector}}{\text{Area of the circle}}$$

$$= \frac{0}{\text{Area of the circle}}$$

$$= 0$$

ATTENTION

Since a point is selected at random, any point in the circle will have the same chance of being selected. We assume that the point will not fall on any of the lines separating the four sectors.

Problem Solving Tip

The number of points in a sector is proportional to the area of the sector, which is proportional to the angle of the sector.

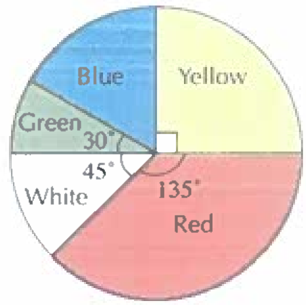
PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 15B Questions 3–4

A circle is divided into sectors of different colours. A point is selected at random in the circle. Find the probability that the point lies in the

(i) red sector,
(ii) blue sector,
(iii) purple sector,
(iv) green or white sector.



Worked Example 8

(Probability involving Algebra)

A box contains x red marbles, $(x + 3)$ yellow marbles and $(4x - 15)$ blue marbles.

(i) Find an expression, in terms of x, for the total number of marbles in the box.

(ii) A marble is drawn at random from the box. Write down an expression, in terms of x, for the probability that the marble is blue.

(iii) Given that the probability in **(ii)** is $\frac{1}{2}$, find the value of x.

Solution:

(i) Total number of marbles $= x + (x + 3) + (4x - 15)$
$= 6x - 12$

(ii) P(drawing a blue marble) $= \dfrac{4x - 15}{6x - 12}$

(iii) Given that $\dfrac{4x-15}{6x-12} = \dfrac{1}{2}$,

$$
\begin{aligned}
2(4x - 15) &= 6x - 12 \\
8x - 30 &= 6x - 12 \\
2x &= 18 \\
x &= 9
\end{aligned}
$$

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 15B Questions 5, 8–13

1. There are 12 green balls and $(x + 2)$ yellow balls in a box.

(i) Find an expression, in terms of x, for the total number of balls in the box.

(ii) A ball is drawn at random from the box. Write down an expression, in terms of x, for the probability that the ball is yellow.

(iii) Given that the probability in **(ii)** is $\frac{2}{5}$, find the value of x.

2. There are 28 boys and 25 girls in a school hall. After y girls leave the hall, the probability of selecting a girl at random becomes $\frac{3}{7}$. Find the value of y.

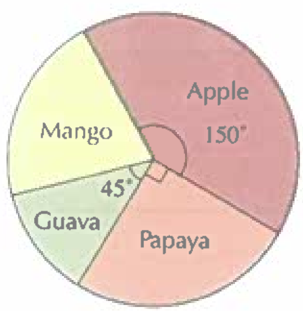
Exercise 15B

BASIC LEVEL

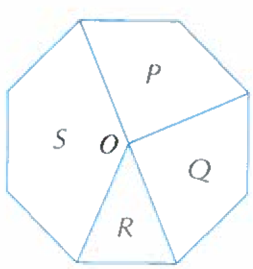
1. A class of 30 students consists of 8 Chinese girls, 3 Malay girls, 1 Indian girl, 11 Chinese boys, 4 Malay boys and 3 Indian boys. If a student is chosen at random to take part in a survey, find the probability that the student is
   (i) a girl,
   (ii) not a Chinese,
   (iii) not an Indian boy,
   (iv) a Eurasian.

2. Shirley has 5 novels and 5 comic books in her bag. Three of her books are in Japanese, 2 of which are comic books. The rest of her books are in English. If a book is chosen at random from her bag, find the probability of choosing
   (i) a book in Japanese,
   (ii) a novel which is in English.

3. A survey is conducted to find out which of the four fruits, apple, papaya, guava and mango, the students in a class prefer. The pie chart shows the results of the survey. A student is selected at random. Find the probability that the student prefers
   (i) apple,
   (ii) mango,
   (iii) papaya or guava.



4. A regular octagon is divided into 4 regions, where O is its centre. A point is selected at random in the octagon. Find the probability that the point lies in
   (i) region R,
   (ii) region S,
   (iii) region P or Q.



5. There are 15 girls and x boys at a school parade square.
   (i) Write down an expression, in terms of x, for the total number of students at the school parade square.
   (ii) A student is selected at random. Write down an expression, in terms of x, for the probability that the student is a girl.
   (iii) Given that the probability in **(ii)** is $\frac{1}{5}$, find the value of x.

INTERMEDIATE LEVEL

6. A class of 38 students went on a short trip to Bangkok. Of the 18 boys, 6 of them checked in their luggage at the airport. 8 of the girls did not check in their luggage. If a student is chosen at random, find the probability that the student
   (i) is a girl who did not check in her luggage,
   (ii) checked in his/her luggage.

7. **(a)** A class has 16 boys and 24 girls. Of the 16 boys, 3 are left-handed. Of the 24 girls, 2 are left-handed. If a student is chosen at random to clean the whiteboard, find the probability that the student is
   (i) a boy,
   (ii) left-handed.

   (b) The student chosen to clean the whiteboard in **(a)** is a girl who is not left-handed. Another student is selected at random from the remaining students to borrow the visualiser from the class next door. Find the probability that the student is
   (i) a boy who is left-handed,
   (ii) a girl who is not left-handed.

8. Santa Claus has $(3h + 11)$ red presents and $(h + 5)$ white presents in his stocking. Ethan selects a present at random from the stocking. Given that the probability that he obtains a red present is $\frac{19}{26}$, find the value of h.

9. Some patients participated in a clinical trial for a new drug to treat osteoporosis. A patient is selected at random. The probability that the patient had no change in his bone mass density is $\frac{7}{13}$, the probability that he had a slight reduction in his bone mass density is $\frac{1}{k}$ and the probability that he had a significant reduction in his bone mass density is $\frac{1}{2k}$. Find the value of k.

10. A carton contains 15 toothbrushes, of which p have soft bristles. After 5 more toothbrushes with soft bristles are added to the carton, the probability of drawing a toothbrush with soft bristles becomes $\frac{3}{4}$. Find the value of p.

11. There are 23 boys and 35 girls on the school's track and field team. After q boys and $(q + 4)$ girls graduate at the end of this year, the probability of selecting a boy at random to represent the school for an event becomes $\frac{2}{5}$. Find the value of q.

ADVANCED LEVEL

12. A bag contains 40 balls, some of which are red, some of which are yellow and the rest are black. The probabilities of drawing a red ball and a yellow ball at random from the bag are $\frac{1}{4}$ and $\frac{2}{5}$ respectively.
    (i) Find the probability of drawing a black ball at random from the bag.

    $(2x + 1)$ red balls and $(x + 2)$ yellow balls are added to the bag while $(x - 3)$ black balls are removed from the bag. The probability of drawing a yellow ball at random from the bag is now $\frac{3}{7}$. Find
    (ii) an expression, in terms of x, for the total number of balls in the bag now,
    (iii) the number of yellow balls in the bag now.

13. There are 50 students in an auditorium, of which $2x$ are boys and y are girls. After $(y - 6)$ boys leave the auditorium and $(2x - 5)$ girls enter the auditorium, the probability of selecting a girl at random becomes $\frac{9}{13}$. Find the value of x and of y.

1. **Probability** is a measure of chance.
2. A **sample space** is the collection of all the possible outcomes of a probability experiment.
3. In a probability experiment with m *equally likely* outcomes, if k of these outcomes favour the occurrence of an event E, then the probability, $P(E)$, of the event happening is given by:

$$P(E) = \frac{\text{Number of favourable outcomes for event } E}{\text{Total number of possible outcomes}} = \frac{k}{m}$$

4. For any event E, $0 \leqslant P(E) \leqslant 1$.
    - $P(E) = 0$ if and only if E is an *impossible* event, i.e. it will *never* occur.
    - $P(E) = 1$ if and only if E is a *certain* event, i.e. it will *definitely* occur.
5. For any event E, $P(\text{not } E) = 1 - P(E)$.

Review Exercise 15

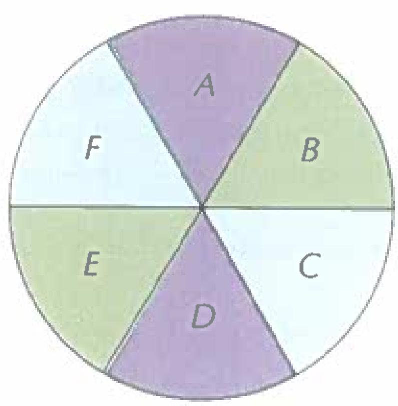
1. Each of the numbers 5, 6 and 8 is written on a card. One or more of these cards are drawn at random to form a one-, two- or three-digit number.
   (a) For this experiment,
      (i) write down the sample space,
      (ii) state the total number of possible outcomes.
   (b) Find the probability that the number formed
      (i) consists of two digits,
      (ii) is a multiple of 5.

2. A 6-sided fair die is rolled once. Find the probability of getting
   (i) an even number,
   (ii) a composite number,
   (iii) a number that is divisible by 4.

3. All the 26 red cards from a standard pack of playing cards are mixed thoroughly. A card is then drawn at random. Find the probability of drawing
   (i) the queen of hearts,
   (ii) the jack of clubs,
   (iii) either the six of hearts or the seven of diamonds,
   (iv) a card which is not a nine.

4. In a shopping mall, if a customer spends a minimum of \$500 in a single receipt, he has the chance to spin a wheel to win a prize. The wheel is divided into 6 equal sectors. The prizes correspond to the letters on the wheel.

   A: \$30 shopping voucher
   B: \$50 supermarket voucher
   C: watch worth \$60
   D: umbrella
   E: \$40 dining voucher
   F: 1 kg cheesecake



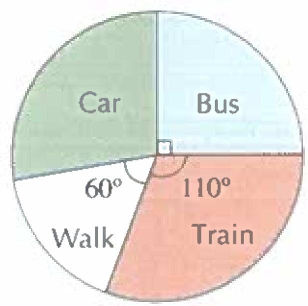
   Find the probability that a customer who spins the wheel wins
   (i) an umbrella,
   (ii) a voucher,
   (iii) \$100 cash.

5. A reel of a fair slot machine has 22 symbols, of which 4 are cherries, 7 are oranges, 9 are peaches and the rest are grapes. Find the probability that the reel will stop to show
   (i) the symbol 'orange',
   (ii) the symbol 'grape',
   (iii) the symbol 'pineapple',
   (iv) either the symbol 'cherry' or the symbol 'peach'.

6. A bag contains 20 sweets, of which 7 are toffee wrapped in green paper, 6 are mints wrapped in green paper, 3 are toffee wrapped in red paper and 4 are mints wrapped in red paper. If a sweet is drawn at random from the bag, find the probability that the sweet is
   (i) a mint wrapped in red paper,
   (ii) a toffee,
   (iii) wrapped in green paper.

7. A bag contains 7 white staplers and 11 orange staplers.
   (a) If a stapler is drawn at random from the bag, find the probability that the stapler is
      (i) green,
      (ii) either white or orange.
   (b) 12 red staplers are added to the bag. A stapler is then drawn at random from the bag. Find the probability that the stapler is
      (i) red,
      (ii) not orange.

8. A survey is conducted to find out how the students in a class travel to school. The students either take the bus, train, the car or walk to school. The pie chart shows the results of the survey. A student is selected at random. Find the probability that the student travels to school
   (i) by car,
   (ii) by train or on foot,
   (iii) by bicycle.



9. A bed of flowers consists of 100 stalks of flowers, of which 20 are lilies, h are roses and the rest are tulips.
   (i) Given that the probability of picking a stalk of tulip at random is $\frac{1}{4}$, find the value of h.
   (ii) 10 stalks of lilies are removed from the bed. A stalk of flower is picked at random from the remaining stalks of flowers. Find the probability that a stalk of rose is picked.

10. In a car park, there are 125 cars, $3p$ motorcycles, $2q$ lorries and 20 buses. One of the vehicles leaves the car park at random.
   (i) Given that the probability that the vehicle is a motorcycle is $\frac{3}{40}$, form an equation in p and q.
   (ii) Given that the probability that the vehicle is a bus is $\frac{1}{10}$, form another equation in p and q.
   (iii) Hence, find the value of p and of q.

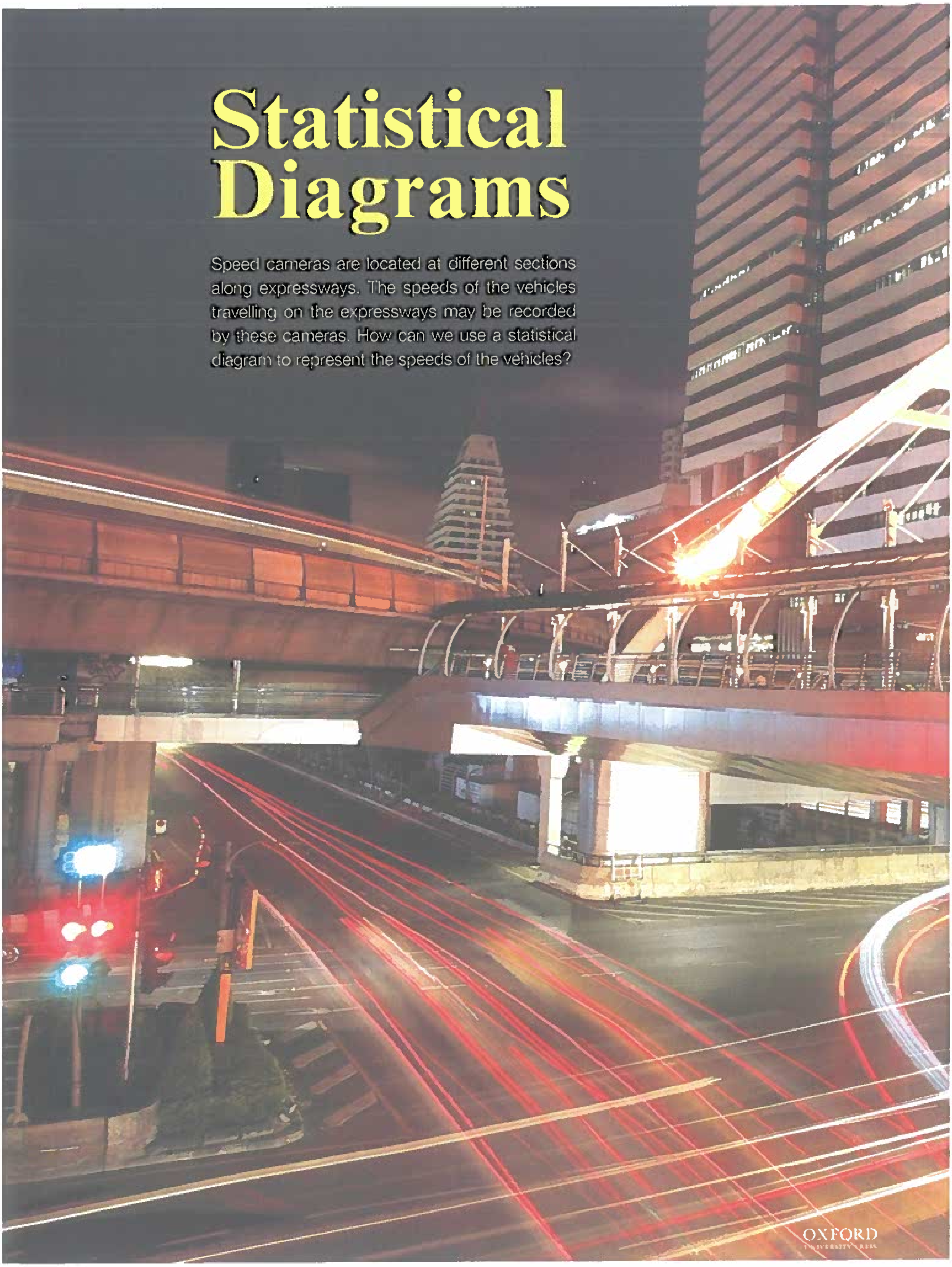
Challenge Yourself

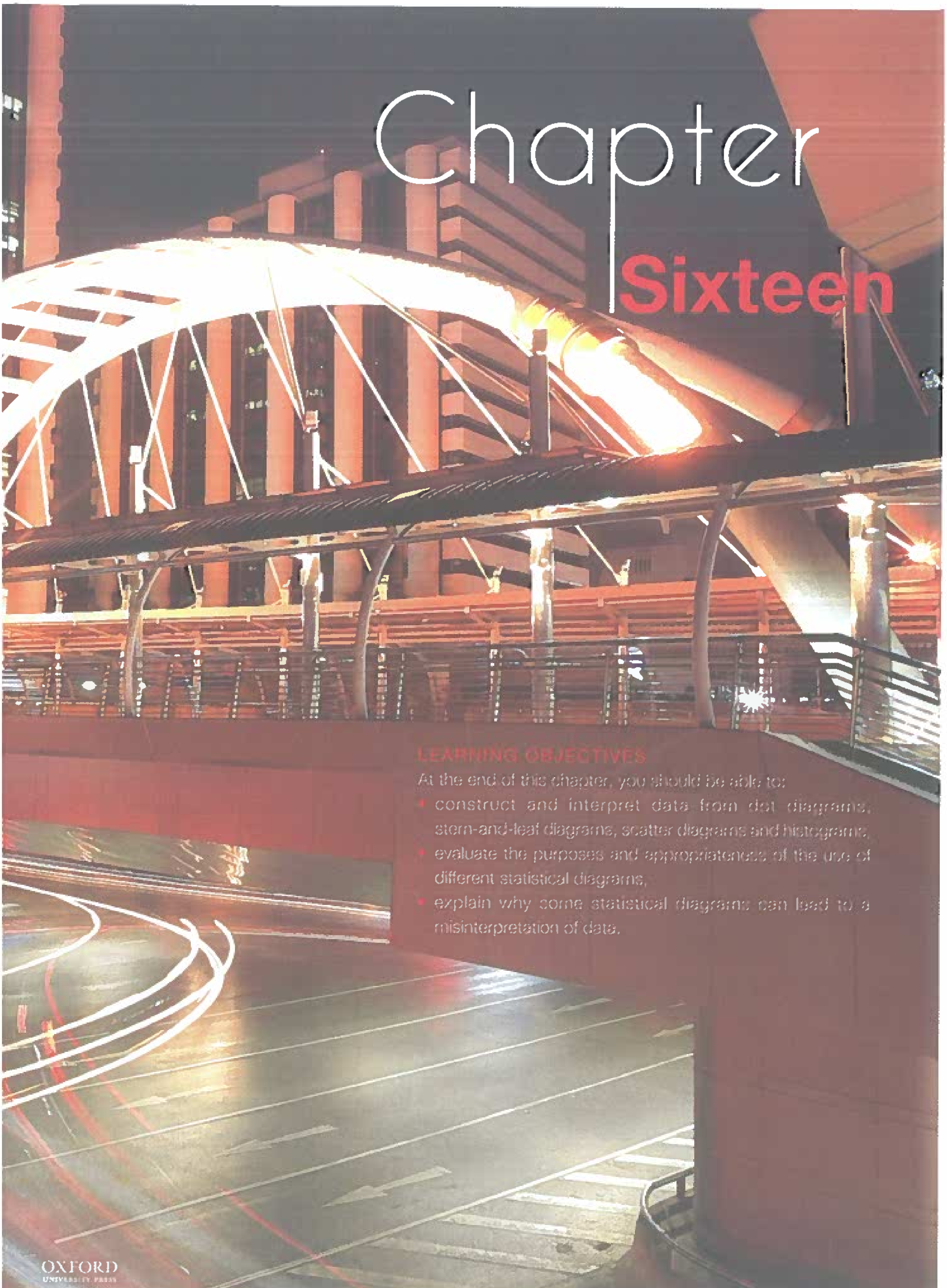
1. A standard pack of 52 playing cards is randomly divided into two unequal piles. Given that the probability of drawing a picture card from the smaller pile is $\frac{4}{11}$ while the probability of drawing a non-picture card from the bigger pile is $\frac{13}{15}$, find the number of cards in each pile.

2. Rui Feng writes 3 letters to 3 of his friends, Farhan, Vishal and Michael. He types each of their addresses on each of the 3 envelopes and puts the letters into the envelopes randomly before he sends them out. Find the probability that
   (i) exactly one of his friends receive the correct letter,
   (ii) exactly two of his friends receive the correct letters,
   (iii) all three of his friends receive the correct letters.

Statistical Diagrams

Speed cameras are located at different sections along expressways. The speeds of the vehicles travelling on the expressways may be recorded by these cameras. How can we use a statistical diagram to represent the speeds of the vehicles?





Chapter Sixteen

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- construct and interpret data from dot diagrams, stem-and-leaf diagrams, scatter diagrams and histograms,
- evaluate the purposes and appropriateness of the use of different statistical diagrams,
- explain why some statistical diagrams can lead to a misinterpretation of data.

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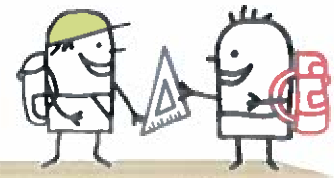
16.1 Statistical Diagrams



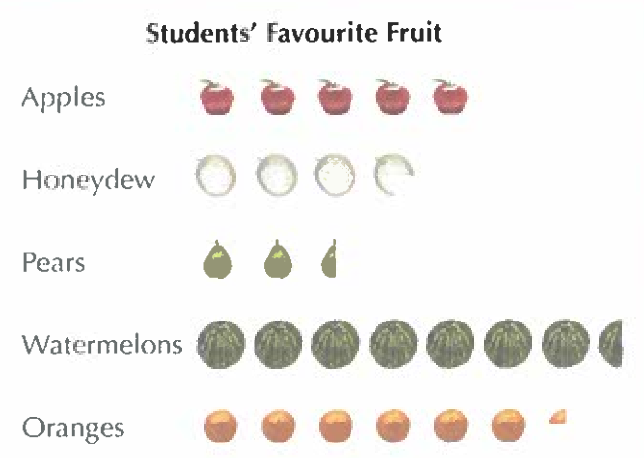
Recap (Pictograms, Bar Graphs, Pie Charts and Line Graphs)

The choice of an *appropriate* statistical diagram depends on the *type of data collected* and the *purpose* of collecting the data.

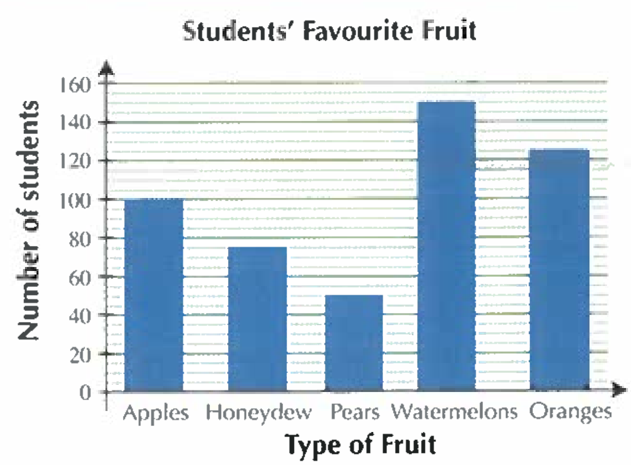
INFORMATION

'Data' is the plural of 'datum'.

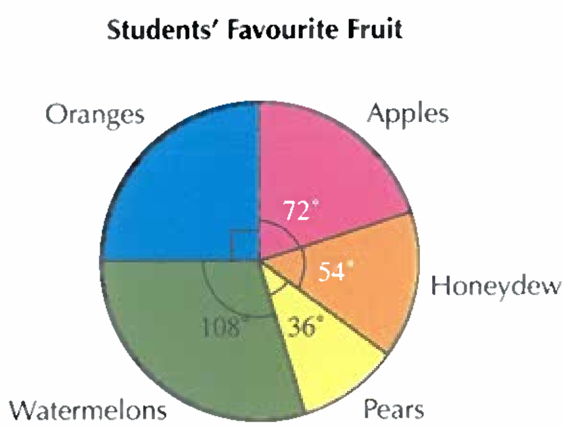
In Book 1, we have learnt the following statistical diagrams as shown in Table 16.1.



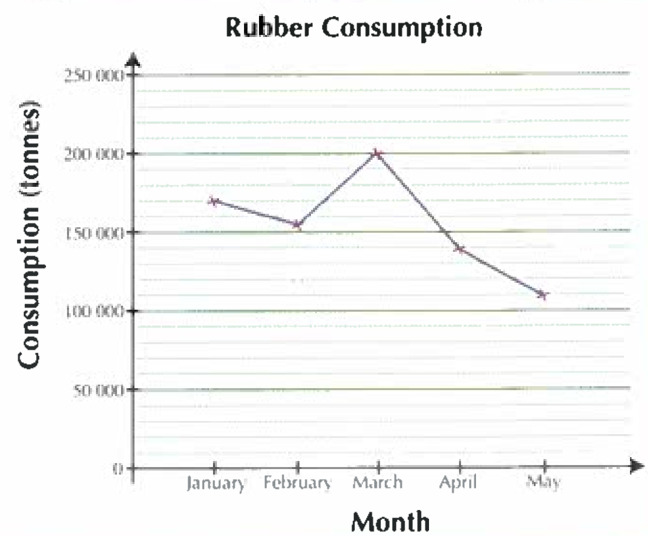
Pictogram



Bar graph



Pie chart

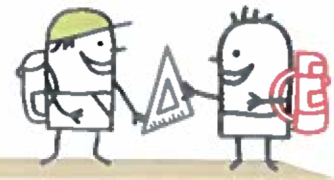


Line graph

Table 16.1

In this chapter, we will learn about dot diagrams, stem-and-leaf diagrams and histograms.

16.2 Dot Diagrams



A **dot diagram** consists of a horizontal number line and dots placed above the number line. The dots represent the values in a set of data.

Worked Example 1

(Dot Diagram)

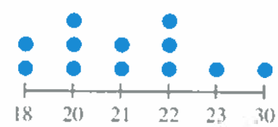
The table shows the time taken, in minutes, for 12 employees of a company to travel from their homes to the office.

20	22	21	21	18	18
30	20	22	23	22	20

(i) Represent the data on a dot diagram.
(ii) Briefly describe the distribution of the data.

ATTENTION

The values on the number line should always be at *equal intervals*. For example, the dot diagram in Worked Example 1 should not be drawn like this:

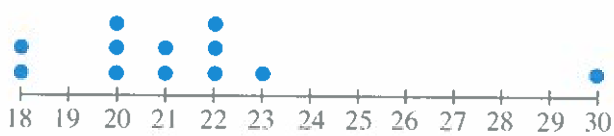


Solution:

(i) **Step 1:** Identify the range of values, i.e. 18 minutes to 30 minutes.

Step 2: Draw a horizontal number line with equal intervals to represent the range of values identified in Step 1.

Step 3: Using the data from the table, plot each datum with a dot over its value on the number line.



(ii) The time taken for the 12 employees to travel from their homes to the office ranges from 18 minutes to 30 minutes. The time taken is clustered around 20 to 23 minutes. There is an extreme value of 30 minutes.

ATTENTION

To describe a distribution, we may consider the following.

- **Range** (variation of data)
- **Clusters** (data that is visually grouped about certain values)
- **Extreme data** (data that deviates significantly from the other data)
- **Symmetry** (equal number of data on each side of a value)

PRACTISE NOW 1

The table shows the marks obtained by 16 students in a class for a Mathematics test.

3	18	20	19	14	9	8	13
16	19	14	10	14	10	12	10

(i) Represent the data on a dot diagram.
(ii) If 50% of the students passed the test, find the passing mark for the test.
(iii) Briefly describe the distribution of the data.

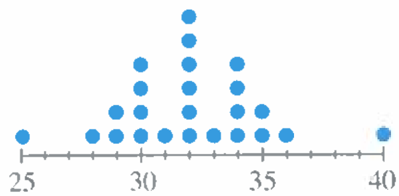
SIMILAR QUESTIONS

Exercise 16A Questions 1, 4, 12

Worked Example 2

(Dot Diagram)

The dot diagram represents the masses, in kg, of 24 students.



ATTENTION

An advantage of a dot diagram is that it is an easy way to display small sets of data that do not contain many distinct values. A disadvantage is that if a set of data is too large, the diagram may appear packed.

(i) What is the most common mass?

(ii) Calculate the percentage of students who have a mass of more than or equal to 32 kg.

(iii) A student is selected at random. The probability that the student has a mass of at least x kg is $\frac{1}{3}$. Find the value of x.

(iv) Briefly describe the distribution of the data.

Solution:

(i) The most common mass is 32 kg.

(ii) Percentage of students who have a mass of more than or equal to 32 kg

$= \frac{15}{24} \times 100\%$

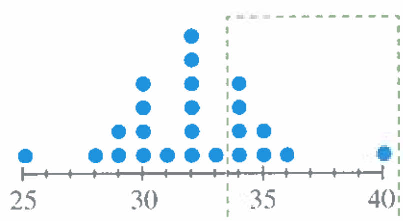
$= 62.5\%$

(iii) Number of students who have a mass of at least x kg

$= \frac{1}{3} \times 24$ (since probability $= \frac{1}{3}$)

$= 8$

Since $4 + 2 + 1 + 1 = 8$,



$\therefore$ Value of $x = 34$

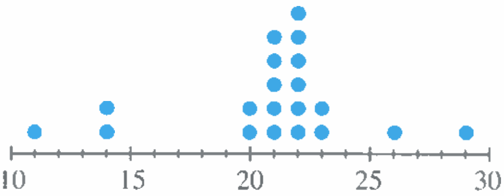
(iv) The masses of the 24 students range from 25 kg to 40 kg. The masses are symmetrical about 32 kg. The extreme masses of 25 kg and 40 kg deviate considerably from the other masses recorded.

PRACTISE NOW 2

SIMILAR QUESTIONS

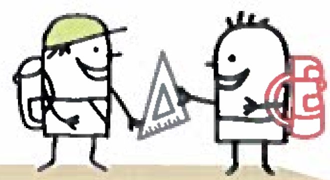
Exercise 16A Questions 2, 5-6

The dot diagram represents the ages, in years, of contestants who join a karaoke contest.



(i) How old is the youngest contestant?
(ii) Find the total number of contestants who join the karaoke contest.
(iii) A contestant is selected at random. The probability that the age of the contestant is less than or equal to x years is $\frac{1}{4}$. Find the value of x.
(iv) Briefly describe the distribution of the data.

16.3 Stem-and-Leaf Diagrams



A **stem-and-leaf diagram** consists of a few stems, each with a different number of leaves. Fig. 16.1 shows an example of a stem-and-leaf diagram.

Stem	Leaf
2	0 4 4 5 5 8 9
3	0 0 0 6 6 7 7 7 7 7 7 8 8 8 9 9
4	0 4 4 8

Key: 2 | 4 means 24

Fig. 16.1

In Fig. 16.1, each stem represents a digit in the tens place and each leaf represents a digit in the ones places (refer to the key). For example, all the numbers 20, 24, 24, 25, 25, 28 and 29 have a common stem 2. What is the common stem of 40, 44, 44 and 48?

ATTENTION

- In a stem-and-leaf diagram, the stems must be arranged in numerical order.
- The leaves must be recorded in ascending order. Do not use commas to separate the leaves.
- The number of leaves must tally with the total number of data collected.
- An advantage of a stem-and-leaf diagram is that the individual data values are retained; and if the appropriate units are used, the shape of the data distribution can be easily observed.
- A disadvantage is that it is not useful if there are extreme values in a set of data as many stems with no leaves will be present.

Worked Example 3

(Stem-and-Leaf Diagram)

The table shows the masses, in g, of 10 salt containers.

56	58	67	72	61
63	76	50	64	79

Represent the data using a stem-and-leaf diagram.

Solution:

arrange the stems in numerical order

Stem	Leaf
5	0 6 8
6	1 3 4 7
7	2 6 9

Key: 5 | 6 means 56 g

arrange the leaves in ascending order

PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 16A Questions 3, 13

The table shows the number of text messages sent by a group of students on a typical weekend.

20	33	30	59	14	41
56	12	37	44	23	35

Represent the data using a stem-and-leaf diagram.

Sometimes, a stem-and-leaf diagram may have more leaves on some stems, such as the leaves corresponding to the stem 3 in Fig. 16.1. When this happens, we can break each stem into two halves – one for the leaves numbered 0 to 4 and the other for the leaves numbered 5 to 9. We will then obtain a **stem-and-leaf diagram with split stems** (see Fig. 16.2).

Stem	Leaf
2	0 4 4
2	5 5 8 9
3	0 0 0
3	6 6 7 7 7 7 7 7 8 8 8 9 9
4	0 4 4
4	8

Key: 2 | 4 means 24

Fig. 16.2

Worked Example 4

(Stem-and-Leaf Diagram with Split Stems)

The stem-and-leaf diagram represents the amount of money, in dollars, collected by a group of students during a charity run.

Stem	Leaf
4	0 4 8 9
5	2 3 4 4 7 9 9 9 9 9
6	0 0 0 5 6 7 9

Key: 4 | 8 means \$48

(i) Construct a stem-and-leaf diagram with split stems.
(ii) What is the amount of money that was most commonly collected?
(iii) Find the ratio of the number of students who collected more than \$40 but less than \$55 to the total number of students.

Solution:

(i) To construct a stem-and-leaf diagram with split stems, we separate the stems into smaller number of equal-sized units.

Stem	Leaf	
4	0 4	(consists of values from 40 – 44)
4	8 9	(consists of values from 45 – 49)
5	2 3 4 4	(consists of values from 50 – 54)
5	7 9 9 9 9 9	(consists of values from 55 – 59)
6	0 0 0	(consists of values from 60 – 64)
6	5 6 7 9	(consists of values from 65 – 69)

Key: 4 | 8 means \$48

(ii) The amount of money that was most commonly collected is \$59.
(iii) Required ratio = 7 : 21
= 1 : 3

PRACTISE NOW 4

SIMILAR QUESTIONS

Exercise 16A Questions 7–9

The stem-and-leaf diagram represents the masses, in kg, of some boxes.

Stem	Leaf
8	0 3 4 6 6 6 7 8 8 9 9
9	0 0 0 0 1 1 2 2 5 6 7 7 9
10	0 1 2 4 4 6 8 8

Key: 8 | 3 means 83 kg

(i) Construct a stem-and-leaf diagram with split stems.
(ii) What is the most common mass?
(iii) Find the percentage of boxes with a mass more than 85 kg but less than 90 kg.

When two sets of data are given, we can use a stem-and-leaf diagram with a common stem to represent the data. This is known as a **back-to-back stem-and-leaf diagram.**

Worked Example 5

Back-to-Back Stem-and-Leaf Diagram

The lengths, in mm, of terrapins from two different habitats are recorded in the table.

Habitat *A*	55	60	71	50	70	69	53
	50	55	62	58	64	65	
Habitat *B*	60	68	63	59	77	73	69
	64	69	75	72	70	55	

(i) Represent these two sets of data by a back-to-back stem-and-leaf diagram.

(ii) Which habitat is more suitable for the growth of terrapins? Explain your answer.

Solution:

(i) A back-to-back stem-and-leaf diagram consists of a stem and leaves on both sides of the stem.

Step 1: Construct a stem-and-leaf diagram for Habitat *A*.

Step 2: Using the same stem, construct the stem-and-leaf diagram for Habitat *B* on the left side of the stem.

Leaves for Habitat *B*	Stem	Leaves for Habitat *A*
9 5	5	0 0 3 5 5 8
9 9 8 4 3 0	6	0 2 4 5 9
7 5 3 2 0	7	0 1

Key: 5 | 5 means 55 mm

(ii) Habitat *B* is more suitable for the growth of terrapins as there are more terrapins with greater lengths (between 70 and 77 mm) in Habitat *B* than in Habitat *A*.

ATTENTION

The leaves corresponding to Habitat *B* are arranged in ascending order from the right to the left.

PRACTISE NOW 5

The times, in minutes, Michael and Khairul spent using their tablet computers for the past 20 days are recorded in the table.

Michael	83	59	83	89	77	72	81	91	98	58
	83	47	70	65	65	84	88	81	89	99
Khairul	41	79	52	90	85	78	82	53	84	49
	43	42	92	49	82	46	52	46	95	78

(i) Represent these two sets of data by a back-to-back stem-and-leaf diagram.

(ii) Who used his tablet computer for the longest time in a day?

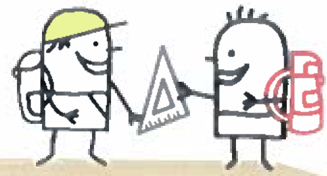
(iii) Who used his tablet computer for the shortest time in a day?

(iv) Who used his tablet computer for a longer duration of time? Explain your answer.

SIMILAR QUESTIONS

Exercise 16A Questions 10–11

16.4 Scatter Diagrams



A **scatter diagram** is commonly used to illustrate the results of a statistical survey or enquiry by comparing the two sets of data. Values of the two sets of data are recorded in pairs and these are plotted on a graph just like plotting a pair of x-y coordinates.

If these pairs of coordinate points tend to lie on a straight line, we can conclude that there is a close relationship or **correlation** between these two sets of data.

The marks of 12 pupils scored in Paper 1 and Paper 2 of a mathematics examination marked out of 60 are shown in the table below.

Paper 1	55	45	20	34	53	58	38	40	15	29	18	27
Paper 2	47	42	17	26	48	51	31	34	12	23	16	24

Fig. 16.3 shows the scatter diagram when the set of data is plotted. We usually plot the independent variable on the horizontal axis and the dependent variable on the vertical axis. In the case above where there are no clear independent and dependent variables, we plot the first variable on the horizontal axis and the second variable on the vertical axis.

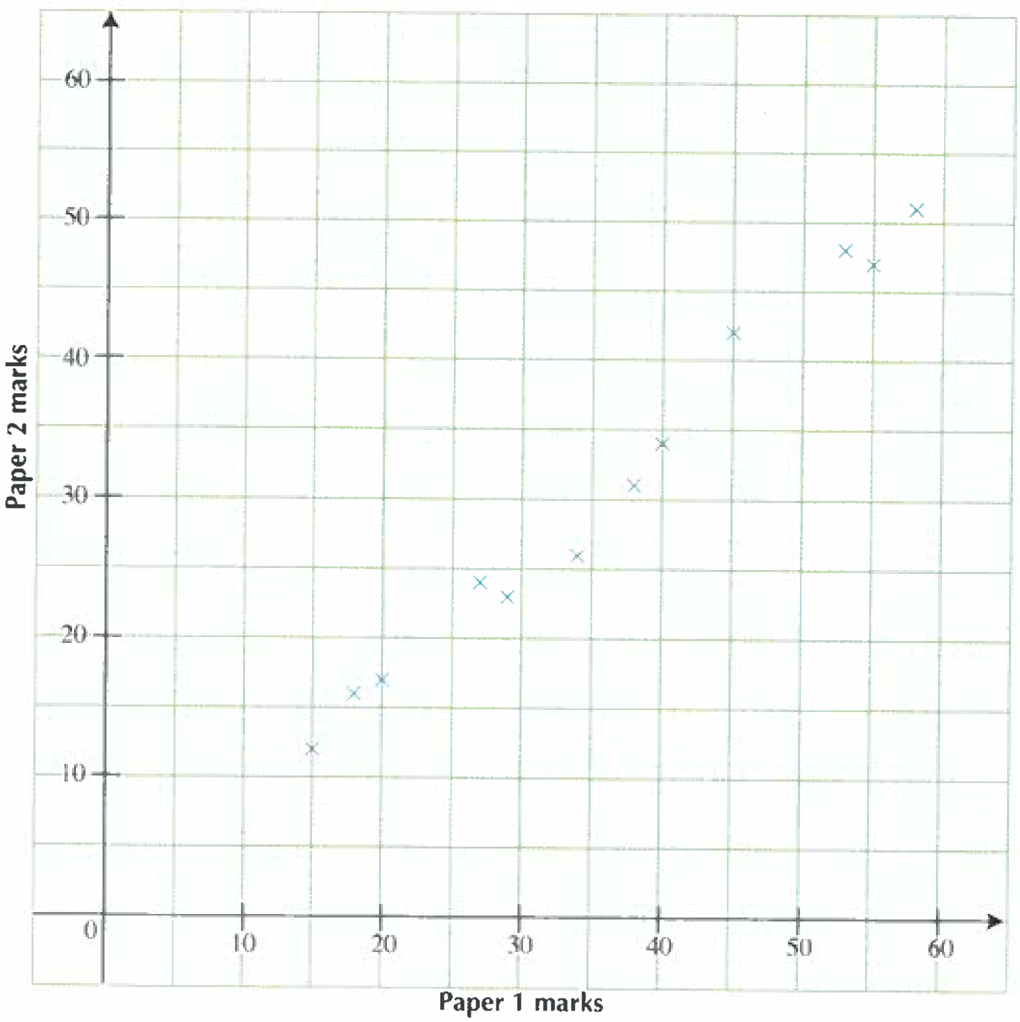


Fig. 16.3

INFORMATION

A scatter diagram is also known as a scatter plot or a scatter graph.

ATTENTION

An advantage of a scatter diagram is that it can be easily drawn to represent large quantities of data between two variables that are being measured. A disadvantage is that it does not give the exact extent of correlation.

Fig. 16.3 suggests that there is a close relationship between the scores for Paper 1 and Paper 2. The higher the Paper 1 marks, the higher the Paper 2 marks. This is called a **positive correlation**. We may also use the term "strong", "moderate" or "weak" to give a clearer indication of the correlation.

We can also show the general trend by drawing a line that will pass through most of the points or as close to most of the points as possible. This line is called the **line of best fit** as shown in Fig. 16.4.

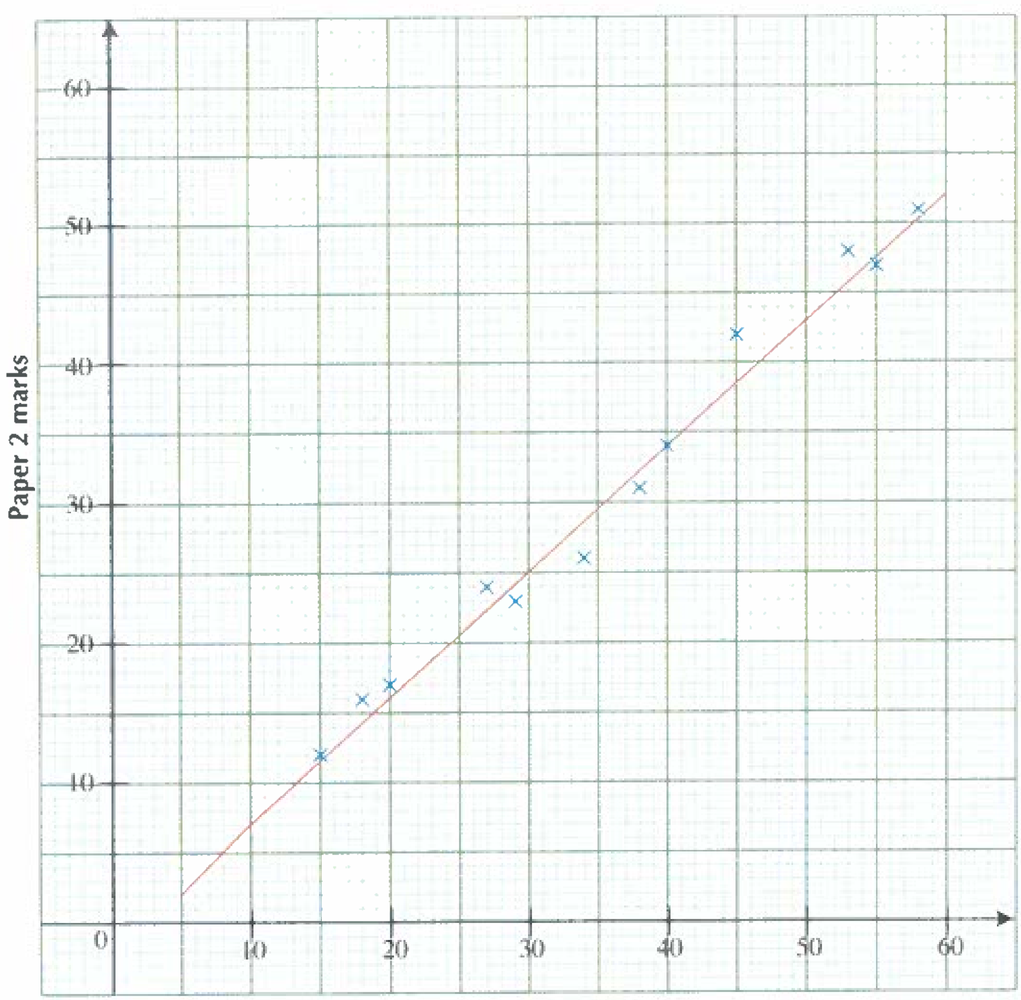


Fig. 16.4

ATTENTION

When data points are more scattered, the correlation between the two variables weakens.

As most of the marks lie very close to the line of best fit, we say that there is **strong positive correlation** between Paper 1 and Paper 2.

We can use the line of best fit to predict the performance of a pupil who was absent from sitting Paper 2. For example, if a pupil scored 48 marks in Paper 1 but was absent for Paper 2, we can use the graph to determine the likely score he might score in Paper 2. Draw a vertical line at 48 marks for Paper 1 to cut the line of best fit and read off the value for Paper 2 by drawing a horizontal line across which reads 41 marks.

Thinking Time

Earlier in chapter 2, we have learnt that the equation of a straight line is $y = mx + c$, where the constant m is the gradient of the line and the constant c is the y-intercept. Can we apply this to find the equation of the line of best fit?

From the thinking time, we can conclude that we can obtain the equation of the line of best fit in the same way that we do for any other straight line.

Worked Example 6

(Scatter Diagram with Positive Correlation)

The table below shows the marks scored by 15 pupils in a Mathematics and Physics examination.

Math	85	67	65	84	53	80	70	87	79	74	90	60	76	57	77
Physics	72	60	54	69	48	64	58	75	68	64	76	51	67	46	63

(i) Draw a scatter diagram for the above data.

(ii) What type of correlation is shown in the scatter diagram?

(iii) Draw a line of best fit on the scatter diagram.

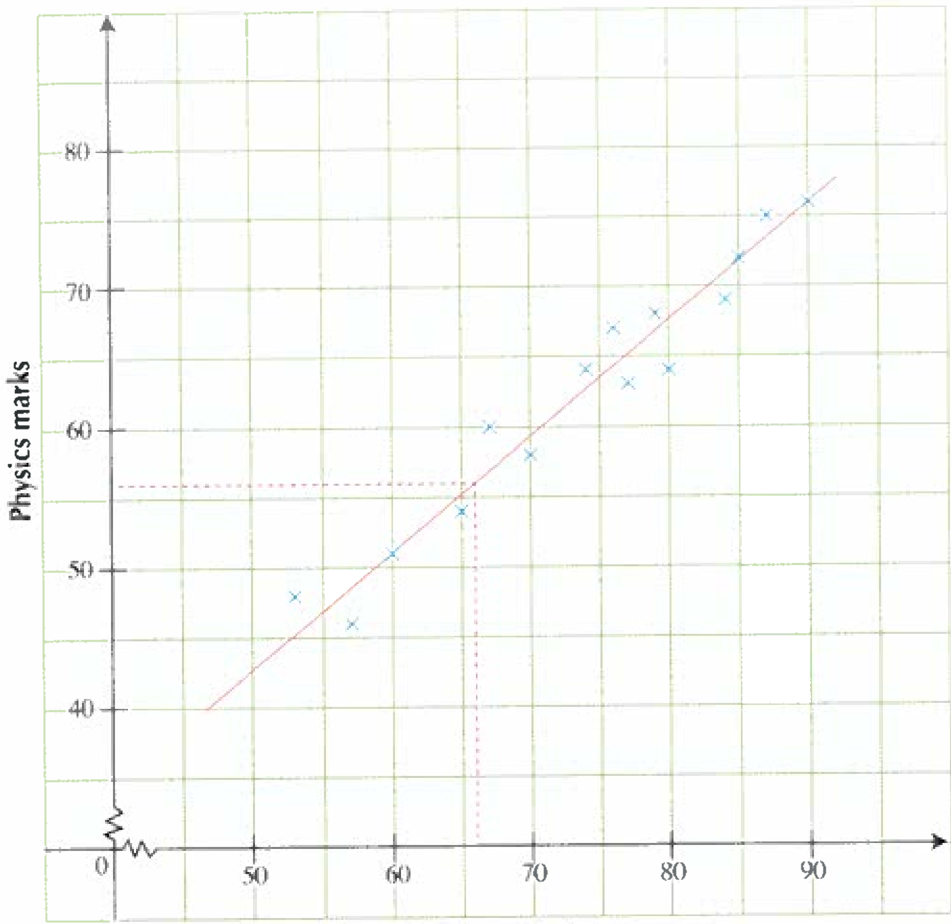
(iv) Use your line of best fit to estimate the number of marks that a pupil who scored 56 in Physics is likely to score in Mathematics.

(v) Given that the equation of the line of best fit is in the form $y = mx + c$, where m is the gradient and c is the y-intercept, find the equation of the line of best fit.

(vi) Would it be reliable to use your line of best fit to estimate the number of marks that a pupil who scored 45 in Mathematics is likely to score in Physics? Explain your answer clearly.

Solution:

(i)



Problem Solving Tip

The jagged lines near the origin are used to indicate a **broken scale**. They are used when values close to 0 are not required. In this case, we start with 50 marks on the horizontal axis and 40 marks on the vertical axis.

(ii) At first glance these values suggest that there is close connection between the Mathematics and Physics marks. The higher the Mathematics marks, the higher the Physics marks. It is a **strong positive correlation**.

(iii) The line of best fit is drawn with approximately half of the plots lying above the line and the other half lying below the line of best fit.

(iv) Using the line of best fit on the scatter diagram, a student who scored 56 in Physics is likely to score 66 in Mathematics.

(v) Take two points on the line and draw dotted lines to form a right-angled triangle.

Vertical change (or rise) = 69 – 35

= 34

Horizontal change (or run) = 82 – 0

= 82

Since the line slopes upwards from the left to the right, its gradient is positive.

$$\text{Gradient} = \frac{\text{rise}}{\text{run}} = \frac{34}{82} = \frac{17}{41}$$

From the diagram, y-intercept = 35

$\therefore$ The equation of the line of best fit is $y = \frac{17}{41}x + 35$.

ATTENTION

Some points to take note of when drawing the line of best fit:

- The line drawn may pass through all the points, some of the points or none of them at all.
- The data points have to be evenly distributed on either sides of the line of best fit.
- Use a transparent ruler to draw the line of best fit so as to gauge the best position to fit the line.

(vi) It would be unreliable since the score of 45 lies outside of the range as no pupil had scored less than 50 marks for Mathematics.

PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 16A Questions 4(b), (c), 5, 14

The fuel consumption of a small truck measured against the total mass of the truck and the mass of goods it carries is shown in the table below.

Mass (tons)	3.00	3.04	2.26	2.70	3.10	2.50	2.80	2.56	2.84	2.30	2.90	2.32	2.44
Litres/ 100 km	14.8	15.2	11.2	13.3	15.1	12.1	13.5	12.7	13.8	11.2	14.5	11.7	11.9

(i) Draw a scatter diagram for the above data.

(ii) What type of correlation is shown in the scatter diagram?

(iii) Draw a line of best fit on the scatter diagram.

(iv) Use your line of best fit to estimate the number of litres of petrol that a truck with a total mass of 2.62 tons will need to travel 100 km.

(v) Given that the equation of the line of best fit is in the form $y = mx + c$, where m is the gradient and c is the y-intercept, find the equation of the line of best fit.

(vi) Would it be reliable to use your line of best fit to estimate the number of litres of petrol that a truck with a total mass of 4.8 tons will need to travel 100 km? Explain your answer clearly.

Worked Example 7

(Scatter Diagram with Negative Correlation)

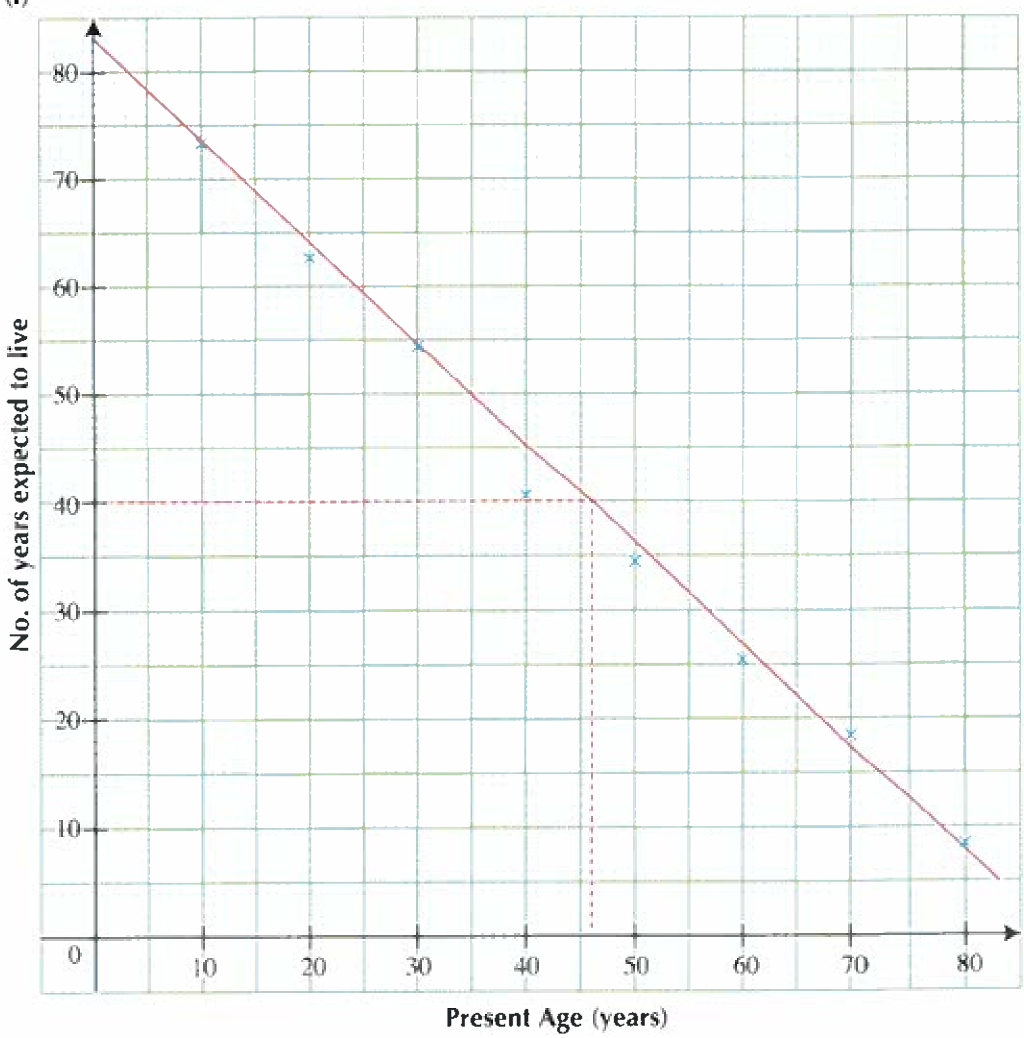
The table below shows the number of years that people of various ages in an Asian country are expected to live out their remaining years in life.

Present Age (years)	10	20	30	40	50	60	70	80
Number of years expected to live	73.2	62.8	54.3	40.7	34.5	25.1	18.7	8.3

(i) Draw a scatter diagram for the above data.
(ii) What type of correlation is shown in the scatter diagram?
(iii) Draw a line of best fit on the scatter diagram.
(iv) Use your line to estimate the number of years that a person aged 46 is expected to live.
(v) Given that the equation of the line of best fit is in the form $y = mx + c$, where m is the gradient and c is the y-intercept, find the equation of the line of best fit.
(vi) Would it be reliable to use your line of best fit to estimate the number of years that the oldest person in the country aged 105 is expected to live? Explain your answer.

Solution:

(i)



(ii) The data shows strong negative correlation.

(iii) The line of best fit is drawn passing through as many points as possible and as close as possible to all the other points.

(iv) Using the line of best fit on the scatter diagram, a person aged 46 is expected to live for another 40 years.

(v) Take two points on the line and draw dotted lines to form the right-angled triangle.

Vertical change (or rise) = 83 – 20

= 63

Horizontal change (or run) = 67 – 0

= 67

Since the line slopes downwards from the left to the right, its gradient is negative.

Gradient $= -\frac{\text{rise}}{\text{run}}$

$= -\frac{63}{67}$

y-intercept = 83

$\therefore$ The equation of the line of best fit is $y = -\frac{63}{67}x + 83$

(vi) No, it would be highly unreliable as it lies outside the range since there are no person with present age of more than 100 in the study.

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 16A Questions 4(a), 15

The table below shows the time taken (in seconds) by 15 pupils aged between 11 and 16 years to run the 100 m race.

Age (years)	13.3	15.3	13.6	15.5	14.0	14.3	15.3	12.8	12.2	14.8	11.7	14.9
Time (s)	13.9	12.3	13.2	11.6	13.1	12.3	12.3	13.9	14.6	13.1	14.7	12.1

Age (years)	12.5	13.2	11.1
Time (s)	14.5	13.3	15.2

(i) Draw a scatter diagram for the above data.

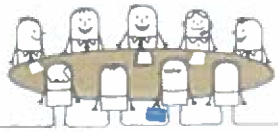
(ii) What type of correlation is shown in the scatter diagram?

(iii) Draw a line of best fit on the scatter diagram.

(iv) Use your line to estimate the time that a pupil aged 14.5 years is expected to complete the 100 m race.

(v) Given that the equation of the line of best fit is in the form $y = mx + c$, where m is the gradient and c is the y-intercept, find the equation of the line of best fit.

(vi) Can we use the above data to predict the time that a pupil aged 17 years old will take to complete the 100 m race? Explain your answer.



Class Discussion

Scatter Diagram with No Correlation

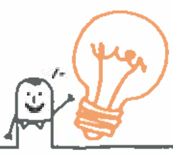
Work in pairs.

The table below shows the height of a group of pupils and the number of marks they scored in an English examination.

Height (cm)	123	124	125	129	135	137	142	145	149	153	153	157	160	164	167	168
English (marks)	62	85	45	67	55	40	52	78	47	38	89	64	45	71	53	42

1. Draw a scatter diagram for the above data.
2. Describe the correlation shown in the scatter diagram.

From the class discussion, we observe that the scatter diagram for two variables with no correlation will have plots with no visible upward or downward trend.



Thinking Time

What sort of correlation (if any) would you expect between each of the following? Explain your assumptions.

1. The number of hours one spends doing mathematical problems and the number of marks in the examination.
2. The number of cars on the road and the number of traffic accidents.
3. The number of hours one spends watching television programme and his Intelligence Quotient score (IQ).
4. The weight of a person and the house number that they live in.
5. The amount of money one has in the bank and the amount of interest he will earn.
6. The price of a shirt and the colour of its buttons.
7. The size of a cup-cake and the number of cup-cakes Michael can eat.
8. The height of a man and the number of fingers that he has.
9. The speed of a sprinter and the number of races he wins.

Journal Writing

If you are required to find the correlation between two quantities, write down the steps that you would need to prepare. Do you have pre-conceived results that you expect to observe? How do you intend to set about finding data to support your investigation? Explain some of the difficulties that you encounter in your investigation and some steps to avoid if you were to carry out another investigation.

Exercise 16A

BASIC LEVEL

1. The table shows the masses, in kg, of 16 bags of rice.

43	30	26	25	36	25	36	36
25	36	30	20	15	20	15	23

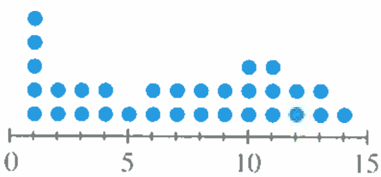
(i) Represent the data on a dot diagram.

(ii) What is the mass of the lightest bag of rice?

(iii) What is the most common mass?

(iv) Bags of rice with a mass of less than 30 kg have to be refilled. How many bags have to be refilled?

2. The dot diagram represents the time taken, in minutes, for a group of people to complete a questionnaire.



(i) What was the most common time taken?

(ii) Find the difference between the shortest and the longest time taken to complete the questionnaire.

(iii) Find the total number of people who completed the questionnaire.

(iv) Find the ratio of the number of people who took 10 minutes or longer to complete the questionnaire to the total number of people who completed the questionnaire.

3. The table shows the time taken, in hours, for 20 students to complete their Social Studies project.

3.5	7.0	4.8	2.8	5.5	6.8	6.6	5.5	4.5	2.5
6.5	3.6	3.3	4.0	3.5	7.5	4.8	6.5	6.4	2.8

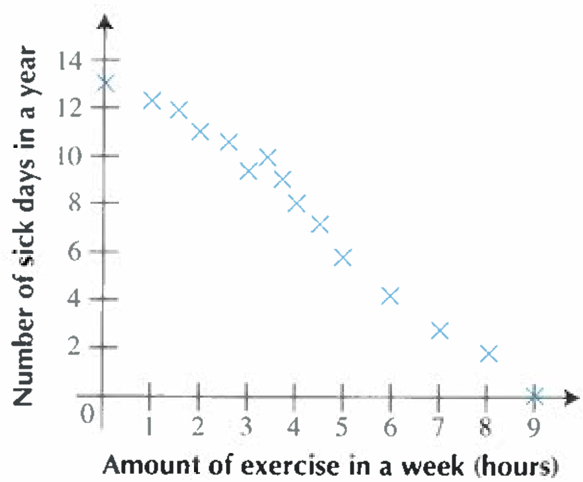
(i) Represent the data using a stem-and-leaf diagram.

(ii) Find the percentage of students who spent less than 3 hours to complete their project.

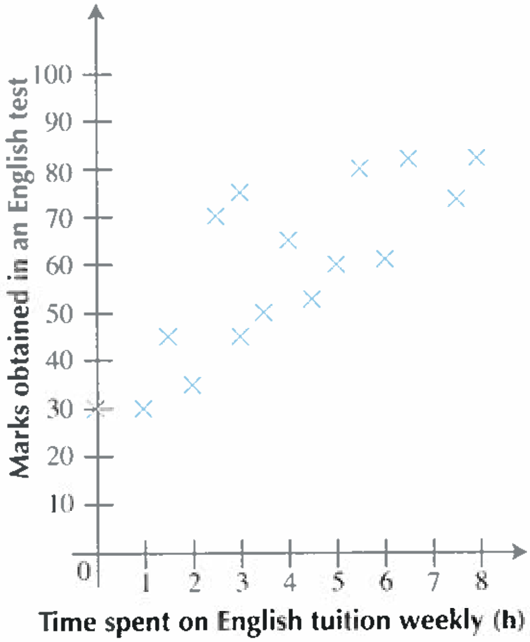
(iii) Students who spent more than 6 hours to complete their project had 2 marks deducted. What fraction of the students had 2 marks deducted?

4. State the type of correlation for each of the scatter diagrams below.

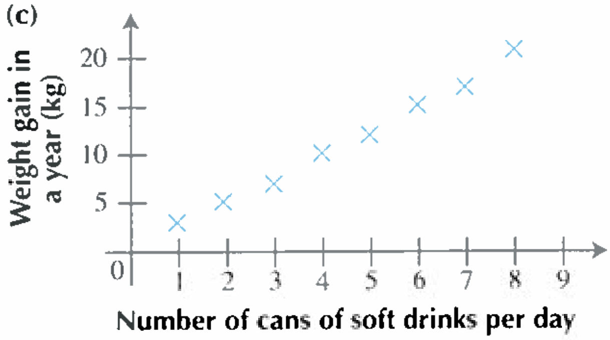
(a)



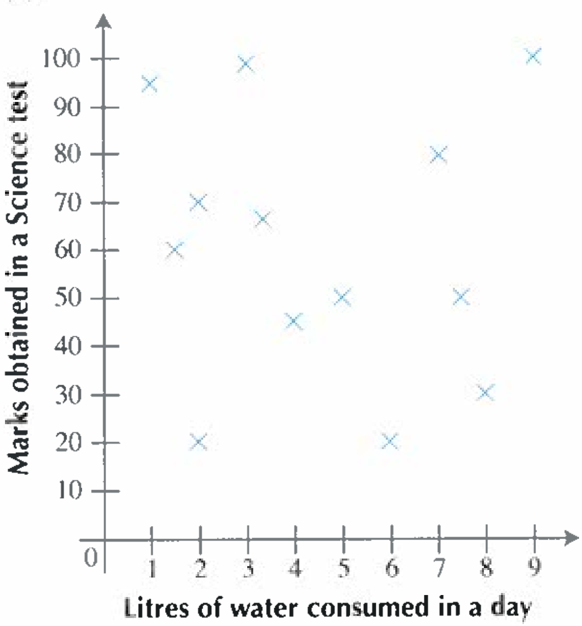
(b)



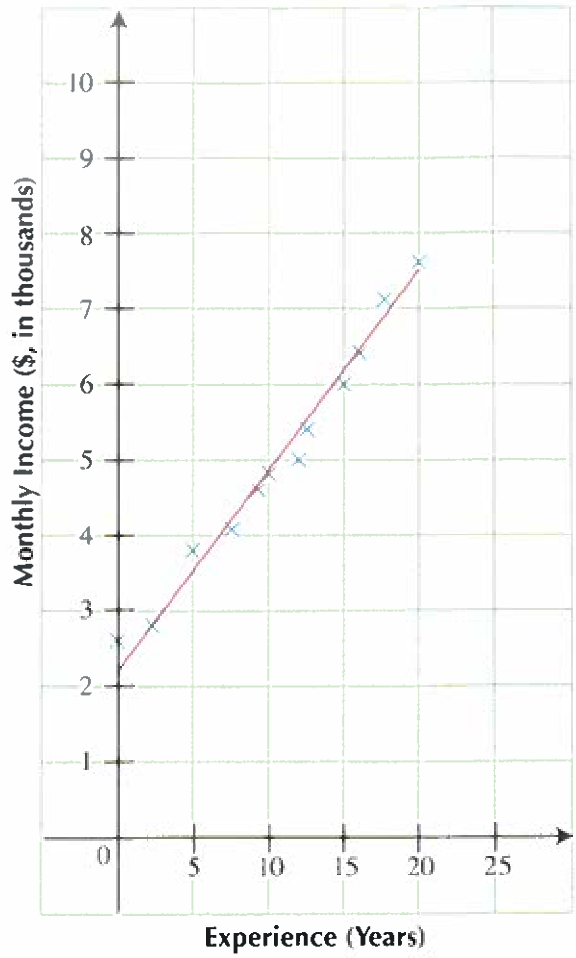
(c)



(d)



5. The scatter diagram shows the monthly income earned by individuals with varying years of experience. The line of best fit for the scatter diagram has been drawn.



Using the line of best fit, estimate the monthly income earned by an individual with 18 years of experience.

INTERMEDIATE LEVEL

6. The average daily temperature (in °C) of a city in autumn is recorded in the table.

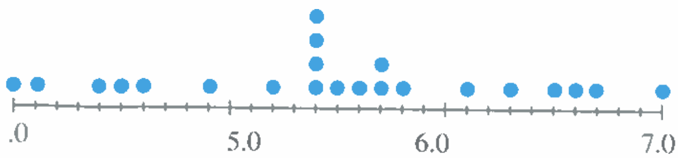
22	20	22	19	23	17	21	18	20	21
20	20	18	21	19	21	19	19	21	19

(i) Represent the data on a dot diagram.

(ii) Find the percentage of temperatures recorded that are at most 20 °C.

(iii) Comment on the distribution of the data.

7. The dot diagram represents the attention span, in minutes, of 22 pre-schoolers.

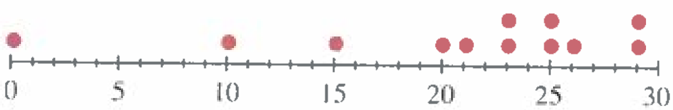


(i) What is the most common attention span?

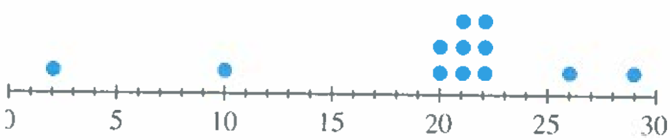
(ii) A pre-schooler is chosen at random. Find the probability that the attention span of the pre-schooler is not more than 5.5 minutes.

(iii) Briefly describe the distribution of the data.

8. The dot diagrams represent the results of a Mathematics test by gender.



Results of the girls



Results of the boys

(a) If the passing mark for the test is 15, how many boys failed the test?

(b) Distinctions are awarded to students who scored 25 marks and above in the test.

(i) Express the number of girls who scored distinctions as a percentage of the total number of girls in the class.

(ii) Express the number of boys who scored distinctions as a percentage of the total number of boys in the class.

(c) Which gender performed better in the test? Explain your answer.

9. The stem-and-leaf diagram represents the number of chocolates in some packets.

Stem	Leaf
2	0 2 4 4 4 5 6 6 6 6 6 7 8 9 9 9
3	1 1 1 2 2 3 4 4 5 5 7 7 8 8
4	0 1 2 3 3 3 5 5 6 8

Key: 2 | 2 means 22

(i) Construct a stem-and-leaf diagram with split stems.

(ii) What is the most common number of chocolates?

(iii) Find the percentage of packets which have more than or equal to 25 chocolates but fewer than 32 chocolates.

10. The times, in minutes, for 20 sportsmen to complete a running test are recorded in the table.

7.9	8.3	7.8	8.5	7.4	8.3	7.0	8.1	8.3	8.5
8.0	9.4	7.4	8.5	7.5	9.8	8.1	7.7	9.4	8.3

(i) Construct a stem-and-leaf diagram with split stems.

(ii) What were the fastest and the slowest time taken?

(iii) Find the percentage of sportsmen who took at least 8.5 minutes to complete the test.

(iv) A sportsman is selected at random. Find the probability that the sportsman failed the test if the passing time is 8.4 minutes.

11. The table shows the unemployment rates in a country from 2002 to 2012.

Year	Unemployment Rate (%)
2002	3.5
2003	4.0
2004	3.5
2005	3.2
2006	2.7
2007	2.0
2008	2.2
2009	2.9
2010	2.2
2011	2.1
2012	2.0

(a) If we say that a year is 'good' when the unemployment rate is less than 2.5%, which years are considered 'good'?

(b) Construct

(i) a dot diagram,

(ii) a stem-and-leaf diagram with split stems.

(c) State an advantage of using

(i) the dot diagram,

(ii) the stem-and-leaf diagram with split stems, to represent the employment rates in the country from 2002 to 2012.

12. The back-to-back stem-and-leaf diagram represents the scores of the students in two different schools for a common Geography quiz. In each school, 29 students sat for the quiz.

Leaves for School *P*	Stem	Leaves for School *Q*
4 0	5	2 6 8 9
9 9 6	6	2 5 8 8 9 9
9 8 5 3 2 0	7	4 6 7 8 8 9
9 9 9 7 7 6 6 4 2	8	0 3 4 4 6 7 7
9 8 8 7 6 6 3 2 0	9	0 2 7 8
	10	0 0

Key: 6 | 9 means 69 marks

(i) Which school had the highest scorer?

(ii) Which school had the lowest scorer?

(iii) Which school performed better in the quiz? Explain your answer.

13. The results of a Mathematics examination of two Secondary Two classes are recorded in the table.

Class *A*	77	70	31	45	39	45	59	63	47	68
	69	50	68	98	66	85	60	55	50	70
Class *B*	68	90	42	80	82	88	69	67	69	75
	55	94	79	92	66	72	78	70	87	65

(i) Represent these two sets of data by a back-to-back stem-and-leaf diagram.

(ii) Find the percentage of students who scored distinctions, i.e. 70 marks and above, in each class.

(iii) Which class performed better in the examination? Explain your answer.

14. The following table shows the age and their systolic blood pressure of 14 women.

Age	45	51	75	33	52	55	68	44	55	48	62	35	65	42
BP(mm Hg)	123	127	151	107	135	131	152	125	134	130	138	113	145	117

(i) Draw a scatter diagram to show the age and blood pressure of the 14 women.

(ii) Describe the type of correlation between the age and the blood pressure of the women in the study.

(iii) Draw a line of best fit for this set of data.

(iv) Mary is 58 years old. What would her blood pressure be?

(v) Mr Cook is 86 years old. Would it be reliable to use the line of best fit to predict his blood pressure? Explain your answer clearly.

15. The table below shows the number of cigarettes smoked per day and the life expectancy of a group of smokers.

Number of cigarettes smoked	42	13	45	23	40	48	16	38	56	27	8	35	5	33
Life expectancy	56	75	53	71	51	50	73	61	44	64	77	55	86	63

(i) Draw a scatter diagram to show the number of cigarettes smoked per day and the life expectancy of this group of smokers.

(ii) Describe the type of correlation between the number of cigarettes smoked per day and the life expectancy of this group of smokers.

(iii) Draw a line of best fit for this set of data.

(iv) Johnny smoked 25 cigarettes per day, what would his life expectancy be?

(v) Given that the equation of the line of best fit is in the form $y = mx + c$, where m is the gradient and c is the y-intercept, find the equation of the line of best fit.

(vi) Would it be reliable to use the line of best fit to predict the life expectancy of Mr Ong who is a non-smoker? Explain your answer clearly.

16. The table below shows the mass of a group of pupils and their respective intelligence quotient (IQ) score.

Mass (kg)	66	62	56	75	50	43	90	56	47	88	67	82	85	72
IQ score	111	95	113	94	107	92	91	91	102	101	90	106	115	104

(i) Draw a scatter diagram for the above data.

(ii) What type of correlation is shown in the scatter diagram?

ADVANCED LEVEL

17. The table shows the total number of goals scored at full-time in each match from the round of 16 to the final in the FIFA World Cup hosted by South Africa in 2010.

	Match	Number of goals scored
Round of 16	Uruguay : Korea Republic	3
	USA : Ghana	2
	Germany : England	5
	Argentina : Mexico	4
	Netherlands : Slovakia	3
	Brazil : Chile	3
	Paraguay : Japan	0
	Spain : Portugal	1
Quarterfinals	Netherlands : Brazil	3
	Uruguay : Ghana	2
	Argentina : Germany	4
	Paraguay : Spain	1
Semifinals	Uruguay : Netherlands	5
	Germany : Spain	1
Match for third place	Uruguay : Germany	5
Final	Netherlands : Spain	0

(i) Represent the data on a dot diagram.

(ii) What are the most common number of goals scored?

(iii) Express the number of matches that have at least 3 goals scored as a percentage of the total number of matches from the round of 16 to the final.

(iv) Consider the matches from the round of 16 to the final of the FIFA World Cup. From the World Cup in 2006 to that in 2010, there was a decrease of 37.5% in the number of matches that have not more than one goal scored at full time (before penalty shootouts). Find the number of matches with more than one goal scored (excluding penalty shootouts) in the 2006 FIFA World Cup.

18. Some plants are grown by a group of students for an experiment. The table shows the heights, in cm, of the plants after 8 weeks.

4.5	8.6	5.6	5.5	8.1	4.4	6.5
8.1	4.7	7.2	6.0	5.3	7.2	7.8

Explain clearly whether you would use a dot diagram or a stem-and-leaf diagram to represent the data.

16.5 Histograms for Ungrouped Data

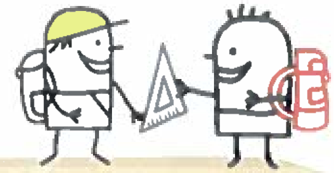


Table 16.2 shows the scores of 40 students in a class for a Mathematics test. The total score of the test is 10 marks.

8	6	4	3	5	5	2	9	2	7
9	3	3	7	7	5	8	3	7	3
4	8	7	8	2	4	6	2	4	1
7	7	6	2	6	4	4	6	10	6

Table 16.2

We arrange the scores in order of magnitude before going through the list of scores and keeping a tally as shown in Table 16.3. The number of times each score appears is called its **frequency**. Table 16.3 is known as a **frequency table** as it shows the frequency of each score. The frequency table also shows the distribution of the scores obtained by the 40 students in the test.

Score	Tally	Frequency
0		0
1	/	1
2	###	5
3	###	5
4	### /	6
5	///	3
6	### /	6
7	### //	7
8	////	4
9	//	2
10	/	1
Total frequency		40

Table 16.3

Based on the information provided in Table 16.3, we can construct a dot diagram as shown in Fig. 16.5.

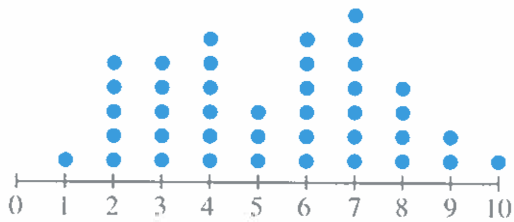


Fig. 16.5

Is there another type of graphical representation which we can use to display the data clearly? We can also construct a **histogram** (see Fig. 16.6) to display the information given in Table 16.3.

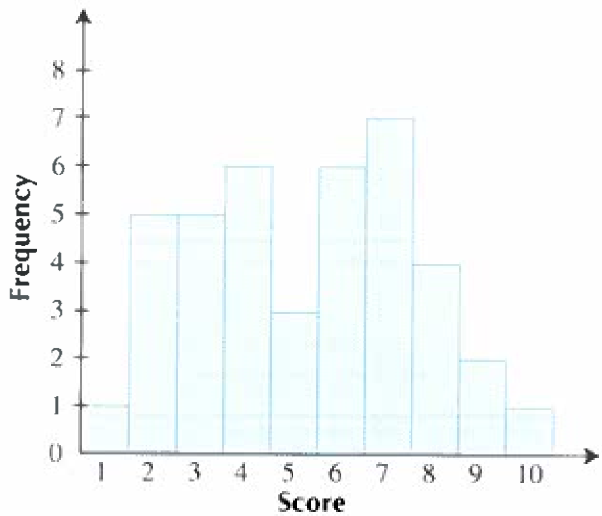


Fig. 16.6

In a histogram, the areas of the rectangles, and not the heights, are proportional to the frequencies they represent. In Fig. 16.6, the bases of the rectangles are equal. Hence, the heights of the rectangles are proportional to the frequencies.

In contrast to a bar graph, there are no spaces between the rectangles in a histogram. Another important difference is that while the categories on the horizontal axis in a bar graph can be arranged in any order, the horizontal axis in a histogram is a number line where the values have to be arranged in a certain order.

Journal Writing

1. Discuss some similarities and differences between a dot diagram and a histogram.

2. When representing ungrouped data,
(a) when is a dot diagram more appropriate than a histogram?
(b) when is a histogram more appropriate than a dot diagram?

Worked Example 8

(Histogram for Ungrouped Data)

The average amount of sleep, in hours, that 30 students in a Secondary Two class get on a weekday is given in the table.

8	7	6	4	8	9
5	3	10	6	6	5
3	8	7	5	7	4
9	8	8	5	5	5
9	4	10	9	6	5

(i) Construct a frequency table for the data.

(ii) Draw a histogram to illustrate the data.

(iii) Calculate the percentage of students who get an average of at least 8 hours of sleep on a weekday.

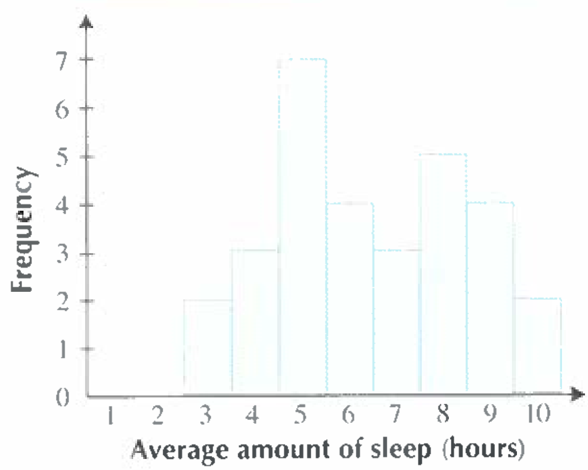
(iv) Huixian says that the most common average amount of sleep the students get on a weekday is 7 hours. Do you agree with her? Explain your answer.

Solution:

(i)

Average amount of sleep (hours)	3	4	5	6	7	8	9	10
Frequency	2	3	7	4	3	5	4	2

(ii)



(iii) Percentage of students who get an average of at least 8 hours of sleep on a weekday

$= \frac{11}{30} \times 100\%$

$= 36\frac{2}{3}\%$

(iv) No, I do not agree with her. The most common average amount of sleep the students get on a weekday is 5 hours.

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 16B Questions 1–2, 16

The number of mistakes made by 30 students in a Primary One class for a vocabulary test is given in the table.

3	4	6	0	2	2	4	3	5	3
4	2	2	3	1	5	3	0	4	5
4	3	4	0	3	2	6	3	1	0

(i) Construct a frequency table for the data.
(ii) Draw a histogram to illustrate the data.
(iii) Among these students, what is the most common number of mistakes made?
(iv) Find the fraction of students who made at most 3 mistakes in the test.

Class Discussion

Evaluation of Statistical Diagrams

Work in pairs.

1. Nora conducted a survey to find out the favourite ice-cream flavours of her classmates. She presented the data on a dot diagram as shown in Fig. 16.7.

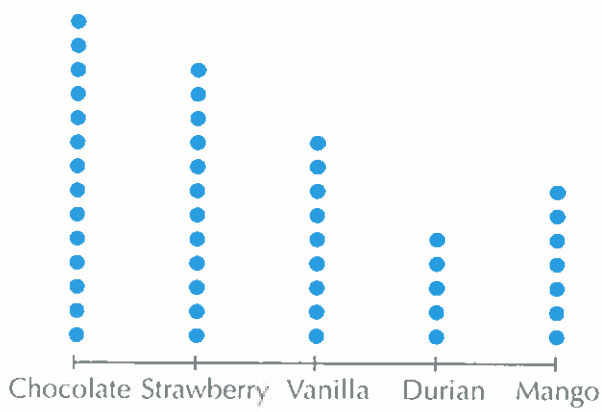


Fig. 16.7

Comment on the suitability of the choice of a dot diagram to present the data collected.

Hint: You may wish to search on the Internet to find out about the two main types of data in Statistics – categorical data and numerical data, and the types of statistical diagrams that are used to display each of these types of data.

2. Fig. 16.8 shows a histogram that illustrates the number of smartphones owned by 525 households in a housing estate.

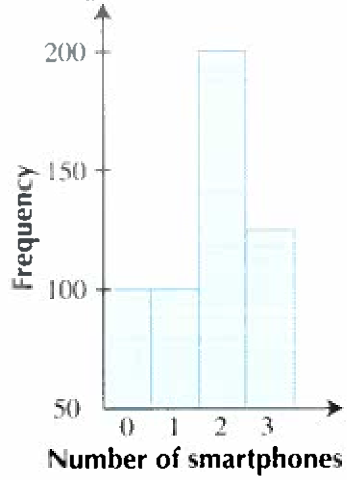
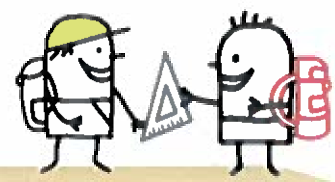


Fig. 16.8

Jun Wei says that there are three times as many households which own 2 smartphones as that which own 0 smartphones and that which own 1 smartphone. He also says that the number of households which own 3 smartphones is half of that which own 2 smartphones. Is he correct? Explain your answer.

16.6 Histograms for Grouped Data



In Section 16.5, we have learnt how ungrouped data can be presented in a histogram. In this section, we will learn how to display grouped data in a histogram to allow the information to be interpreted meaningfully.

Histograms for Grouped Data with Equal Class Intervals

Table 16.4 shows the approximate lengths, in mm, of 40 leaves taken from different plants of a certain species.

40	54	25	50	58	45	47	49	30	28
52	31	52	41	47	44	46	39	41	59
49	38	43	48	43	43	40	51	40	56
31	53	44	37	35	37	33	38	46	36

Table 16.4

These measurements can be confusing and may require an orderly arrangement for interpretation. One way to do so is to group the data into **class intervals.** If possible, all the intervals should be of equal size.

Step 1: Group the data based on the class intervals and set up a frequency table as shown in Table 16.4.

Length (x mm)	Tally	Frequency
$25 \leqslant x < 30$	//	2
$30 \leqslant x < 35$	////	4
$35 \leqslant x < 40$	~~////~~ //	7
$40 \leqslant x < 45$	~~////~~ ~~////~~	10
$45 \leqslant x < 50$	~~////~~ ///	8
$50 \leqslant x < 55$	~~////~~ /	6
$55 \leqslant x < 60$	///	3
Total frequency		40

Table 16.4

ATTENTION

The interval $25 \leqslant x < 30$ includes the length 25 mm but does not include the length 30 mm.

Step 2: Draw a histogram (see Fig. 16.9) with the frequency as the vertical axis and the lengths of the leaves as the horizontal axis.

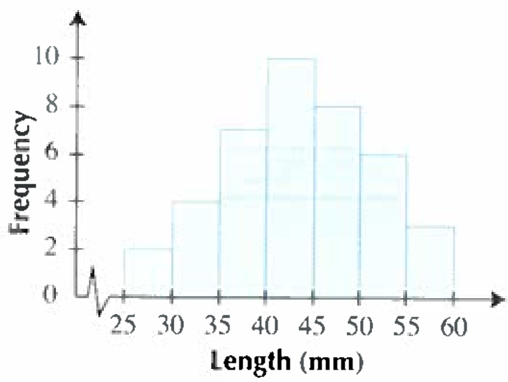


Fig. 16.9

If we construct a stem-and-leaf diagram with split stems to display the information given in Table 16.4, what will we notice about the shape of the stem-and-leaf diagram with split stems and that of the histogram in Fig. 16.9?

ATTENTION

- An advantage of using a histogram to present ungrouped or grouped data is that the data sets with the lowest and the highest frequencies can be easily identified while a disadvantage is that it is difficult to compare more than two histograms that display different sets of data.
- Another advantage of using a histogram to present grouped data is that it can be used to display a large set of grouped data clearly while another disadvantage is that it is not possible to obtain the exact values of the original data.

Journal Writing

When representing grouped data,

(a) when is a stem-and-leaf diagram more appropriate than a histogram?

(b) when is a histogram more appropriate than a stem-and-leaf diagram?

Worked Example 9

(Histogram for Grouped Data with Equal Class Intervals)

The table shows the lengths, x mm, of 30 leaves taken from various rubber trees.

137	152	147	134	147	141	157	132	153	166
147	136	146	142	162	169	149	135	166	148
157	141	146	147	163	133	148	150	136	127

(i) Using the class intervals $120 \leqslant x < 130$, $130 \leqslant x < 140$ and so on, construct a frequency table for the data.

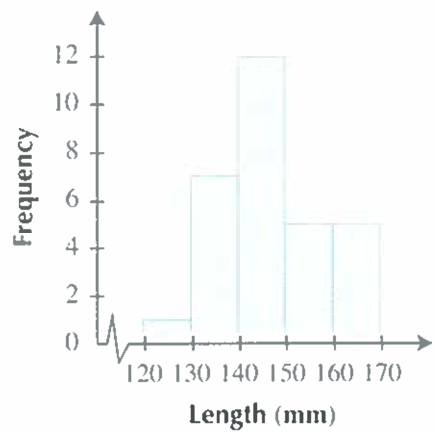
(ii) Draw a histogram to illustrate the data.

Solution:

(i)

Length (x mm)	Tally	Frequency
$120 \leqslant x < 130$	/	1
$130 \leqslant x < 140$	卌 //	7
$140 \leqslant x < 150$	卌 卌 //	12
$150 \leqslant x < 160$	卌	5
$160 \leqslant x < 170$	卌	5
Total frequency		30

(ii)



PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 16B Questions 3–6, 10–13

The table shows the waiting time, x minutes, for 36 patients at the pharmacy of a hospital.

12	17	25	34	46	18	11	8	13
14	22	45	48	35	52	23	37	16
51	27	19	23	29	32	31	43	16
44	31	16	9	10	14	16	39	29

(i) Using the class intervals $5 \leqslant x < 10$, $10 \leqslant x < 15$ and so on, construct a frequency table for the data.

(ii) Draw a histogram to illustrate the data.

Performance Task

1. Measure the length, correct to the nearest cm, of the right shoe worn by each of your classmates.
2. Record your findings and arrange them in the form of a frequency table.
3. Display your data using a histogram.
4. Give an interpretation of your data.
5. Present your findings to the class.

We have learnt how to draw histograms for grouped data with *equal* class intervals.
We will now learn how to draw a histogram for grouped data with *unequal* class intervals.

Histograms for Grouped Data with Unequal Class Intervals

Class Discussion

A survey was conducted to determine the income distribution of the people in a certain town.

Table 16.5 shows the annual income of 100 households in 2013 and Fig. 16.10 shows the histogram for the distribution. Notice that there are gaps along this distribution.

Income ($\$x$, in thousands)	Frequency
$0 \leqslant x < 20$	2
$20 \leqslant x < 40$	0
$40 \leqslant x < 60$	4
$60 \leqslant x < 80$	7
$80 \leqslant x < 100$	30
$100 \leqslant x < 120$	34
$120 \leqslant x < 140$	20
$140 \leqslant x < 160$	0
$160 \leqslant x < 180$	0
$180 \leqslant x < 200$	3

Table 16.5

If we combine some of the class intervals to remove the gaps, we will obtain the distribution in Table 16.6. Does Fig. 16.11 or Fig. 16.12 give a more accurate representation of the data?

Income ($\$x$, in thousands)	Frequency
$0 \leqslant x < 60$	6
$60 \leqslant x < 80$	7
$80 \leqslant x < 100$	30
$100 \leqslant x < 120$	34
$120 \leqslant x < 140$	20
$140 \leqslant x < 200$	3

Table 16.6

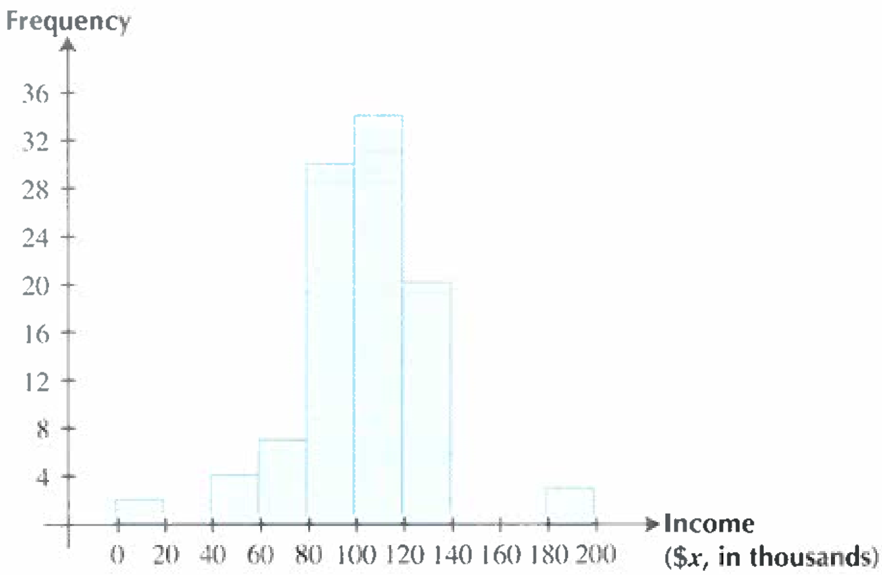


Fig. 16.10

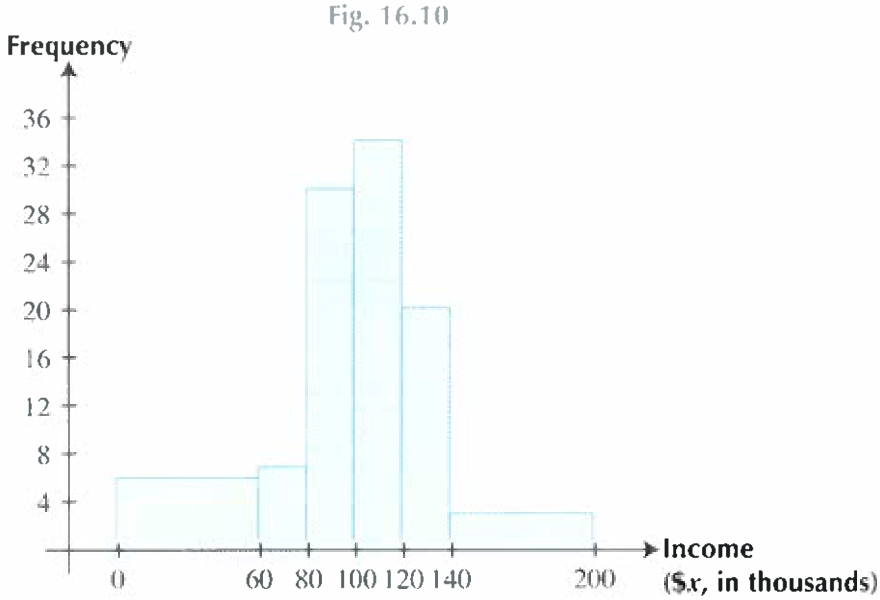


Fig. 16.11

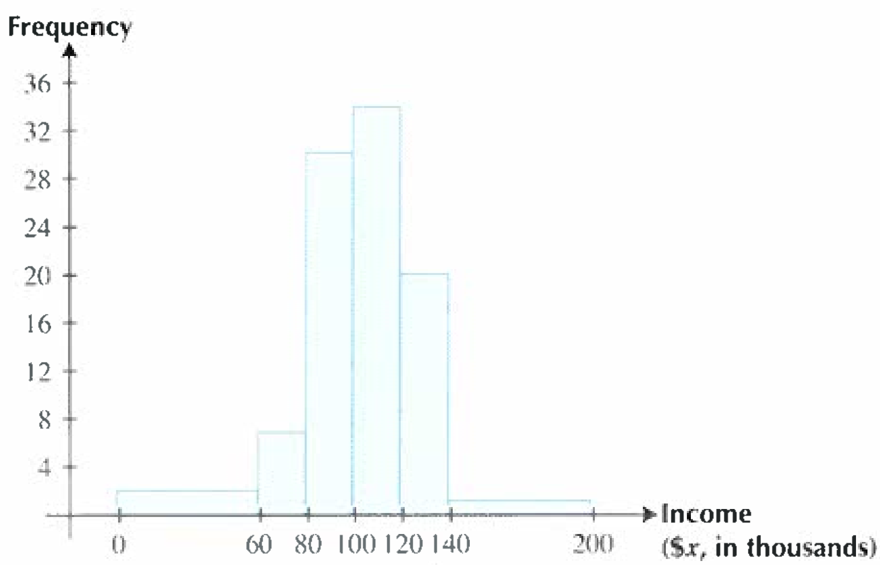


Fig. 16.12

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How are the heights of the rectangles in Fig. 16.12 obtained?

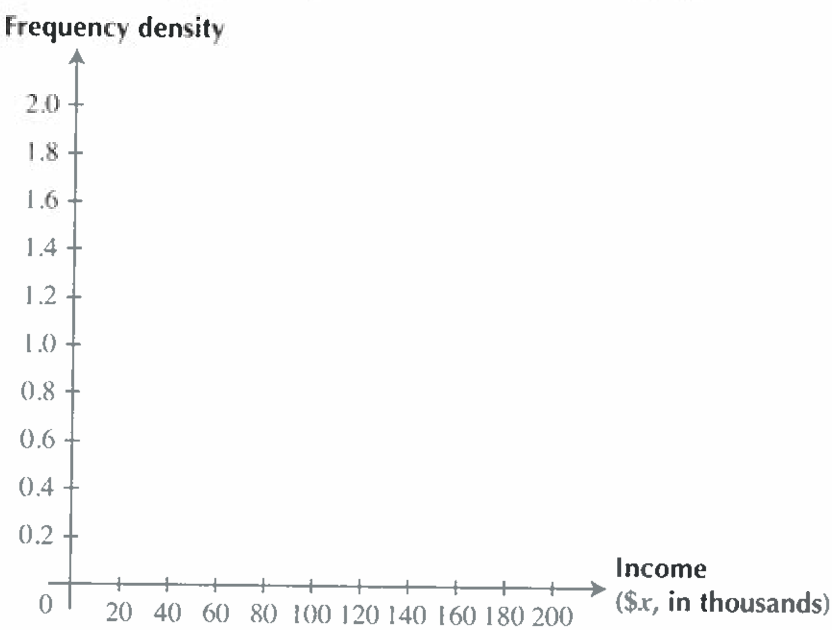
Recall that in a histogram, the area (and *not* the height) of each rectangle is **proportional** to the frequency. A more common approach to constructing a histogram with unequal class intervals is to use **frequency densities** for the heights of the rectangles as shown on the next page.

1. Complete Table 16.7 to calculate the frequency densities for each class interval.

Income ($\$x$, in thousands)	**Frequency**	**Frequency density =** $\dfrac{\textbf{Frequency}}{\textbf{Size of class interval}}$
$0 \leqslant x < 60$	6	$\dfrac{6}{60} = 0.1$
$60 \leqslant x < 80$	7	$\dfrac{7}{20} = 0.35$
$80 \leqslant x < 100$	30	
$100 \leqslant x < 120$	34	
$120 \leqslant x < 140$	20	
$140 \leqslant x < 200$	3	

Table 16.7

2. Draw a histogram with frequency density as the vertical axis to illustrate the data.



3. Find the area of each rectangle in the histogram in Question 2. What do you notice about the frequencies in Table 16.7 and the areas of the rectangles?
4. Suggest why the histogram with unequal class intervals as drawn in Question 2 is preferred over the histogram with equal class intervals as shown in Fig. 16.10.
5. Discuss with your classmates other examples where unequal class intervals are used in histograms.

Worked Example 10 illustrates how to construct a histogram with unequal class intervals using the heights of the rectangles and frequency densities.

Worked Example 10

(Histogram for Grouped Data with Unequal Class Intervals)

The table below shows the heights of 44 students.

Height (cm)	$130 < x \leqslant 145$	$145 < x \leqslant 150$	$150 < x \leqslant 155$	$155 < x \leqslant 160$	$160 < x \leqslant 170$	$170 < x \leqslant 180$
Frequency	9	7	8	6	20	4

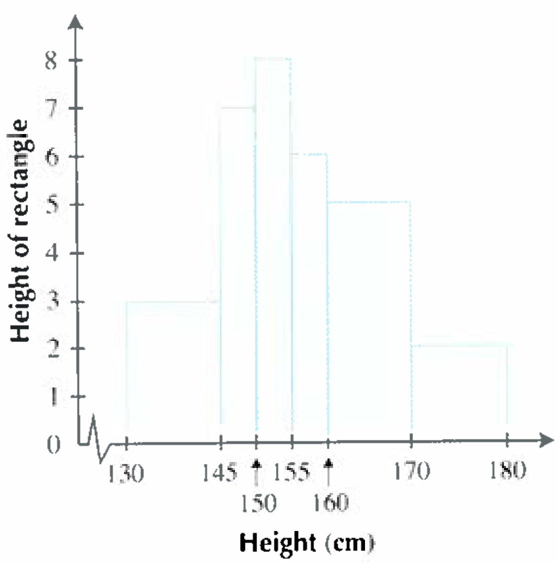
Construct a histogram representing this distribution.

Solution:

Method 1 (Using the heights of rectangles):

The table below shows the calculation of the heights of the rectangles.

Height	Class width		Frequency	Rectangle's height
$130 < x \leqslant 145$	15	3 × standard	9	$9 \div 3 = 3$
$145 < x \leqslant 150$	5	1 × standard	7	$7 \div 1 = 7$
$150 < x \leqslant 155$	5	1 × standard	8	$8 \div 1 = 8$
$155 < x \leqslant 160$	5	1 × standard	6	$6 \div 1 = 6$
$160 < x \leqslant 170$	10	2 × standard	10	$10 \div 2 = 5$
$170 < x \leqslant 180$	10	2 × standard	4	$4 \div 2 = 2$



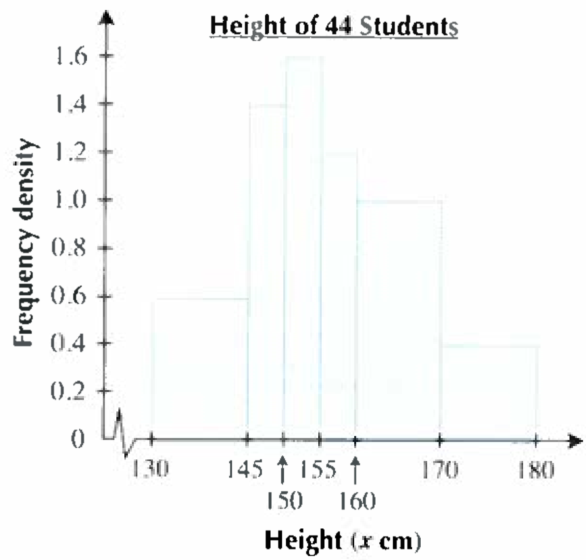
ATTENTION

In examining the sizes of the classes, we find that the interval size of 5 is the smallest. Three class intervals are of this size: $145 < x \leqslant 150$, $150 < x \leqslant 155$ and $155 < x \leqslant 160$. The class intervals $160 < x \leqslant 170$ and $170 < x \leqslant 180$ are each of size 10 and therefore they contain 2 class intervals of size 5 grouped together. The size of the class interval $130 < x \leqslant 145$ is 15, i.e. there are 3 class intervals of size of 5 grouped together

Method 2 (Using frequency densities):

The table below shows the calculation of the frequency density in the distribution of the heights of 44 students.

Height (cm)	Frequency	Class width	Frequency density = $\frac{\text{Frequency}}{\text{Class width}}$
$130 < x \leq 145$	9	15	$9 \div 15 = 0.6$
$145 < x \leq 150$	7	5	$7 \div 5 = 1.4$
$150 < x \leq 155$	8	5	$8 \div 5 = 1.6$
$155 < x \leq 160$	6	5	$6 \div 5 = 1.2$
$160 < x \leq 170$	10	10	$10 \div 10 = 1$
$170 < x \leq 180$	4	10	$4 \div 10 = 0.4$



PRACTISE NOW 10

SIMILAR QUESTIONS

Exercise 16B Questions 7–9

The circumferences, in cm, of the trunks of 100 beech trees from a certain forest are measured.

Circumference (x cm)	Number of trees
$40 < x \leq 70$	33
$70 < x \leq 80$	27
$80 < x \leq 100$	30
$100 < x \leq 110$	6
$110 < x \leq 120$	4

Construct a histogram representing this information.

Frequency Polygons

The value mid-way between the class boundaries of a class is called the **class mark**, or the **mid-value**, of the class.

Table 16.8 shows the fluoride levels of treated drinking water and Fig. 16.13 shows the histogram for the distribution. The table shows the mid-value for each class interval in the frequency distribution of the fluoride levels.

Fluoride levels (PPM)	Class boundaries	Mid-value	Tally	Frequency
0.71 – 0.75	0.705 – 0.755	0.73	//	2
0.76 – 0.80	0.755 – 0.805	0.78	𝍸	5
0.81 – 0.85	0.805 – 0.855	0.83	𝍸 ////	9
0.86 – 0.90	0.855 – 0.905	0.88	𝍸 /	6
0.91 – 0.95	0.905 – 0.955	0.93	𝍸	5
0.96 – 1.00	0.955 – 1.005	0.98	//	2
1.01 – 1.05	1.005 – 1.055	1.03	/	1
Total frequency				30

Table 16.8

In general, the mid-value of a class or the class mark is given by

$$\frac{\text{Lower class limit} + \text{Upper class limit}}{2} \text{ or } \frac{\text{Lower class boundary} + \text{Upper class boundary}}{2}.$$

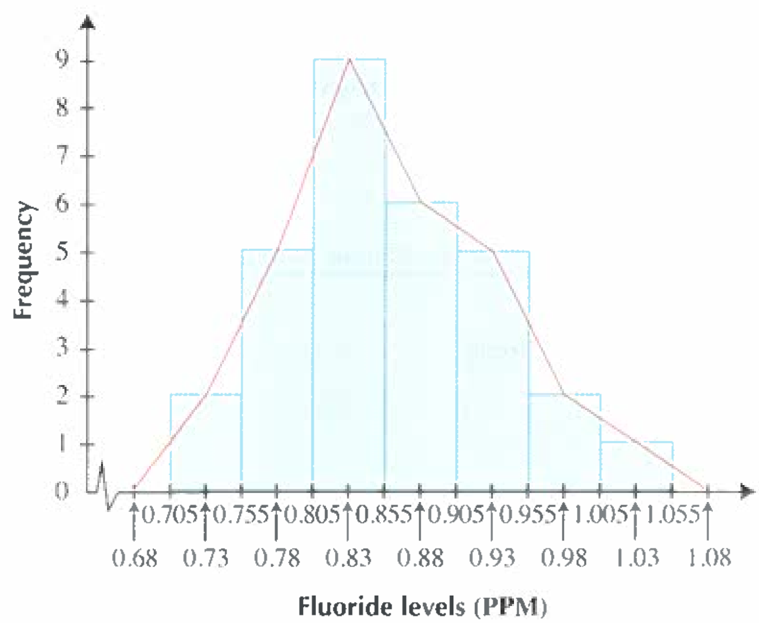


Fig. 16.13

A **frequency polygon** is drawn by joining all the mid-points at the top of each rectangle. The mid-points at both ends are joined to the horizontal axis to accommodate the end points of the polygon. This makes the graph neater with the end points falling off to zero on the horizontal axis.

Worked Example 11

(Frequency Polygon)

The table shows the masses, in kg, of 50 boys.

Mass (x kg)	$40 < x \leq 45$	$45 < x \leq 50$	$50 < x \leq 55$	$55 < x \leq 60$	$60 < x \leq 65$	$65 < x \leq 70$	$70 < x \leq 75$
Frequency	4	5	10	14	8	6	3

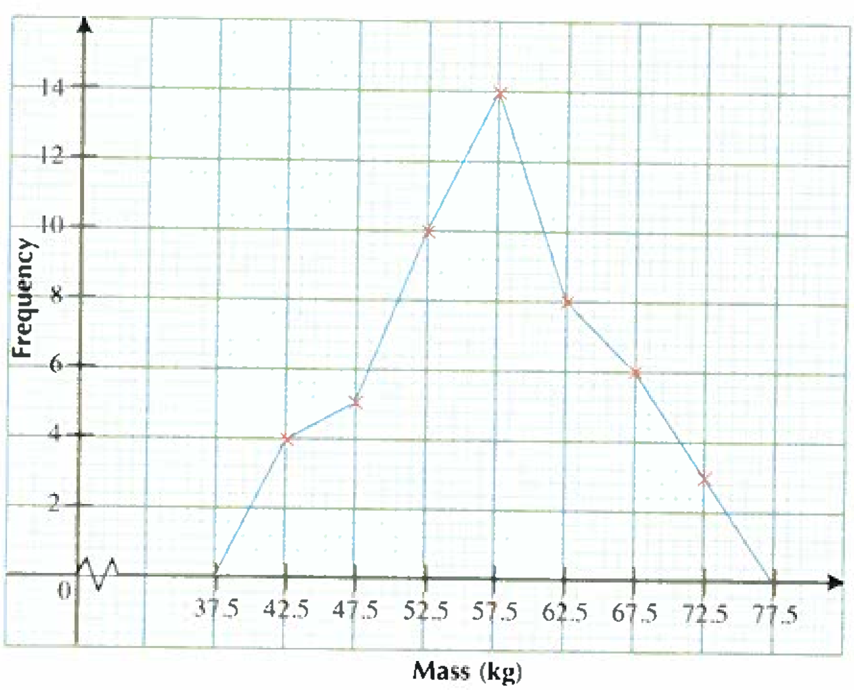
Draw a frequency polygon of the distribution.

Solution:

The table below shows the mid-value of each class.

Mass (x kg)	Mid-value	Frequency
$40 < x \leq 45$	42.5	4
$45 < x \leq 50$	47.5	5
$50 < x \leq 55$	52.5	10
$55 < x \leq 60$	57.5	14
$60 < x \leq 65$	62.5	8
$65 < x \leq 70$	67.5	6
$70 < x \leq 75$	72.5	3

The points to be plotted are (37.5, 0), (42.5, 4), (47.5, 5), (52.5, 10), (57.5, 14), (62.5, 8), (67.5, 6), (72.5, 3) and (77.5, 0).



Problem Solving Tip

We can draw the frequency polygon of a distribution without first drawing the histogram. We plot each class frequency against the mid-value of the class or the class mark to obtain points which we join by straight lines to form the frequency polygon.

ATTENTION

The first and last points are added to give a complete polygon.

INFORMATION

A frequency polygon is often useful when we wish to observe trends. For example, we can answer questions such as

- What do you notice from the frequency polygon about the relationship between the masses and the number of boys?
- Does the number of boys increase as the mass increases? If yes, is this trend maintained throughout? If not, after what mass does the number of boys start to decline?

A frequency polygon is also useful when we wish to compare two distributions by displaying two overlapping frequency polygons on the same set of axes. For example, we may want to compare the mass distributions among boys and girls of a particular age group in this way.

PRACTISE NOW 11

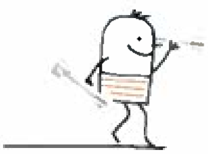
SIMILAR QUESTIONS

Exercise 16B Questions 14, 15

The following table gives the frequency distribution of the marks obtained by 40 students in an English test.

Mark (x)	Number of students
$20 < x \leqslant 30$	2
$30 < x \leqslant 40$	3
$40 < x \leqslant 50$	8
$50 < x \leqslant 60$	9
$60 < x \leqslant 70$	11
$70 < x \leqslant 80$	5
$80 < x \leqslant 90$	2

(a) Write down the mid-value of the class interval $40 < x \leqslant 50$.
(b) Draw the frequency polygon of the distribution.



Exercise 16B

BASIC LEVEL

1. A survey is conducted to find out the average number of hours the teachers in a school spend on marking their students' assignments each day. The data collected is given in the table.

6	4	3	1	2	2	3	1	4
1	2	5	3	4	5	2	2	3
3	1	2	2	3	1	4	2	

(i) Construct a frequency table for the data.
(ii) Draw a histogram to illustrate the data.
(iii) What is the most common average number of hours the teachers spend on marking their students' assignments each day?
(iv) Find the number of teachers who participated in the survey.
(v) Find the fraction of teachers who spend an average of 3 hours marking their students' assignments each day.

2. The number of fire incidents that occurred in a city each day over a period of 60 days was recorded.

Number of fire incidents	0	1	2	3	4	5	6	7
Number of days	16	12	11	9	6	2	2	2

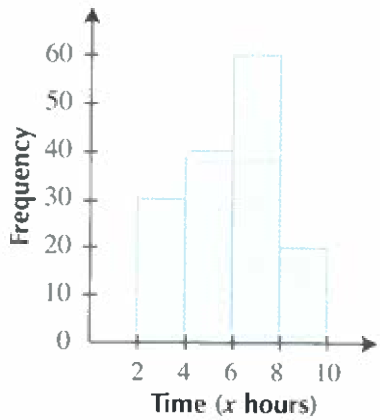
(i) Draw a histogram to illustrate the data.
(ii) Find the ratio of the number of days which had no fire incidents to the number of days which had more than 5 fire incidents.
(iii) Rui Feng says that the most common number of fire incidents is 16. Do you agree with him? Explain your answer.

3. The number of laptops sold each day over a period of 19 days was recorded.

Number of laptops (x)	Frequency
$0 \leqslant x < 5$	9
$5 \leqslant x < 10$	5
$10 \leqslant x < 15$	2
$15 \leqslant x < 20$	2
$20 \leqslant x < 25$	1

(i) Draw a histogram to illustrate the data.
(ii) On how many days were the number of laptops sold more than or equal to 15?

4. The histogram shows the number of hours spent on revising Mathematics by all the Secondary Two students in a school during the week before the common test.



(i) Using the class intervals $0 \leqslant x < 2$, $2 \leqslant x < 4$ and so on, construct a frequency table for the data.
(ii) State the number of students who spent more than or equal to 4, but less than 6 hours, on revising Mathematics in that week.
(iii) Find the total number of Secondary Two students in the school.
(iv) Find the percentage of students who spent less than 6 hours on revising Mathematics for the common test.

5. The following table shows the distribution of marks of some students who took part in a Science quiz.

Marks	Tally	Lower class boundary	Upper class boundary	Frequency
56–60	~~////~~ //			
61–65	~~////~~ //			
66–70	~~////~~			
71–75	~~////~~ ~~////~~			
76–80	~~////~~			
81–85	~~////~~			
86–90	//			
91–95	///			
96–100	///			

(a) Copy and complete the table.
(b) To which classes do the marks 90.9, 66.2 and 81.5 belong?
(c) Draw a histogram to represent this distribution.

6. In a survey on the different prices of an article sold in the shops of a certain city, the following results were obtained.

Cost (x cents)	Frequency
$90 \leqslant x < 95$	4
$95 \leqslant x < 100$	11
$100 \leqslant x < 105$	15
$105 \leqslant x < 110$	24
$110 \leqslant x < 115$	18
$115 \leqslant x < 120$	9
$120 \leqslant x < 125$	3

(a) How many shops were included in the survey?
(b) Construct a histogram to represent the data.

7. **(a)** Copy and complete the following frequency distribution table.

Class interval	Class width		Frequency	Height of rectangle
10 – 14	5	1 × standard	5	$5 \div 1 = 5$
15 – 24			8	
25 – 29			6	
30 – 34			11	
35 – 39			13	
40 – 54			3	
55 – 64	10	2 × standard	4	$4 \div 2 = 2$

(b) Construct a histogram to represent the distribution.

8. **(a)** Copy and complete the following table.

Class interval	Class width	Frequency	Frequency density = $\frac{\text{Frequency}}{\text{Class width}}$
$0 < x \leqslant 20$	20	4	$4 \div 20 = 0.2$
$20 < x \leqslant 30$		12	
$30 < x \leqslant 40$	10	14	$14 \div 10 = 1.4$
$40 < x \leqslant 50$		11	
$50 < x \leqslant 70$		8	
$70 < x \leqslant 100$		6	

(b) Draw a histogram to represent the distribution.

9. **(a)** Draw a histogram representing the distribution in Question 7 using frequency densities.
(b) Draw a histogram representing the distribution in Question 8 using the heights of the rectangles.

INTERMEDIATE LEVEL

10. The table shows the average amount, $\$x$, Singaporeans spend on taxi fares in a week. The amounts are given to the nearest dollar.

19	56	67	25	60	99	108	100	24	11
15	88	68	120	97	42	94	82	27	58
28	74	79	36	77	65	28	45	40	52
95	53	63	38	44	80	101	108	30	38

(i) Using the class intervals $0 \leqslant x < 20$, $20 \leqslant x < 40$ and so on, construct a frequency table for the data.

(ii) Draw a histogram to illustrate the data.

11. The table shows the gold-medal times (in seconds) for the women's swimming 100-m freestyle event in the Olympic Games from 1920 to 2012.

Year	Time	Year	Time
1920	73.6	1972	58.59
1924	72.4	1976	55.65
1928	71.0	1980	54.79
1932	66.8	1984	55.92
1936	65.9	1988	54.93
1948	66.3	1992	54.64
1952	66.8	1996	54.50
1956	62.0	2000	53.83
1960	61.2	2004	53.84
1964	59.5	2008	53.12
1968	60.0	2012	53.00

(i) Complete the frequency table for the data.

Time (x seconds)	Frequency
$50 \leqslant x < 55$	
$55 \leqslant x < 60$	
$60 \leqslant x < 65$	
$65 \leqslant x < 70$	
$70 \leqslant x < 75$	

(ii) Draw a histogram to illustrate the data.

(iii) Express the number of women who won the gold medal with times of less than 55 seconds in the swimming 100-m freestyle event as a percentage of the number of women who won the gold medal with times of at least 65 seconds in the same event in the Olympic Games from 1920 to 2012.

(iv) Comment on the trend in the gold-medal times from 1920 to 2012. Provide a reason for this trend.

12. The daily wages of 50 workers, in dollars, are given below. Construct a frequency table with class intervals 10 – 14, 15 – 19, 20 – 24, and so on. Draw a histogram to represent the data.

12	21	13	17	29	33	26	47	10	17	36	31	32	27	25	16	36	29	22	24	21	25	45	18	37
42	35	28	20	44	34	43	22	36	34	20	15	26	17	21	25	30	27	32	26	28	30	38	19	26

13. The table shows the waiting time, x minutes, for 60 patrons at a certain restaurant.

25	12	53	8	26	5	19	73	67	18	87	42	6	21	14	19	12	15	13	36
36	16	72	36	13	37	11	51	39	32	30	47	6	22	68	25	98	23	45	22
79	26	35	27	48	58	56	29	20	32	62	80	41	58	17	54	15	14	74	25

(a) Using the class intervals $0 < x \leq 10$, $10 < x \leq 20$, $20 < x \leq 30$ and so on, construct a frequency table for the data.

(b) Draw a histogram to illustrate the data.

14. 120 teachers have been in the teaching service for 10 years or longer. The following table gives the frequency distribution of the length of service of these 120 teachers.

Length of service (x years)	$10 \leq x < 15$	$15 \leq x < 20$	$20 \leq x < 25$	$25 \leq x < 30$	$30 \leq x < 40$	$40 \leq x < 50$
Number of teachers	32	40	25	12	7	4

(a) Draw a histogram for the frequency distribution.

(b) Using the same diagram, draw the frequency polygon of the distribution.

15. Copy and complete the following table which gives the frequency distribution of the lengths of 40 fishes of a certain species, measured to the nearest mm. Draw the frequency polygon of the distribution.

Length (mm)	Mid-value	Frequency
25 – 29	27	2
30 – 34		4
35 – 39		7
40 – 44		10
45 – 49		8
50 – 54		6
55 – 59		3

ADVANCED LEVEL

16. 100 crates of oranges imported from Country A are inspected. The number of rotten oranges in the crates is recorded.

Rotten oranges	0	1	2	3	4	5	6	7	8	9
Number of crates	4	9	12	28	22	15	5	2	2	1

Another 100 crates of oranges imported from Country B are also inspected. The number of rotten oranges in the crates is recorded.

Rotten oranges	0	1	2	3	4	5	6	7	8
Number of crates	51	30	8	4	1	2	2	1	1

(i) Draw a histogram to illustrate each set of data.

(ii) State the largest number of rotten oranges found in a crate from each country.

(iii) Find the total number of rotten oranges from each country.

(iv) A crate is selected at random from the crates of oranges imported from Country A. The probability that the crate contains no fewer than p rotten oranges is $\frac{3}{4}$. Find the value of p.

Summary

1. Data in a **dot diagram** are represented by dots above a horizontal number line.
2. The values in **stem-and-leaf diagrams** are split into two parts, the stems and the leaves. There are variations of the stem-and-leaf diagrams such as those with **split stems** and the **back-to-back stem-and-leaf diagrams**.
3. Data in a **scatter diagram** are represented by plotting pairs of coordinates on a Cartesian plane. A **line of best fit** is often drawn, passing through the middle of all the data points, on the scatter diagram to determine the strength of **correlation**.
4. In a **histogram**, the areas of the rectangles, and not the heights, are proportional to the frequencies they represent. If the bases of the rectangles in a histogram are equal, then the heights of the rectangles are proportional to the frequencies.
5. If the mid-points of the tops of the consecutive bars in a histogram are joined by line segments and the mid-points at both ends are joined to the horizontal axis, a **frequency polygon** is obtained.
6. The choice of an appropriate statistical diagram depends on the type of data collected and the purpose of collecting the data.
   - A dot diagram is most suitable when we want to display a small set of data that does not contain many distinct values.
   - A stem-and-leaf diagram is most suitable when we need to know the exact values of the original data.
   - A scatter diagram is most suitable when we need to determine the correlation between any two sets of data.
   - For ungrouped data, a histogram is most suitable when we are interested to know the exact values of the frequencies of the data sets and we want to identify the data sets with the lowest and the highest frequencies.
   - For grouped data, a histogram is most suitable when we want to clearly display a large set of data and when we do not need to know the exact values of the original data.

Review Exercise 16

1. The table shows the time taken, measured to the nearest minute, for 30 students to read a passage.

2	1	4	2	7	1	1	5	3	4
1	2	1	4	2	3	3	2	2	1
2	1	5	4	1	3	2	2	1	4

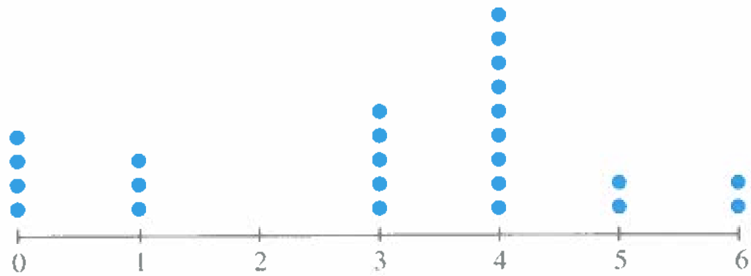
(i) Represent the data on a dot diagram.

(ii) What were the most common time taken?

(iii) Find the percentage of students who took at most 3 minutes to read the passage.

(iv) Briefly describe the distribution of the data.

2. The dot diagram represents the results of a survey conducted on a group of primary school children to find out the average number of sweets they eat in a week.



(i) How many sweets do most children eat in a week?

(ii) What is the greatest number of sweets the children eat in a week?

(iii) Find the fraction of children who do not eat sweets in a week.

3. The table shows the lengths, in cm, of some metal rods in a factory.

146	130	158	165	133	153	145	151	132	145
141	150	130	145	152	147	165	131	146	141

(i) Represent the data using a stem-and-leaf diagram.

(ii) What is the length of the longest metal rod?

(iii) What is the most common length?

(iv) Find the ratio of the number of metal rods that are shorter than 146 cm to the number of metal rods that are at least 152 cm long.

4. The stem-and-leaf diagram with split stems represents the speeds, in km/h, of some vehicles as they drove past a particular section of an expressway. The speeds of the vehicles were recorded by a speed camera located at that section.

Stem	Leaf
5	3
5	5 7 9
6	0 1 2 3
6	6 8 8 9
7	0 2 2 3 4 4
7	5 5 5 6 6 8 8 9 9
8	0 1 2 2 2 2 2 2 2 3 3 3 3 4 4
8	5 5 5 6 6 6 7 7 7 7 7 8 9
9	0 0 1 2
9	9

Key: 5 | 2 means 52 km/h

(i) What was the most common speed?

(ii) The highest speed limit for vehicles on an expressway is 90 km/h. Find the percentage of vehicles that exceeded the speed limit.

(iii) A vehicle is selected at random. The probability that the speed of the vehicle exceeds x km/h is $\frac{3}{10}$. Find the value of x.

5. The average amounts of money, in dollars, saved by two groups of students in a month are recorded in the table.

Group A	73	70	30	36	68	55	42	66
	28	64	96	74	68	85	51	30
Group B	39	80	42	27	60	59	35	59
	62	38	45	61	46	32	73	36

(i) Represent these two sets of data by a back-to-back stem-and-leaf diagram.

(ii) What is the greatest average amount of money saved by the students in each group?

(iii) Which group of students saves more money in a month? Explain your answer.

6. The following table shows the number of rainy days, x, for different months and the number of road accidents, y, for each of the months.

x	2	5	8	10	14	18	20	22	24
y	2	3	5	5	7	8	11	12	13

(a) On a sheet of graph paper, using a scale of 2 cm to represent 2 days on the horizontal axis and 2 cm to represent 1 accident on the vertical axis, plot the values given in the table above.

(b) Draw a line of best fit for the scatter diagram.

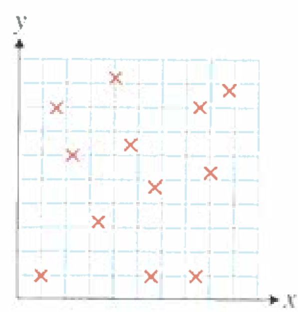
(c) Using the line of best fit drawn, estimate the number of road accidents on a month with

(i) 17 rainy days, **(ii)** 27 rainy days.

(d) Find the equation of the line of best fit drawn.

(e) Describe the correlation between rain and road accidents.

7. Describe the correlation in the following scatter diagram and suggest a possible scenario to produce such a result.



8. The number of hours 60 factory workers worked in a particular week is recorded.

Number of hours	41	42	43	44	45	46	47	48	49	50
Number of people	2	3	4	10	18	11	3	5	2	2

(i) Draw a histogram to illustrate the data.

(ii) Find the fraction of workers who worked no more than 45 hours in that week.

(iii) The table shows the hourly wages of each worker.

For the first 42 hours	\$6
Thereafter	\$9

Calculate the total salary paid to the workers who worked more than 48 hours in that week.

9. The table shows the amount, $\$x$, spent by some families at a shopping mall over a particular weekend.

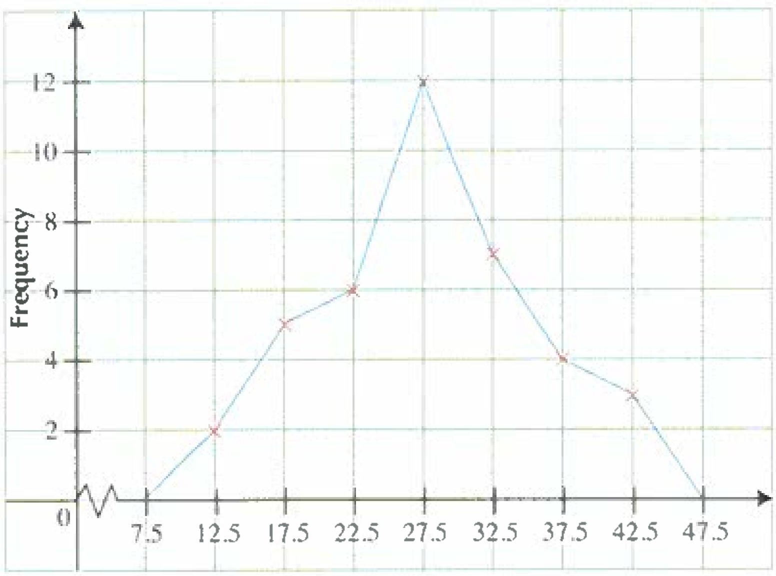
74	85	65	88	80	54	78
58	73	70	72	67	75	75
66	70	80	74	100	69	80
80	80	75	63	62	90	60
78	55	68	95	85	71	62

(i) Using the class intervals $50 \leqslant x < 60$, $60 \leqslant x < 70$ and so on, construct a frequency table for the data.

(ii) Draw a histogram to illustrate the data.

(iii) Families which spent more than \$80 were given a chance to participate in a lucky draw. If a family is selected at random, find the probability that the family selected is eligible for the lucky draw.

10. The heights of high-rise flats (in metres) in a certain housing estate are represented by the frequency polygon below:



(a) Copy and complete the following table.

Height (x m)	Number of flats
$10 < x \leqslant 15$	2
$15 < x \leqslant 20$	5
	6
$25 < x \leqslant 30$	

(b) Find the total number of flats in the distribution.

(c) Using the graph, construct a frequency table for the distribution.

(d) Draw the corresponding histogram, using a separate diagram.

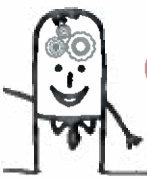
11. The time, in minutes, taken by an auditing firm to audit each of the 60 accounts is given below:

51	35	52	36	36	57	57	35	49	42
21	31	41	25	44	41	33	42	39	48
52	41	41	45	22	28	38	46	42	35
43	53	35	39	44	31	47	47	27	25
31	54	42	47	32	43	46	35	42	31
58	41	31	44	39	42	42	31	33	38

(a) Construct a grouped frequency table for the information using a class width of 5, the first class having a lower limit of 20.

(b) Draw a histogram to represent the information.

(c) Using a separate diagram, draw the frequency polygon representing the data.



Challenge Yourself

The marks obtained in a Science quiz by 36 students in a class were recorded. The total score of the quiz is 10.

Marks obtained	0	1	2	3	4	5	6	7	8	9	10
Number of students	0	0	1			$x - 1$	x	$x + 3$	$x + 5$		4

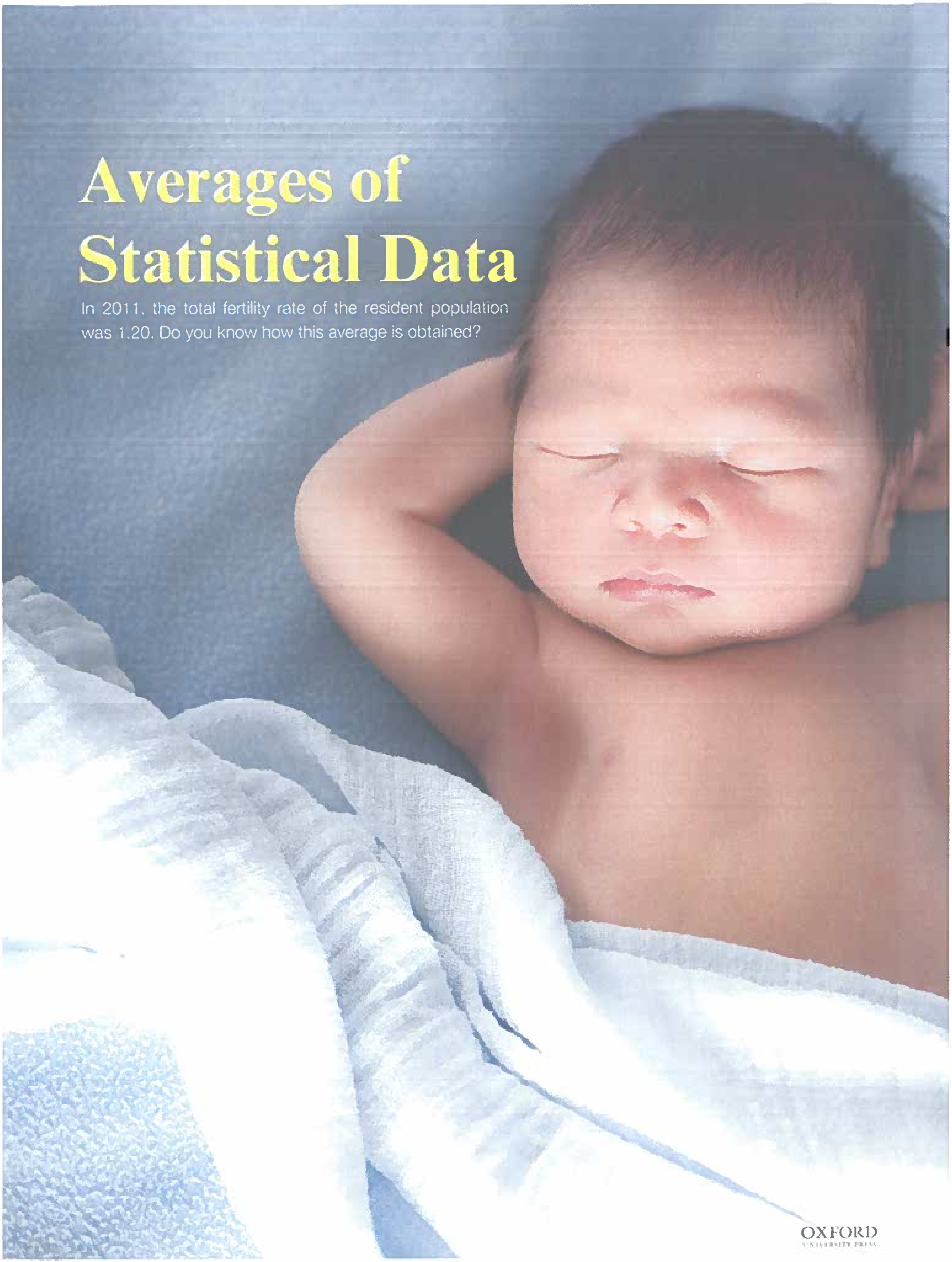
Additional information is given as follows:

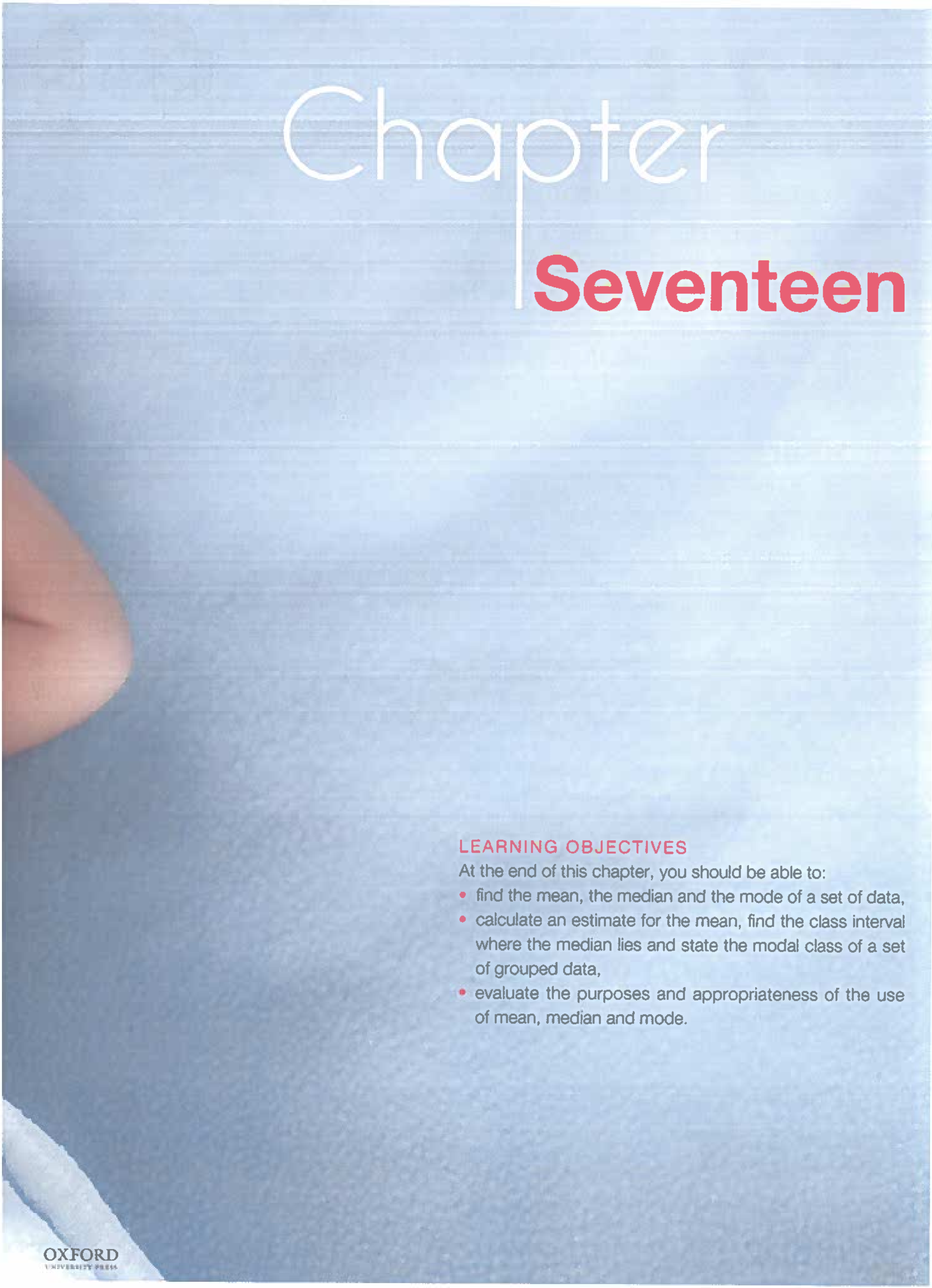
- 25% of the students in the class scored at least 9 marks.
- If a student is selected at random, the probability that the student obtained more than 4 marks but not more than 9 marks is $\frac{2}{3}$.
- The average marks obtained by the students who scored more than or equal to 4 marks but less than or equal to 6 marks is 4.8.

Use the above information to draw a histogram to represent the data.

Averages of Statistical Data

In 2011, the total fertility rate of the resident population was 1.20. Do you know how this average is obtained?





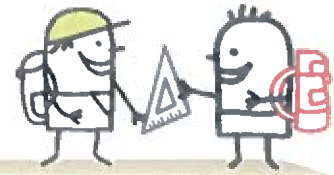
Chapter Seventeen

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- find the mean, the median and the mode of a set of data,
- calculate an estimate for the mean, find the class interval where the median lies and state the modal class of a set of grouped data,
- evaluate the purposes and appropriateness of the use of mean, median and mode.

17.1 Mean



Averages of Statistical Data

In Book 1 and in Chapter 16, we have learnt how to use statistical diagrams to display a set of data. A set of data can also be summarised using numerical measures.

In this chapter, we will learn about the three most common numerical measures, i.e. **mean**, **median** and **mode**. These measures are also known as **averages** in statistics. They are indicative of the central tendency or centre of a set of data.

Mean

Consider the heights of 6 students:

1.56 m, 1.67 m, 1.49 m, 1.55 m, 1.71 m, 1.68 m

Recall that in primary school, if we are required to find the average height of the students, we would divide the sum of the heights of the students by the number of students. In statistics, this average is called the **mean** and is given by:

$$\text{Mean} = \frac{\text{Sum of data}}{\text{Number of data}}$$

ATTENTION

The idea of the mean is to even out the data. If we replace each height of the students by their mean height, the sum of their heights remains the same.

Worked Example 1

Finding the Mean of Ungrouped Data

The heights of 6 students are 1.56 m, 1.67 m, 1.49 m, 1.55 m, 1.71 m and 1.68 m. Calculate the mean height of the students.

Solution:

$$\begin{aligned}\text{Mean height} &= \frac{\text{Sum of heights}}{\text{Number of students}}\\ &= \frac{1.56 + 1.67 + 1.49 + 1.55 + 1.71 + 1.68}{6}\\ &= \frac{9.66}{6}\\ &= 1.61 \text{ m}\end{aligned}$$

PRACTISE NOW 1

The scores of 8 students in an English test are 79, 58, 73, 66, 50, 89, 91 and 58. Find their mean score.

SIMILAR QUESTIONS

Exercise 17A Questions 1-2

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Worked Example 2

(Finding an Unknown Given the Mean)

The mean of \$10, \$15, \$12, \$20 and \$$x$ is \$13.
Calculate the value of x.

Solution:

$$\frac{10 + 15 + 12 + 20 + x}{5} = 13$$

$$10 + 15 + 12 + 20 + x = 65$$

$$57 + x = 65$$

$$\therefore x = 8$$

PRACTISE NOW 2

The mean of 44, 47, y, 58 and 55 is 52. Find the value of y.

SIMILAR QUESTIONS

Exercise 17A Question 3

Worked Example 3

(Problem involving Mean)

The mean of 6 numbers is 17. Four of the numbers are 15, 17, 20 and 22. The remaining two numbers are each equal to x. Calculate

(i) the sum of the 6 numbers,

(ii) the value of x.

Solution:

(i) Since mean $= \dfrac{\text{sum of the 6 numbers}}{6}$,

then sum of the 6 numbers $= 6 \times$ mean

$= 6 \times 17$

$= 102$

(ii) $15 + 17 + 20 + 22 + x + x = 102$

$74 + 2x = 102$

$2x = 28$

$x = 14$

PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 17A Questions 4-5, 9-11, 17

1. The mean of 7 numbers is 11. Five of the numbers are 3, 17, 20, 4 and 15. The remaining two numbers are each equal to y. Find
   (i) the sum of the 7 numbers,
   (ii) the value of y.
2. The mean height of 20 boys and 14 girls is 161 cm. If the mean height of the 14 girls is 151 cm, find the mean height of the 20 boys.
3. Given that the mean of 16, w, 17, 9, x, 2, y, 7 and z is 11, find the mean of w, x, y and z.

Worked Example 4

(Finding the Mean from a Frequency Table)

The weekly salaries of the employees in a company are recorded.

Weekly salary ($)	1000	1100	1200	2100	2500
Number of employees	20	8	10	7	5

Calculate

(i) the total number of employees in the company,

(ii) the total salary paid to the employees in a week,

(iii) the mean weekly salary of the employees.

Solution:

(i) Total number of employees = 20 + 8 + 10 + 7 + 5
= 50

(ii) Total salary paid to the employees in a week
= 20 × \$1000 + 8 × \$1100 + 10 × \$1200 + 7 × \$2100 + 5 × \$2500
= \$68 000

(iii) Mean weekly salary $= \dfrac{\$68\ 000}{50}$
= \$1360

PRACTISE NOW 4

The amount of money spent by the visitors at a carnival was recorded.

Amount spent ($)	40	60	80	100	160	200
Number of visitors	12	32	54	68	18	16

Find

(i) the total number of visitors who were at the carnival,

(ii) the total amount of money spent by the visitors at the carnival,

(iii) the mean amount of money spent by the visitors at the carnival.

SIMILAR QUESTIONS

Exercise 17A Questions 6-7, 12

In general, a set of data $x_1, x_2, x_3, \ldots, x_n$ with the corresponding frequencies $f_1, f_2, f_3, \ldots, f_n$ is usually displayed in the form of a frequency table as shown in Table 17.1.

x	x_1	x_2	x_3	...	x_n
f	f_1	f_2	f_3	...	f_n

Table 17.1

The mean of the distribution is given by:

$$\text{Mean} = \frac{f_1x_1 + f_2x_2 + f_3x_3 + \ldots + f_nx_n}{f_1 + f_2 + f_3 + \ldots + f_n}$$

If we write the sum of fx as Σfx and the sum of f as Σf, then we have

$$\bar{x} = \frac{\Sigma fx}{\Sigma f},$$

where $\bar{x}$ is the mean of a set of data.

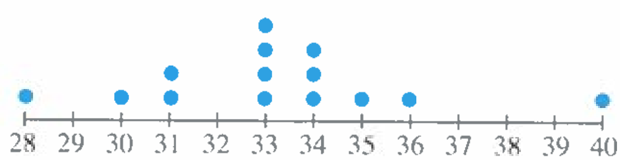
INFORMATION

Σ (sigma) is the uppercase of the eighteenth letter of the Greek alphabet. In Mathematics, Σ is used as the symbol for the summation operator.

Worked Example 5

(Finding the Mean from a Dot Diagram)

14 boxes are to be shipped from Korea to Taiwan. The dot diagram represents the masses, in kg, of the boxes.



Find the mean mass of the boxes.

Solution:

$$\text{Mean mass} = \frac{\Sigma fx}{\Sigma f}$$

$$= \frac{1 \times 28 + 1 \times 30 + 2 \times 31 + 4 \times 33 + 3 \times 34 + 1 \times 35 + 1 \times 36 + 1 \times 40}{14}$$

$$= \frac{465}{14}$$

$= 33.2$ kg (to 3 s.f.)

Problem Solving Tip

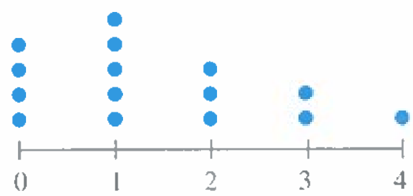
Since the question states that there are 14 boxes, $\Sigma f = 14$. There is no need to count the number of dots in the dot diagram to obtain the value of Σf.

PRACTISE NOW 5

SIMILAR QUESTIONS

Exercise 17A Questions 8(a)-(e)

1. The dot diagram represents the number of siblings 15 students have.



Find the mean number of siblings the students have.

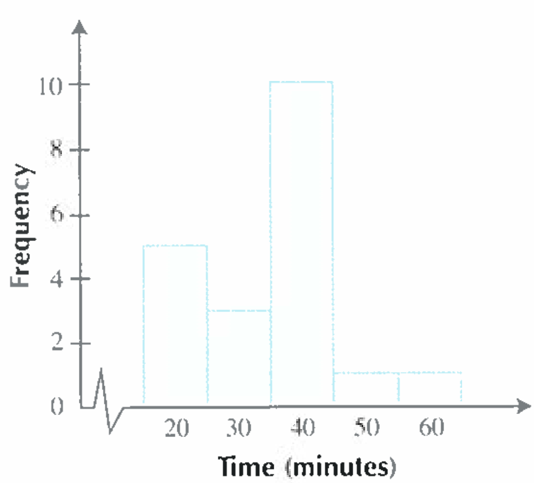
2. In Chemistry, pH is the measure of the acidity of a solution. The stem-and-leaf diagram represents the pH values of 20 acidic solutions.

Stem	Leaf
1	4 4 9
2	3 6 7 7 7 8
3	3 6 6 6 9
4	1 2 2 5 5 9

Key: 1 | 9 means 1.9

Find the mean pH value of the solutions.

3. The histogram shows the time taken, in minutes, for a group of students to pack for a camp.



Find the mean time taken by the group of students to pack for the camp.

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In Chapter 16, we have learnt how to group a set of data into class intervals. We shall now learn how to *estimate* the mean of a set of grouped data.

The number of magazines sold in different shops in a week was recorded.

Number of magazines sold (x)	Number of shops
$70 \leqslant x < 75$	4
$75 \leqslant x < 80$	11
$80 \leqslant x < 85$	15
$85 \leqslant x < 90$	24
$90 \leqslant x < 95$	18
$95 \leqslant x < 100$	9
$100 \leqslant x < 105$	3

Table 17.2

We are only able to estimate the mean of the number of magazines sold in the shops. This is because, for example, we only know there were 4 shops which sold more than or equal to 70 but fewer than 75 magazines, but from Table 17.2, we do not know the exact number of magazines sold in each shop.

To estimate the mean of a set of grouped data, we need to represent the data in each of the class intervals by a value that best represents the class interval. For example, in the class interval $70 \leqslant x < 75$, we cannot use 70 because most of the data could be more than 70. Hence, a value that best represents the class interval $70 \leqslant x < 75$ is its mid-value, i.e. $\frac{70 + 75}{2} = 72.5$.

To estimate the mean number of magazines sold in the shops, we follow the steps shown:

Step 1: Find the mid-value x and fx for each class interval.

Number of magazines sold (x)	Frequency (f)	Mid-value (x)	fx
$70 \leqslant x < 75$	4	72.5	290
$75 \leqslant x < 80$	11	77.5	852.5
$80 \leqslant x < 85$	15	82.5	1237.5
$85 \leqslant x < 90$	24	87.5	2100
$90 \leqslant x < 95$	18	92.5	1665
$95 \leqslant x < 100$	9	97.5	877.5
$100 \leqslant x < 105$	3	102.5	307.5
	$\Sigma f = 84$		$\Sigma fx = 7330$

Table 17.3

Step 2: Find Σf.

Step 3: Find Σfx.

Step 4: Estimated mean number of magazines sold $= \frac{7330}{84}$

$= 87.3$ (to 3 s.f.)

In general, the estimated mean $\bar{x}$ of a set of grouped data is

$$\bar{x} = \frac{\sum fx}{\sum f},$$

where x is the mid-value and f is the frequency of the class interval.

Worked Example 6

(Finding the Mean of Grouped Data)

The time taken, in minutes, for a group of teenagers to cycle between two parks is recorded.

Time taken (*x* minutes)	$15 < x \leq 20$	$20 < x \leq 25$	$25 < x \leq 30$	$30 < x \leq 35$	$35 < x \leq 40$
Number of teenagers	5	10	20	15	18

Calculate an estimate for the mean time taken by the teenagers to cycle between the two parks.

Solution:

Time (*x* minutes)	Frequency (*f*)	Mid-value (*x*)	*fx*
$15 < x \leq 20$	5	17.5	87.5
$20 < x \leq 25$	10	22.5	225
$25 < x \leq 30$	20	27.5	550
$30 < x \leq 35$	15	32.5	487.5
$35 < x \leq 40$	18	37.5	675
	$\sum f = 68$		$\sum fx = 2025$

Estimated mean time taken $= \frac{2025}{68}$

$= 29.8$ minutes (to 3 s.f.)

PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 17A Questions 13-16, 18

The ages, in years, of the employees of an advertising firm are recorded.

Age (*x* years)	$20 < x \leq 30$	$30 < x \leq 40$	$40 < x \leq 50$	$50 < x \leq 60$	$60 < x \leq 70$
Number of employees	12	10	20	15	18

Calculate an estimate for the mean age of the employees.

Exercise 17A

BASIC LEVEL

1. The number of passengers on coaches travelling along 12 popular scenic routes are 29, 42, 45, 39, 36, 41, 38, 37, 43, 35, 32 and 40. Find the mean number of passengers on the coaches.

2. Consider the prices, in \$, of various computing books at a bookstore:

19.90, 24.45, 34.65, 26.50, 44.05,
38.95, 56.40, 48.75, 29.30, 35.65

Find the mean price of the books.

3. The mean of 7 cm, 15 cm, 12 cm, 5 cm, h cm and 13 cm is 10 cm. Find the value of h.

4. The mean mass of 5 boys is 62 kg. When the mass of a boy is excluded, the mean mass of the remaining 4 boys becomes 64 kg. Find the mass of the boy that has been excluded.

5. The mean of 8 numbers is 12. Five of the numbers are 6, 8, 5, 10 and 28. The remaining three numbers are each equal to k. Find
(i) the sum of the 8 numbers,
(ii) the value of k.

6. The number of goals scored per match by a team during a soccer league season was recorded.

Number of goals scored per match	0	1	2	3	4	5	6
Number of matches	6	8	5	6	2	2	1

Find
(i) the total number of matches played,
(ii) the total number of goals scored,
(iii) the mean number of goals scored per match by the team.

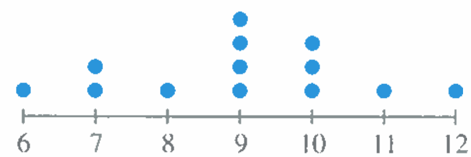
7. The number of days the students in a class were absent from school in a term was recorded.

Number of days absent	0	1	2	3	4	5	6	9
Number of students	23	4	5	2	2	1	2	1

Find the mean number of days of absence.

8. For each of the following, find the mean of the distribution.

(a)

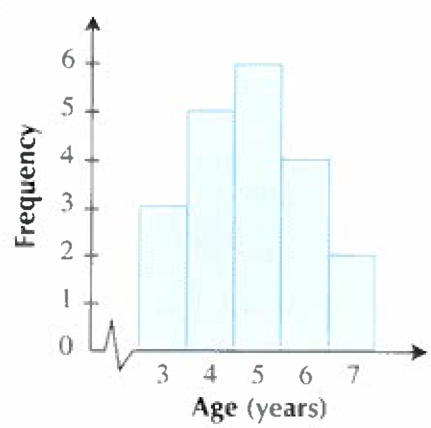


(b)

Stem	**Leaf**
7	2 3 5 5
8	2 7 8 8 9 9 9
9	1 3 7 7
10	2 7 8

Key: 7 | 3 means \$7.30

(c)



INTERMEDIATE LEVEL

9. The mean of 10 numbers is 14. Three of the numbers have a mean of 4. The remaining seven numbers are 15, 18, 21, 5, m, 34 and 14. Find
   (i) the sum of the remaining seven numbers,
   (ii) the value of m.

10. The mean monthly wage of 7 experienced and 5 inexperienced workers is \$1000. If the mean monthly wage of the 5 inexperienced workers is \$846, find the mean monthly wage of the 7 experienced workers.

11. The heights of three plants A, B and C in a garden are in the ratio 2 : 3 : 5. Their mean height is 30 cm.
   (i) Find the height of Plant B.
   (ii) If another Plant D is added to the garden such that the mean height of the four plants is now 33 cm, find the height of Plant D.

12. The marks obtained in an English and a Mathematics test by 40 students were recorded. The total score of each test is 10.

Marks obtained	0	1	2	3	4	5	6	7	8	9	10
Number of students (English)	0	1	6	14	4	8	2	4	0	1	0
Number of students (Mathematics)	0	4	1	6	5	10	3	5	3	1	2

   (a) Find the mean mark of the students for each subject.
   (b) Given that the percentage passing mark for each subject was 50%, find the percentage of students who
      (i) passed English,
      (ii) did not pass Mathematics.

13. The heights, measured to the nearest cm, of the plants in a nursery are recorded.

Height (x cm)	$0 < x \leqslant 10$	$10 < x \leqslant 20$	$20 < x \leqslant 30$	$30 < x \leqslant 40$	$40 < x \leqslant 50$
Number of plants	4	6	14	6	10

   (i) Calculate an estimate for the mean height of the plants.
   (ii) A plant is selected at random. Find the probability that the plant is not taller than 40 cm.

14. 100 lorries are required to transport raw materials from a quarry to a construction site. The time taken, in minutes, for the lorries to travel from the quarry to the construction site using a fixed route was recorded.

Time taken (t minutes)	**Number of lorries**
$116 \leqslant t < 118$	1
$118 \leqslant t < 120$	6
$120 \leqslant t < 122$	23
$122 \leqslant t < 124$	28
$124 \leqslant t < 126$	27
$126 \leqslant t < 128$	9
$128 \leqslant t < 130$	5
$130 \leqslant t < 132$	1

   (i) Calculate an estimate for the mean travelling time of the lorries.
   (ii) Find the fraction of lorries which took less than 124 minutes to travel from the quarry to the construction site.

15. The speeds, in km/h, of 100 vehicles are recorded.

Speed (x km/h)	$30 < x \leqslant 40$	$40 < x \leqslant 50$	$50 < x \leqslant 60$	$60 < x \leqslant 70$	$70 < x \leqslant 80$
Number of vehicles	16	25	35	14	10

(i) Calculate an estimate for the mean speed of the vehicles.

(ii) Find the ratio of the number of vehicles which travel at a speed of not more than 40 km/h to the number of vehicles which travel at a speed of more than 60 km/h.

16. The table shows the mean distances, d million km, of 20 moons from Jupiter.

21.03	23.55	22.93	23.12	23.22
23.03	21.11	23.22	23.28	21.31
21.28	23.58	21.27	23.40	21.17
21.15	23.62	23.18	22.25	23.36

(i) Complete the frequency table for the data.

Mean distance (d million km)	Frequency
$21.0 \leqslant d < 21.5$	
$21.5 \leqslant d < 22.0$	
$22.0 \leqslant d < 22.5$	
$22.5 \leqslant d < 23.0$	
$23.0 \leqslant d < 23.5$	
$23.5 \leqslant d < 24.0$	

(ii) Using the frequency table in **(i)**, calculate an estimate for the mean of the mean distances of the moons from Jupiter.

ADVANCED LEVEL

17. The mean of three numbers x, y and z is 6 and the mean of five numbers x, y, z, a and b is 8. Find the mean of a and b.

18. The table shows the lifespans, x hours, of 30 light bulbs.

167	171	179	167	171	165	175	179	169	168
171	177	169	171	177	173	165	175	167	174
177	172	164	175	179	179	174	174	168	171

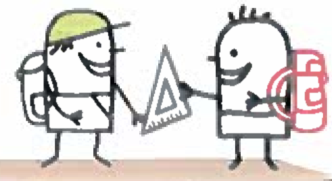
(i) Find the mean lifespan of the light bulbs by dividing the sum of the lifespans of the light bulbs by 30.

(ii) Using the class intervals $164 \leqslant x < 167$, $167 \leqslant x < 170$ and so on, construct a frequency table for the data.

(iii) Using the frequency table in **(ii)**, calculate an estimate for the mean lifespan of the light bulbs.

(iv) Comment on your answers in **(i)** and **(iii)**.

17.2 Median



In a survey to find out the living standards of the people in a city, the monthly incomes of 9 randomly chosen households were obtained as shown in Table 17.4.

Household	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Income ($)	3720	3940	3960	4030	4050	4250	4400	4550	24 250

Table 17.4

The mean income of these 9 households is $6350. However, we observe that 8 out of 9 households have incomes well below $6350. This value gives a distorted picture of the standard of living of the people in the city because household *I* has an extremely high income of $24 250.

To obtain a more accurate representation of the data, one way is to use another type of average called the **median**. The median is the *middle value* when the data is rearranged in ascending order.

The data in Table 17.4 has already been arranged in ascending order and the median is $4050. This is more accurate as most incomes are between $3720 and $4550.

Worked Example 7

(Finding the Median when the Number of Data is Odd)

Find the median of the following set of data.

20, 25, 21, 24, 22, 26, 20

Solution:

Total number of data = 7 (odd)

$$\text{Middle position} = \frac{7+1}{2}$$

$$= 4^{\text{th}} \text{ position}$$

Rearranging the data in ascending order, we have:

20, 20, 21, (22), 24, 25, 26

↑ 4^{th} position

$\therefore$ Median = data in the 4^{th} position

= 22

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 17B Questions 1(a), (c)

Find the median of each of the following sets of data.

(a) 20, 16, 9, 3, 18, 11, 15 **(b)** 11.2, 15.6, 30.2, 17.3, 18.2

Worked Example 8

(Finding the Median when the Number of Data is Even)

Find the median of the following set of data.

12, 8, 19, 30, 14, 21, 9, 5

Solution:

Total number of data = 8 (even)

$$\text{Middle position} = \frac{8+1}{2}$$

$$= 4.5^{\text{th}} \text{ position}$$

Rearranging the data in ascending order, we have:

5, 8, 9, 12, 14, 19, 21, 30

↑

4.5^{th} position

∴ Median = mean of the data in the 4^{th} and the 5^{th} position

$$= \frac{12+14}{2}$$

$$= 13$$

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 17B Questions 1(b), (d), 8

Find the median of each of the following sets of data.

(a) 32, 15, 20, 15, 25, 12 **(b)** 8, 7.3, 8.9, 6.8, 8.8, 8.9, 10, 6.7

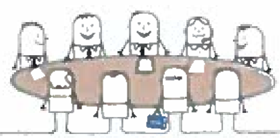
Thinking Time

In Worked Example 8, would we obtain a different median if the data is arranged in descending order instead? Explain your answer.

In general, to obtain the median of a set of data, we arrange the data in *ascending* or *descending order* first.

If the total number of data is *odd*, the median is the *middle value* of the arranged data.

If the total number of data is *even*, the median is the *mean of the two middle values* of the arranged data.



Class Discussion

Creating Sets of Data with Given Conditions

Create as many sets of data as possible that satisfy all the following conditions:

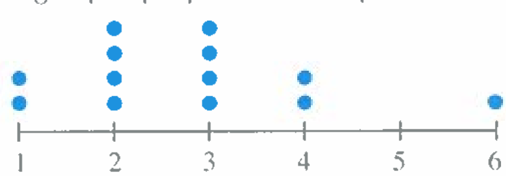
- Each set of data consists of 7 values.
- The difference between the minimum and the maximum value is 10.
- The mean of the data is greater than its median.

Are your answers the same as those obtained by your classmates?

Worked Example 9

Finding the Median from a Dot Diagram

The dot diagram represents the average number of times a group of people visit the supermarket in a week.



Find the median of the distribution.

Solution:

Total number of data = 13 (odd)

Middle position $= \dfrac{13+1}{2}$

$= 7^{th}$ position

$\therefore$ Median = data in the 7^{th} position

$= 3$

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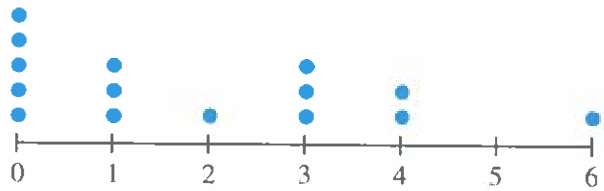
PRACTISE NOW 9

SIMILAR QUESTIONS

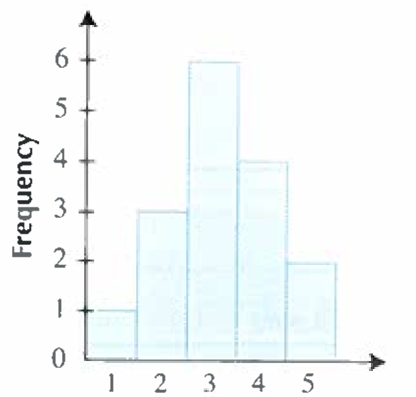
Exercise 17B Questions 2(a)–(b)

1. The dot diagram represents the average number of magazines a group of Secondary Two students buy in a month.



Find the median of the distribution.

2. The histogram shows the number of concert tickets sold by the members of the Guitar Club in a school in a morning. Find the median of the distribution.



Worked Example 10

(Finding the Median from a Stem-and-Leaf diagram)

The stem-and-leaf diagram represents the volumes, in m*l*, of chemical solution in 20 bottles.

Stem	Leaf
2	6 6 8 9
3	0 1 1 2 2 2 3 4 5 7 8
4	3 4 9
5	5 7

Key: 2 | 8 means 28 m*l*

Find the median volume of the chemical solution in the bottles.

Solution:

Total number of data = 20 (even)

Middle position $= \dfrac{20 + 1}{2}$

$= 10.5^{\text{th}}$ position

∴ Median volume = mean of the data in the 10^{th} and the 11^{th} position

$= \dfrac{32 + 33}{2}$

$= 32.5$ m*l*

ATTENTION

The data in a stem-and-leaf diagram has already been arranged in ascending order.

PRACTISE NOW 10

SIMILAR QUESTIONS

Exercise 17B Question 2(c)

The stem-and-leaf diagram represents the distance covered, in km, by a group of athletes during a training session.

Stem	Leaf
3	5 6 6 7 8
4	0 1 2 3 4 7 8 9
5	0 1 1 1 4
6	1 2

Key: 3 | 6 means 3.6 km

Find the median distance covered by the athletes.

Worked Example 11

(Finding the Median from a Frequency Table)

The heights, in cm, of 18 students are recorded.

Height (cm)	152	154	156	158	160
Number of students	2	2	5	1	8

Find the median height of the students.

ATTENTION

Do *not* take the value in the middle column, i.e. 156 cm, as the median height just because the 3rd column is in the middle of the 5 columns.

3rd column

Height (cm)	152	154	156	158	160
Number of students	2	2	5	1	8

Solution:

Total number of data = 18 (even)

Middle position $= \dfrac{18+1}{2}$

$= 9.5^{\text{th}}$ position

$\therefore$ Median height = mean of the data in the 9^{th} and the 10^{th} position

$= \dfrac{156+158}{2}$

$= 157$ cm

PRACTISE NOW 11

SIMILAR QUESTIONS

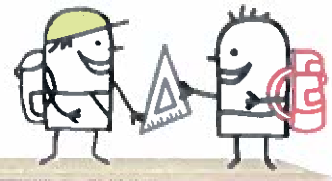
Exercise 17B Question 2(d)

The time taken, in minutes, for a group of children to complete a puzzle is recorded.

Time taken (minutes)	5	6	7	8	9
Number of children	8	4	3	10	3

Find the median time taken by the group of children to complete the puzzle.

17.3 Mode



In statistics, the value which occurs *most frequently* in a set of data is known as the **mode** of the set of data. For example, the mode is important to manufacturers who would like to produce most of their goods (e.g. shoes, shorts, skirts, etc.) in the most popular sizes so as to gain a higher percentage of the market share. Are you able to come up with another example where the mode is important?

From the dot diagram in Fig. 17.1, we can see that the most frequently occurring data is 4. Hence, the mode is 4.

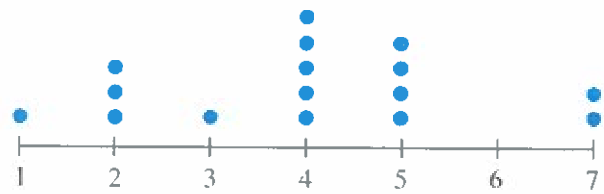


Fig. 17.1

Worked Example 12

(Finding the Mode of a Set of Ungrouped Data)

The scores of 10 students for a Mathematics test are 8, 9, 10, 10, 3, 5, 6, 10, 6 and 1.

(i) State the modal score.

(ii) If 2 more students' scores of 6 and 8 are added, what will be the new modal scores?

Solution:

(i) Modal score = 10 (since it occurs most frequently, i.e. 3 times)

(ii) New modal scores = 6 and 10 (since they occur most frequently, i.e. 3 times each)

INFORMATION

- We say that the distribution of the scores in Worked Example 12(ii) is **bimodal** since there are two modal scores.
- If there is no value in a set of data that occurs more frequently than others, we say that the distribution has no mode.

PRACTISE NOW 12

The lengths of 10 ribbons are 100 cm, 110 cm, 95 cm, 60 cm, 20 cm, 60 cm, 110 cm, 88 cm, 102 cm and 120 cm.

(i) State the modal lengths of the ribbons.

(ii) If a ribbon of length 110 cm is removed, what will be the new modal length of the ribbons?

SIMILAR QUESTIONS

Exercise 17B Questions 3(a)-(b), 5

Thinking Time

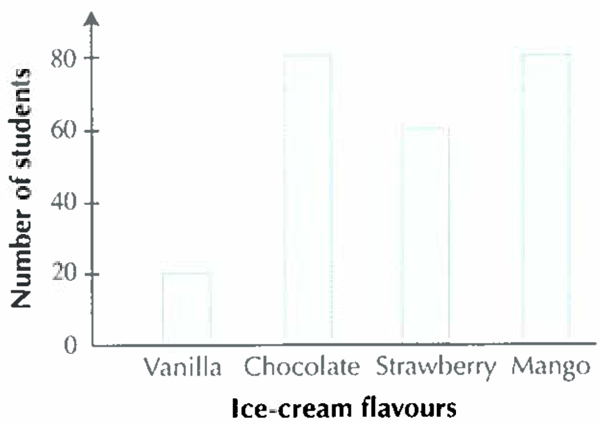
Design a set of data such that the mean is 55, the mode is larger than the mean and the median is larger than the mode.

Worked Example 13

Identifying the Modes from Statistical Diagrams and a Frequency Table

For each of the following, state the mode(s) of the distribution.

(a)



(b)

Stem	Leaf
1	0 1 2 3 4 4 5 5 5 6 7 8 9
2	0 0 1 3 3 5 7 7 7 7 7 7 8 8 9
3	0 4 4 8

Key: 1 | 1 means 1.1

(c)

Commission ($)	1000	1200	1500	2000
Number of salesmen	2	5	3	1

Solution:

(a) The modal ice-cream flavours are chocolate and mango (since they have the highest frequency, i.e. 80 students like each of the flavours).

(b) Mode = 2.7 (since it occurs most frequently, i.e. 6 times).

(c) Modal commission = $1200 (since it has the highest frequency, i.e. 5 salesmen receive $1200 in commission).

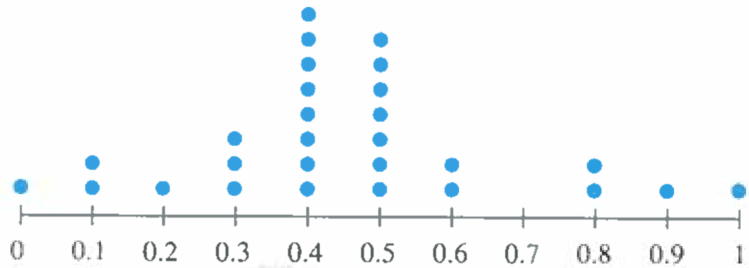
PRACTISE NOW 13

SIMILAR QUESTIONS

Exercise 17B Questions 4(a)-(e)

For each of the following, state the mode(s) of the distribution.

(a)

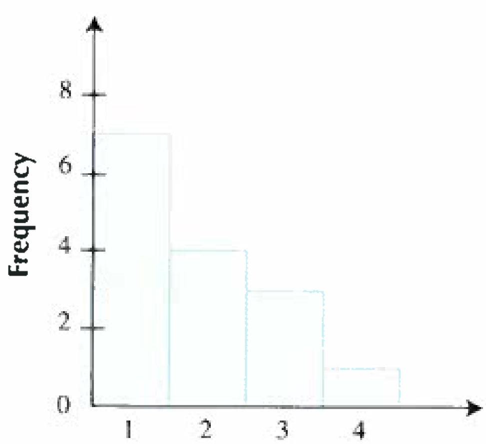


(b)

Stem	Leaf
2	6 6 7 8
3	1 1 2 2 2 3 6 7 7 7
4	0 2 5

Key: 2 | 7 means 27

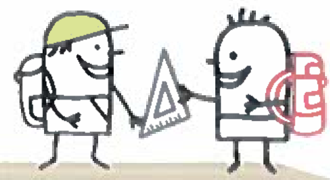
(c)



(d)

Monthly salary ($)	2000	3000	3500	4500	5000
Number of employees	10	56	42	25	17

17.4 Mean, Median and Mode



In the previous sections, we have learnt about the three most common averages in statistics – mean, median and mode.

Worked Example 14

(Problem involving Mean, Median and Mode)

The number of spelling errors a group of students made in an essay is recorded.

Number of spelling errors	0	1	2	3	4	5
Number of students	4	8	x	6	5	4

(a) If the mean number of spelling errors the students made is 2.4, calculate the value of x.

(b) If the median of the distribution is 3, find the possible values of x.

(c) If the modal number of spelling errors the students made is 2, state the smallest possible value of x.

Solution:

(a)
$$\frac{4 \times 0 + 8 \times 1 + 2x + 6 \times 3 + 5 \times 4 + 4 \times 5}{4 + 8 + x + 6 + 5 + 4} = 2.4$$

$$\frac{2x + 66}{x + 27} = 2.4$$

$$2x + 66 = 2.4(x + 27)$$

$$2x + 66 = 2.4x + 64.8$$

$$0.4x = 1.2$$

$$x = 3$$

(b) We write the data as follows:

$$\underbrace{0, \ldots, 0}_{4}, \underbrace{1, \ldots, 1}_{8}, \underbrace{2, \ldots, 2}_{x}, \underbrace{3, \ldots, 3}_{6}, \underbrace{4, \ldots, 4}_{5}, \underbrace{5, \ldots, 5}_{4}$$

The greatest value of x occurs when the median is here.

$\therefore\ 4 + 8 + x = 5 + 5 + 4$

$12 + x = 14$

$x = 2$

$\therefore$ Greatest value of $x = 2$

The smallest value of x occurs when the median is here (if possible).

$\therefore\ 4 + 8 + x + 5 = 5 + 4$

$17 + x = 9$

$x = -8$ (not possible since $x \geq 0$)

$\therefore$ Smallest value of $x = 0$

$\therefore$ Possible values of $x = 0$, 1 or 2

(c) Smallest possible value of $x = 9$ (since $x > 8$)

PRACTISE NOW 14

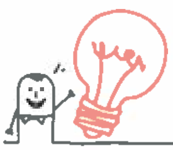
SIMILAR QUESTIONS

Exercise 17B Questions 6-7, 9-11, 12(a)-(b), 15-16, 17(a)-(c), 18

The number of books a group of students borrowed from the school library is recorded.

Number of books	0	1	2	3	4
Number of students	2	x	3	4	1

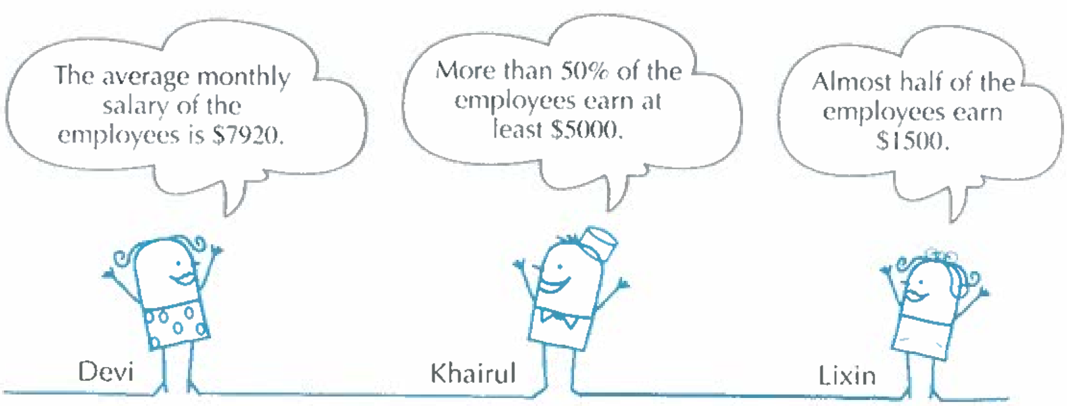
(a) If the mean number of books the students borrowed is 1.8, find the value of x.
(b) If the median of the distribution is 2, find the possible values of x.
(c) If the modal number of books the students borrowed is 3, state the greatest possible value of x.



Thinking Time

The monthly salaries of 25 employees in a company are recorded.

Monthly salary ($)	1500	5000	10 000	15 000	25 000	50 000
Number of employees	12	5	2	4	1	1



Devi, Khairul and Lixin seem to make statements that contradict each other. Who gives the best picture of how much money the employees in the company earn? Explain your answer.

Comparison of Mean, Median and Mode

Class Discussion

Comparison of Mean, Median and Mode

Work in pairs.

1. Consider the time taken, in minutes, for a group of students to complete a History quiz: 15, 17, 13, 18, 20, 19, 15

   (i) Find the mean, median and mode of this distribution.

Mean	Median	Mode

   (ii) If another student's time of 55 minutes is added, do you think the mean, median and mode will be the same? Find the new mean, median and mode.

Mean	Median	Mode

   (iii) Which average is most affected by the addition of a large number? Explain your answer.

   (iv) When a set of data is affected by extreme values, which average is the most appropriate measure to be used? Explain your answer.

2. The sizes of shoes sold at a shop on a weekday morning are given as follows:
   6, 7, 8, 8, 7, 9, 5, 6, 6

   (i) Find the mean, median and mode of this distribution.

Mean	Median	Mode

   (ii) Which average best represents the sizes of shoes sold that morning? Explain your answer.

3. The number of children some families have are recorded as shown.
   2, 3, 1, 4, 5, 1, 2, 2, 1, 1

   (i) Find the mean, median and mode of this distribution.

Mean	Median	Mode

   (ii) Fill in the blank.
   Even though the mean is not an integer, it still has a physical meaning, i.e. 2.2 children per family is equivalent to _____ children in 10 families.

4. Briefly explain when an average, mean, median or mode, is preferred over another.

SIMILAR QUESTIONS

Exercise 17B Questions 13-14

From the class discussion and from the previous sections, we can draw the following conclusions:

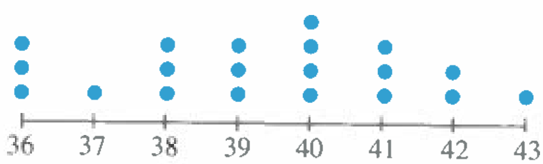
1. The **mean** is usually preferred over the median and the mode because *all* the values in a set of data are used in the calculation of the mean. It is the most reliable measure when there are no extreme values in the set of data.
2. The **median** is preferred for describing economic, sociological and educational data. It is popular in the study of the social sciences because most sets of data in the social sciences contain extreme values.
3. The **mode** is useful in business planning as a measure of popularity as it reflects the opinion of the masses.

Exercise 17B

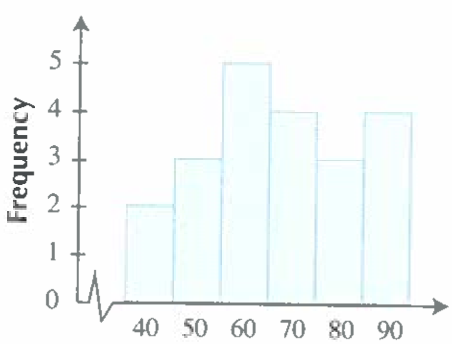
BASIC LEVEL

1. Find the median of each of the following sets of numbers.
   (a) 5, 6, 1, 5, 3, 5, 6
   (b) 30, 33, 37, 28, 29, 25
   (c) 1.2, 1.1, 4.1, 3.2, 4.1, 1.6, 2.8
   (d) 39.6, 12, 13.5, 22.6, 31.3, 8.4, 5.5, 4.7

2. For each of the following, find the median of the distribution.
   (a)



   (b)



   (c)

Stem	Leaf
3	0 1 4
4	2 5 8 9
5	1 7 8 9 9 9
6	3 7 7
7	4

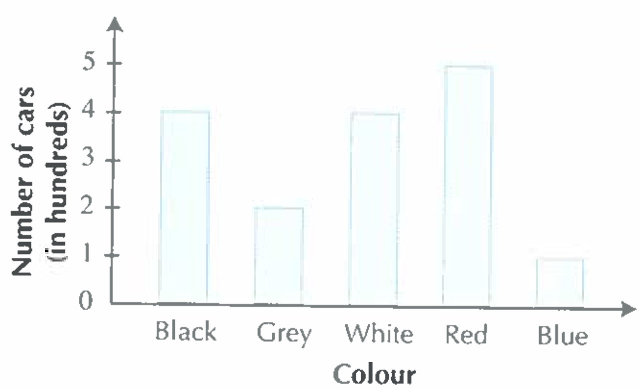
Key: 3 | 1 means 3.1

   (d)

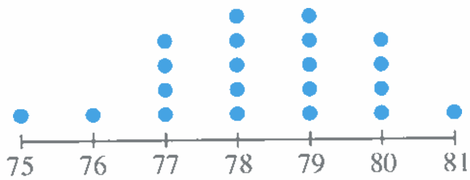
x	30	35	40	45	50	55	60
Frequency	5	6	10	8	7	5	1

3. Find the mode(s) of each of the following sets of numbers.
   (a) 2, 5, 8, 3, 7, 5, 3, 9, 7, 3
   (b) 8.1, 7.7, 7.8, 9.3, 6.4, 7.7, 9.3, 8.7

4. For each of the following, state the mode(s) of the distribution.
   (a)



(b)

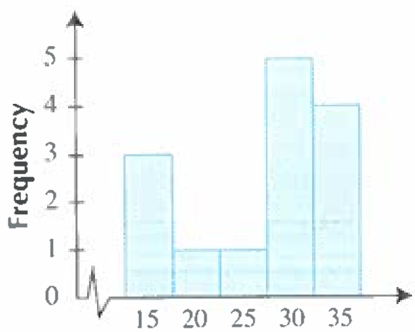


(c)

Stem	Leaf
2	1 1 2 3 3
3	7 8 9 9
4	2 3 5 5 7 7
5	3 3 7
6	0 0 0 1

Key: 2 | 2 means 22

(d)



(e)

x	6	7	8	9	10
Frequency	1	1	1	1	1

5. The temperatures, taken at midnight, of 6 consecutive nights in Singapore are given as follows:

22 °C, 27 °C, 26 °C, 28 °C, 27 °C, 23 °C

(i) State the modal temperature.

The temperature, taken at midnight, of the 7th day in Singapore was 22 °C.

(ii) If 22 °C is added to the above set of data, what will be the new modal temperatures?

6. The table shows the number of goals scored by a soccer team in 20 matches.

4	1	2	1	0	0	1	2	3	3
1	2	1	3	4	3	2	2	4	2

(a) Represent the data on a dot diagram.

(b) Find

(i) the mean,

(ii) the median,

(iii) the mode,

of the distribution.

7. In a science experiment, two groups of students, Group *A* and *B*, were required to measure the length of their pendulums. The lengths of the pendulums measured by both groups are shown in the back-to-back stem-and-leaf diagram.

Group A	Stem	Group B
9	4	
3 3 1	5	3
4 2	6	5 6 6 7
6 5	7	2 3 3
5 3	8	2 3

Key: 4 | 9 means 49 cm

(a) Compare and comment on the lengths of the pendulums measured by Group *A* and Group *B* students.

Then they were required to find the time taken for the pendulum to swing forwards and backwards (1 oscillation). The time taken for the pendulum to make 1 oscillation depends on the length of the pendulum. The longer the length of the pendulum, the longer it takes to make 1 oscillation.

(b) State, with a reason, whether you agree or disagree with the following statement: Group *A* claimed that the average time taken for 1 oscillation in their experiment is longer than that of Group *B*.

INTERMEDIATE LEVEL

8. The median of 8 numbers is 4.5. Given that seven of the numbers are 9, 2, 3, 4, 12, 13 and 1, find the eighth number.

9. The stem-and-leaf diagram represents the distance covered, in km, by a group of amateur runners during a training session.

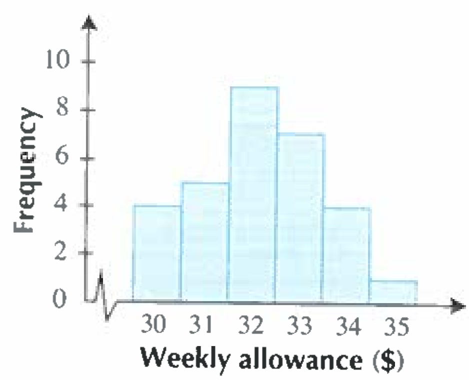
Stem	Leaf
2	3 4 6 9
3	0 1 2 2 2 4 4 5 8
4	2 2

Key: 2 | 4 means 24 km

(a) Find
   (i) the mean distance,
   (ii) the median distance,
   (iii) the modal distance,
   covered by the runners.

(b) A runner is selected at random. Find the probability that the distance covered by the runner is a prime number.

10. The histogram shows the allowance, in dollars, a group of Secondary Two students receive a week.



(a) Find
   (i) the mean allowance,
   (ii) the median allowance,
   (iii) the modal allowance,
   the students receive a week.

(b) Find the fraction of students who receive an allowance of at most \$32 a week.

11. The number of pets 40 students own is recorded.

Number of pets	2	4	6	8	10
Number of students	x	2	y	6	14

(a) (i) Show that $x + y = 18$.
   (ii) If the mean of the distribution is 6.4, show that $x + 3y = 30$.
   (iii) Hence, find the value of x and of y.

(b) Using your answers in **(a)(iii)**, find
   (i) the median,
   (ii) the mode,
   of the distribution.

12. The number of magazines read by a group of women in a week is recorded.

Number of magazines	0	1	2	3
Number of women	5	2	1	x

(a) If the median of the distribution is 2, find the value of x.

(b) If the median of the distribution is 1, find the possible values of x.

13. Jun Wei and Raj were playing golf. Their scores on the first nine holes are shown in the table. In golf, the lower the score, the better it is.

Hole	1	2	3	4	5	6	7	8	9	**Total**
Jun Wei	3	2	5	7	3	2	2	4	17	45
Raj	4	4	6	8	3	3	2	6	6	42

On the ninth hole, Jun Wei hit his golf ball into a sand trap and lost the game.

(i) Find the mean score on the nine holes for each player.

(ii) Which player scored better on most of the holes? Do the mean scores indicate this?

(iii) Find the median score for each player.

(iv) State the modal score of each player.

(v) Which average, i.e. the mean, the mode or the median, gives the best comparison of the abilities of Jun Wei and Raj? Explain your answer.

14. Two classes, each with 21 students, took a physical fitness test. The number of pull-ups done by each student in 30 seconds was recorded.

Secondary 2A	Number of pull-ups	≤ 5	6	7	8	9	≥ 10
	Number of students	3	7	4	4	2	1

Secondary 2B	Number of pull-ups	≤ 5	6	7	8	9	≥ 10
	Number of students	3	4	4	7	2	1

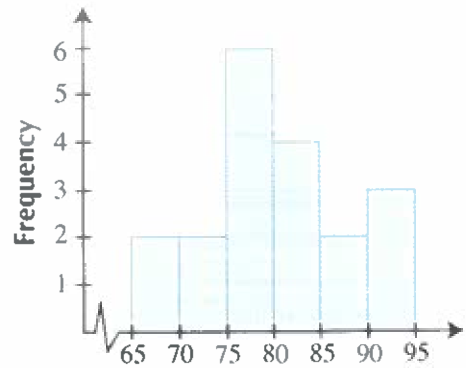
(i) Explain why we are unable to calculate the mean number of pull-ups done by the students in each class.

(ii) Find the median number of pull-ups done by the students in each class.

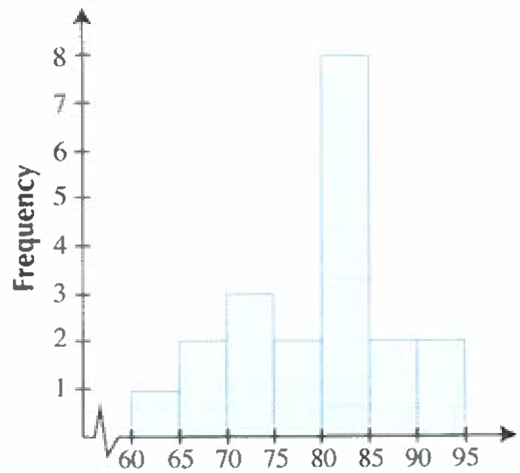
(iii) State the modal number of pull-ups done by the students in each class.

(iv) Does the median or the mode give a better comparison of the number of pull-ups done by the students in the two classes? Explain your answer.

15. The histograms show the number of SMS messages sent by the same group of 20 students in the months of June and July.



Number of SMS sent in June



Number of SMS sent in July

(i) Compare the number of SMS messages sent by the students in the months of June and July.

(ii) In June, one student sent 25 SMS messages only. If this data is included, will the mean or the median of the distribution be affected more? Explain your answer.

ADVANCED LEVEL

16. The number of major hurricanes, which strike the Atlantic coast each year over a period of 50 years, was recorded.

Number of major hurricanes	0	1	2	3	4	5	6
Number of years	5	13	15	x	1	y	2

(a) If the mean of the distribution is 2.18, find the value of x and of y.

(b) Using your answers in **(a)**, find

(i) the median,

(ii) the mode,

of the distribution.

(c) There are at most p major hurricanes striking the Atlantic coast each year in 36% of the years. Using your answers in **(a)**, find the value of p.

17. The number of social networking accounts a group of students own is recorded.

Number of social networking accounts	0	1	2	3	4	5
Number of students	4	6	3	x	3	2

(a) If the mean number of social networking accounts the students own is 2.2, find the value of x.

(b) If the median of the distribution is 2, find the greatest possible value of x.

(c) If the modal number of social networking accounts the students own is 3, state the smallest possible value of x.

18. The number of books read by a group of students in a week is recorded.

Number of books	0	1	2	3	4	5	6
Number of students	$x + 1$	$x - 2$	$x + 2$	x	$x - 2$	$x - 4$	$x - 3$

(i) Given that the mean and the mode of the distribution are equal, find the value of x.

(ii) Using your answer in **(i)**, find the median of the distribution.

	Mean	Median	Mode
Definition	• The mean $\bar{x}$ of a set of data is given by $\bar{x} = \frac{\text{Sum of data}}{\text{Number of data}}$ $= \frac{\sum fx}{\sum f}$, where $\sum fx$ represents the sum of fx and $\sum f$ represents the sum of f. • The estimated mean $\bar{x}$ of a set of grouped data is $\bar{x} = \frac{\sum fx}{\sum f}$, where x is the mid-value and f is the frequency of the class interval.	• The median is the *middle value* when the data is rearranged in ascending or descending order.	• The mode is the value which occurs most *frequently* in a set of data.
Properties	• Takes into consideration *all* values in a set of data • Affected by extreme values, thus may give a misleading interpretation of the distribution of the data	• Takes into consideration the middle value (when the number of data is odd) or two middle values (when the number of data is even) • Not affected by extreme values	• Takes into consideration the most common value in a set of data • Not affected by extreme values

Review Exercise 17

1. For each of the following sets of numbers, determine the mean, the median and the mode.
   (a) 8, 11, 14, 13, 14, 9, 15
   (b) 88, 93, 85, 98, 102, 98

2. The mean of a set of 6 numbers is 2 and the mean of another set of 10 numbers is x. If the mean of all 16 numbers is 7, find the value of x.

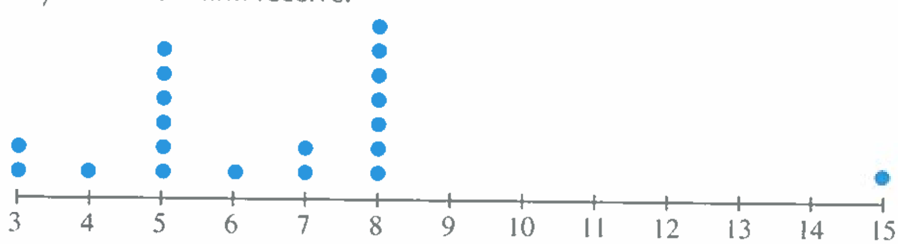
3. The numbers 24, 22, 34, 28, 29, 24, 25, 29, a and b have a median of 27 and a mode of 29. Given that $a < b$, state the value of a and of b.

4. 2000 people responded to the blood bank's appeal for blood donors. The blood type, i.e. A, B, AB or O, of each donor is recorded.

Blood type	A	B	AB	O
Number of donors	h	800	200	k

   (a) (i) Write down an equation in terms of h and k.
   (ii) Given that $9h = k$, find the value of h and of k.
   (b) Using your answers in (a)(ii),
   (i) draw a bar graph to illustrate the data.
   (ii) state the mode of the distribution.

5. The dot diagram represents the annual increment, in percentage points, 20 lawyers of a law firm receive.



   Find
   (i) the mean annual increment,
   (ii) the median annual increment,
   (iii) the modal annual increment,
   the lawyers receive, in percentage points.

6. The stem-and-leaf diagram represents the current, in Amperes (A), flowing through 25 electrical conductors.

Stem	Leaf
5	2
6	8 9 9
7	0 0 0 1 2 2 2 2 3 4 4 6 6 7 7 7
8	4 4 5
9	0 3

Key: 6 | 8 means 68 A

(a) Find
  (i) the mean current,
  (ii) the median current,
  (iii) the modal current,
  flowing through the electrical conductors.
(b) Briefly describe the distribution of the data.

7. A box contains five cards numbered 1, 2, 3, 4 and 5. A card is drawn from the box, its number noted and then replaced. The process is repeated 100 times. The table shows the resulting frequency distribution.

Card	1	2	3	4	5
Frequency	21	x	y	18	17

(a) **(i)** Show that $x + y = 44$.
  (ii) If the mean of the distribution is 2.9, show that $2x + 3y = 112$.
  (iii) Hence, find the value of x and of y.
(b) Using your answers in **(a)(iii)**, find
  (i) the median,
  (ii) the mode,
  of the distribution.

8. The number of songs downloaded each hour from an online music store is recorded.

Number of songs downloaded	10	15	20	25	30
Frequency	5	12	4	m	5

(a) If the mean number of songs downloaded is 20.25, find the value of m.
(b) If the median of the distribution is 20, find the smallest possible value of m.
(c) If the modal number of songs downloaded is 15, state the greatest possible value of m.

9. During a camp, each student was tasked to build the tallest possible structure with blocks. The table shows the heights, x cm, of the structures they built.

122	144	136	136	140	139	126	132	125	129
127	116	132	138	124	135	122	137	135	129
133	130	128	118	131	127	128	147	133	119

(a) Using the class intervals $110 \leqslant x < 120$, $120 \leqslant x < 130$ and so on, construct a frequency table for the data.

(b) Using the frequency table in (a),

(i) draw a histogram to illustrate the data,

(ii) calculate an estimate for the mean height of the structures,

(iii) find the percentage of structures which are shorter than 140 cm.

10. Three netball teams each played five games. The table shows the scores of the games played by the teams.

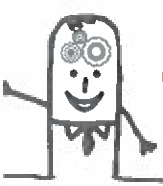
	Game 1	Game 2	Game 3	Game 4	Game 5
Cheetah	65	95	32	96	88
Jaguar	50	90	65	87	87
Puma	90	85	46	44	80

(a) Suppose you want to join the team that is doing the best so far.

(i) By considering the mean score of each team, which team would you join? Justify your answer by showing your working clearly.

(ii) If, instead of using mean scores, you decide to base your decision on the median score of each team, which team would you join? Justify your answer by showing your working clearly.

(b) Suppose you are the coach of Team Jaguar and you are being interviewed about your team for the local newspaper. Would it be better to report the mean score or the median score of your team? Explain your answer.



Challenge Yourself

1. Four whole numbers have a mean of $x + y + 5$, a median of $x + y$ and a mode of x. Find the numbers in terms of x and y.

2. Given that

$$a + \frac{1}{2}b = 13 - \frac{1}{2}e,$$

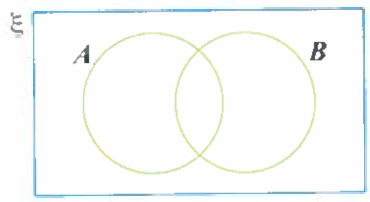
$$c + \frac{1}{2}e + f = 8 - \frac{1}{2}b - d,$$

find the mean of a, b, c, d, e and f.

3. The mean, the median, and the mode of the distribution of the heights of 9 boys from the school basketball team are all equal to 183 cm. Three of the boys have a height of 183 cm and the tallest boy has a height of 187 cm. Given that the heights of all the boys are integers, find the least possible height of the shortest boy.

D1 Revision Exercise

1. $\xi = \{x : x \text{ is an integer and } 20 < x < 35\}$
   $A = \{x : x \text{ is a multiple of } 4\}$
   $B = \{x : 3x - 6 < 80\}$
   (a) List the elements in the set $A' \cup B'$.
   (b) Insert the elements of the above sets in the Venn diagram below.



   (c) Shade the region which represents $A \cap B'$.

2. A two-digit number is chosen at random. Find the probability that the number is
   (i) an even number,
   (ii) a prime number less than 18,
   (iii) a multiple of 10,
   (iv) divisible by both 3 and 4.

3. All the hearts are removed from a standard pack of 52 playing cards. A card is drawn at random from the remaining cards. Find the probability of drawing
   (i) the ten of hearts,
   (ii) a red card,
   (iii) either a king or an ace,
   (iv) a card which is not a two.

4. The table shows the number of times the students in a class fell ill last year.

8	5	7	5	3	6	7
7	10	9	6	8	10	2
5	4	6	2	6	0	5
2	4	5	9	8	6	6
5	5	5	3	2	3	7

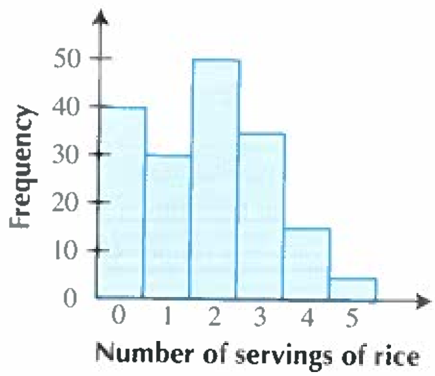
   (i) Represent the data on a dot diagram.
   (ii) What was the most common number of times the students in the class fell ill?
   (iii) Find the percentage of students who did not fall ill last year.

5. The table records the amount of time spent on playing video games daily and the test scores (marks) of some students.

Time spent on playing video games daily (hours)	Test scores (Marks)
0	85
1	80
2	72
3	50
4	42
5	34
6	22
7	15

(a) On a sheet of graph paper, using a scale of 2 cm to represent 1 hour on the horizontal axis and 2 cm to represent 10 marks on the vertical axis, plot the values given in the table above.
(b) Draw a line of best fit for the scatter diagram.
(c) Using the line of best fit drawn, estimate the results of a student who spends
(i) 1.5 hours, (ii) 7.5 hours,
playing video games daily.
(d) Find the equation of the line of best fit drawn, giving the gradient correct to 3 significant figures.
(e) Describe the correlation between playing video games and test scores.

6. The histogram shows the number of servings of rice the Secondary Two students in a school consume each day.



(i) Find the total number of Secondary Two students in the school.
(ii) Find the ratio of the number of students who consume 1 serving of rice each day to the number of students who consume 2 servings of rice each day.
(iii) A student is selected at random. Find the probability that the student consumes at least 3 servings of rice each day.

7. During a Science enrichment programme, a group of lower primary school students learnt how to make a temporary magnet. The number of paper clips each magnet was able to attract was recorded.

Number of paper clips	10	11	12	13	14	15	16
Number of magnets	6	10	20	22	14	6	2

Find
(i) the mean, (ii) the median, (iii) the mode,
of the distribution.

8. The lifespans, in days, of the bees in a colony are recorded.

Lifespan (x days)	$0 < x \leqslant 10$	$10 < x \leqslant 20$	$20 < x \leqslant 30$	$30 < x \leqslant 40$	$40 < x \leqslant 50$
Number of bees	6	11	22	29	32

(i) Calculate an estimate for the mean lifespan of the bees.
(ii) Find the fraction of bees whose lifespans are at most 40 days.

D2 Revision Exercise

1. It is given that

   $\xi = \{x : x \text{ is a positive integer and } 10 < 5x < 48\}$,

   $A = \{x : x \text{ is a multiple of } 4\}$

   and $B = \{x : x \text{ is divisible by } 3\}$.

   (i) Draw a Venn diagram to illustrate this information.

   (ii) Write down $n(A \cap B)$.

   (iii) List the elements contained in the set $A \cup B$.

2. 28 students from the Ecology Club are planning to visit the Marine Life Park during the December holidays. Of the 16 girls, 2 of them have visited the park. 6 boys have not visited the park. If a student is chosen at random, find the probability that the student

   (i) is a boy who has visited the park,

   (ii) has not visited the park.

3. There are 48 boys and 2 girls on the country's wushu team. After a sports fair, x boys and $2x$ girls join the team. Given that the probability of selecting a boy at random to represent the country for a competition is now $\frac{2}{5}$, find the number of girls who join the team.

4. The times, in seconds, for 30 students to complete a 40-metre shuttle run are recorded in the table.

11.8	10.5	10.9	14.0	10.8	11.2	11.2	10.8	10.7	11.6
10.7	11.3	10.6	11.2	10.9	11.4	11.5	10.8	12.3	11.9
12.2	11.8	10.4	11.2	10.3	12.6	10.9	10.7	10.8	11.1

   (i) Construct a stem-and-leaf diagram with split stems.

   (ii) To obtain a grade A for the shuttle run, a student must take less than 11.5 seconds to complete the run. Find the percentage of students who scored an A for the run.

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5. A survey was conducted to find out the number of times a group of Singaporeans withdrew money from automated teller machines (ATMs) last week. The data collected is given in the table.

Number of times (x)	Number of people
$0 \leq x < 2$	14
$2 \leq x < 4$	30
$4 \leq x < 6$	15
$6 \leq x < 8$	12
$8 \leq x < 10$	9

(i) On a sheet of graph paper, draw a histogram to illustrate the data.

(ii) $\frac{33}{40}$ of the people withdrew money from ATMs at least p times last week. Find the value of p.

6. The mean of a, b, c and d is 8 and the mean of a, b, c, d, e and f is 7.6. Find the mean of e and f.

7. The marks obtained by 36 students in a mathematics quiz are recorded.

Marks obtained	3	4	5	6	7	8	9
Number of students	3	4	x	7	y	5	4

(a) If the mean of the distribution is 6, find the value of x and of y.

(b) Using your answers in **(a)**, find

(i) the median,

(ii) the mode,

of the distribution.

PROBLEM 1: Swimming Competition

Three students, Kate, Lixin and Nora, have been shortlisted to represent their school in a 100 m freestyle swimming competition. Each student is given three attempts and their timings (in minutes and seconds) are recorded in the table.

Student	Kate	Lixin	Nora
1st attempt	2:21	1:58	1:59
2nd attempt	1:50	2:11	2:01
3rd attempt	2:31	2:20	2:00

Based on the information given, which student should be selected to participate in the swimming competition? Show how you arrive at your decision and justify your solution.

PROBLEM 2: Louvre Pyramid

The Louvre Museum in Paris, France, is one of the largest museums in the world. The Louvre Pyramid, which serves as the entrance to the museum, was completed in 1989. It reaches a height of 20.6 m and has a square base of length 35 m. The apex of the pyramid is vertically above the centre of the square base.

The pyramid is constructed using 603 rhombus-shaped and 70 triangular glass panels. The photograph on the left shows the side of the pyramid with the entrance. The pyramid has only one entrance and it does not contain any glass panel. The photograph on the right shows one side of the pyramid without the entrance, which contains 153 rhombus-shaped glass panels and 18 triangular glass panels at the bottom. Find

(i) the number of rhombus-shaped and triangular glass panels on the side of the pyramid with the entrance,

(ii) the volume of the pyramid,

(iii) the total area of the glass panels on each side of the pyramid without the entrance.

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PROBLEM 3: Lucky Draw Scams

The table shows the number of lucky draw scams in Singapore from 2010 to 2012. In such a scam, the victim is informed by an anonymous caller that he or she has won an overseas lucky draw, but in order to claim the prize, he or she has to first pay "processing fees" or "taxes". Once the victim transfers money to the caller, the caller uses other excuses to cheat the victim of more money.

Year	2010	2011	2012
Number of Reported Cases	346	298	327
Number of Cases where Victims were Cheated	175	183	181
Total Amount of Money Cheated Per Case	\$3.8 million	\$6.4 million	\$7.4 million

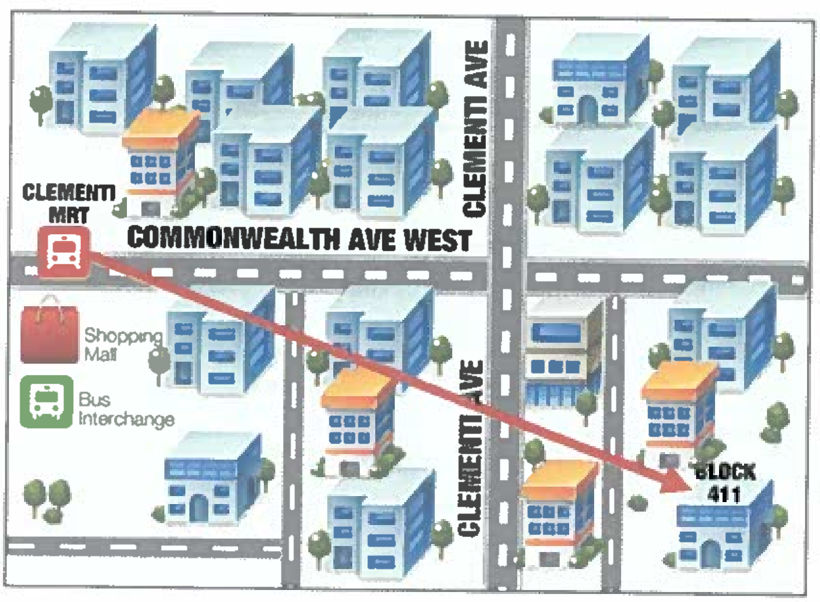
(a) For each of the given years, find the percentage of the reported cases in which the victims were cheated.

(b) Estimate the mean amount of money cheated per case for each of the given years, giving your answer to the nearest thousand.

(c) Should lucky draw scams be a concern for the police? Justify your answer based on the data given and computed.

PROBLEM 4: How Long Do I Need to Get Home?

Vishal lives at Block 411 Commonwealth Avenue West. He claims that he only needs 5 minutes to walk from the Clementi Mass Rapid Transit (MRT) Station to his block if he takes the route as indicated by the arrow in the map. Verify if his claim is true.



PROBLEM 5: Conical Wine Glass

The photograph shows a conical glass.

A problem commonly faced by bartenders is when their customers request for only half a glass of cocktail.

Bartender Ethan: "I always fill the cocktail to half the depth of the glass."

Bartender Jun Wei: "I disagree. I always fill the cocktail to two-thirds of the depth of the glass."

Bartender Michael: "I believe that the cocktail should be poured to three-quarters of the depth of the glass."

Bartender Rui Feng: "No, I think that the cocktail should be poured to four-fifths of the depth of the glass."

(i) Which of the bartenders' method is obviously wrong? Explain your answer.

(ii) Make a guess whose method is the closest to getting half a glass of cocktail.

(iii) Formulate the problem mathematically to determine which of the bartenders' method is the most accurate.

PROBLEM 6: Braking Distance

It is important for a driver to know how far it takes for his or her vehicle to come to a stop after applying the brakes. The table shows the braking distance of a car at different speeds just before the brakes are applied.

Speed (km/h)	0	10	20	30	40	50	60	70	80	90	100
Braking distance (m)	0	1	2	5	8	12	17	24	31	38	45

(a) If the speed of the car just before the brakes are applied doubles, does the braking distance always double? Justify your answer by using at least two sets of values from the table.

Open a spreadsheet and type the headers and values given in the table. Select the entire table and insert a scatterplot with only the markers (or points). Right click on the points and add a trendline to model or best fit the data: choose a linear function before choosing a polynomial of degree 2, i.e. a quadratic function. Set the intercept at 0, i.e. the trendline must pass through the origin, and choose to display the equation of the trendline on the scatterplot.

(b) Use both trendlines to estimate the braking distance of the car when its speed just before the brakes are applied is

(i) 45 km/h,

(ii) 5 km/h.

Alternatively, you may wish to use the equations of the trendlines to find the braking distance of the car at each of the speeds just before the brakes are applied.

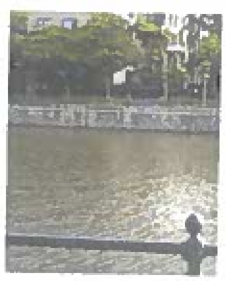
(c) Which trendline provides a better model of the braking distance of the car? By looking at both trendlines and your observation in **(a)**, explain your answer.

The two-second rule is a guide to help drivers keep a safe distance from the vehicle in front. When the vehicle in front passes a landmark, such as a lamppost, the driver will start to count 2 seconds by saying 'One thousand, two thousand'. If the driver drives past the landmark before the 2 seconds are up, he is considered to be too close to the vehicle in front.

(d) Find the distance travelled in 2 seconds for each of the speeds just before the brakes are applied. Will the car be able to come to a stop in time if the driver follows the two-second rule, regardless of the speed it is travelling at just before the brakes are applied?

PROBLEM 7: Singapore River

The photograph shows part of the Singapore River from Robertson Quay. Suppose you are an engineer tasked to build a bridge across this part of the river. Your first job is to determine the width of this stretch of the river. (Your teacher may choose a suitable river or canal near your school for you to measure its width.)



Guiding Questions

(a) Are there any assumptions that you may need to make?

(b) How do you represent the problem mathematically by drawing a model of the method of solution using similar triangles? Solve the problem to estimate the width of this stretch of the river.

(c) Is your mathematical solution the same as the solution for the real-world problem? Explain your answer.

(d) Are there alternative methods of solution? Discuss some methods of solution which may not be feasible.

Practise Now Answers

Chapter 1

Practise Now 1

(a) \$15.95
(b) 8 kg

Practise Now 2

1. **(i)** $y = 5x$
(ii) 50
(iii) 12
2. 17.5
3. 8, 9.5; 24, 42

Practise Now 3

(i) $C = \frac{5}{3}d$
(ii) \$75
(iii) 72 km

Practise Now 4

(i) \$13 200
(ii) 380
(iii) $C = 41n + 5000$
(iv) No

Practise Now 5

(a) y and x^2
(b) $\sqrt{y}$ and x^3

Practise Now 6

1. **(i)** $y = 2x^2$
(ii) 50
(iii) ±4
2. 84
3. 2.5, 7; 36, 225

Practise Now 7

(i) $l = 24.5T^2$
(ii) 612.5 cm
(iii) 2 s

Practise Now 8

40 minutes

Practise Now 9

(a) 10.5 hours
(b) 21 minutes

Practise Now 10

1. **(i)** 1.25
(ii) $y = \frac{10}{x}$
(iii) 1
2. 6
3. 1, 5; 8, $1\frac{1}{3}$

Practise Now 11

(i) 2 A
(ii) 2 Ω

Practise Now 12

(a) y and x^2
(b) y^2 and $\sqrt[3]{x}$
(c) y and $x + 2$

Practise Now 13

1. **(i)** $\frac{1}{2}$
(ii) $y = \frac{32}{x^2}$
(iii) ±2
2. 3.6
3. 0.25, 36; 4, 2

Practise Now 14

(i) 1.6 N
(ii) 1.26 m

Chapter 2

Practise Now 1

(a) 3
(b) $-\frac{1}{2}$
(c) $1\frac{1}{2}$
(d) −6

Practise Now 2

(a) 20 minutes
(b) 9 km
(c) **(i)** $\frac{9}{10}$
(ii) 0
(iii) $-\frac{4}{5}$
(iv) 0
(v) $-\frac{5}{7}$

Practise Now (Page 56)

(a) $y = 1, y = -3.5$

Practise Now (Page 58)

(a) $x = 4, x = -1.2$

Practise Now 3

(a) 7
(c) 4
(d) **(ii)** 0.5

Practise Now 4

1. $x = 1, y = 2$
2. $x = -1, y = 2$

Practise Now 5

1. **(a)** $x = 4, y = 1$
(b) $x = 3, y = -1$
(c) $x = 1, y = -1$
(d) $x = 3, y = -1$
2. $x = -3, y = 5$

Practise Now 6

(a) $x = 3, y = 4$
(b) $x = 3, y = -1$

Practise Now 7

(a) $x = 1, y = -2$
(b) $x = 1, y = -3$

Practise Now 8

$x = 10, y = 3$

Practise Now 9

$x = -1, y = 2$

Practise Now 10

$x = 2, y = -1$

Practise Now 11

(a) $x = 9, y = 15$
(b) $x = -3, y = 6$

Practise Now 12

1. 13.5, 22.5
2. 34°, 146°
3. 30 cm

Practise Now 13

$\frac{7}{9}$

Practise Now 14

1. 10, 40
2. \$76

Practise Now 15

65

Chapter 3

Practise Now (Page 93)

(a) $-2x^2$
(b) $-6x^2$
(c) $2x^2$
(d) $6x^2$
(e) $x^2 - 2$
(f) $7x^2 - 4x - 2$

Practise Now (Page 94)

(a) $-2x^2 - x - 1$
(b) $2x^2 + x - 1$
(c) $-2x^2 - 3x + 3$
(d) $3x^2 - 7x + 7$

Practise Now (Page 96)

(a) $-4x^2 + 2x - 2$
(b) $3x^2 - 6x + 9$
(c) $7x^2 - 3x - 13$
(d) $x^2 + 8x - 16$

Practise Now 1

(a) $13x^2 - 5x$
(b) $11x^2 - 13x + 2$
(c) $-3x^2 - 13x + 1$
(d) $-2x^2 + x + 13$

Practise Now (Page 98)

(a) $6x + 3$
(b) $-6x + 3$
(c) $-2x^2 + 3x$
(d) $-2x^2 + 6x$

Practise Now 2

(a) $12x + 3$
(b) $35x - 14$
(c) $10x^2 - 15x$
(d) $-16x^2 + 6x$

Practise Now 3

(a) $-x - 32$
(b) $9x^2 + 4x$

Practise Now 4

(a) $x^2 + 6x + 8$
(b) $15x^2 - 38x + 24$
(c) $10 - 13x - 3x^2$
(d) $-77x^2 + 39x - 4$

Practise Now 5

$-3x^2 + 29x - 5$

Practise Now (Page 107)

(a) $(x + 1)(x + 5)$
(b) $(x - 1)(x - 5)$
(c) $(x + 2)(x + 6)$
(d) $(x - 2)(x - 6)$

Practise Now 6

(a) $(x + 1)(x + 7)$
(b) $(x - 4)(x - 7)$
(c) $(x - 1)(x + 2)$
(d) $(x + 1)(x - 8)$

Practise Now 7

(a) $(2x + 3)(x + 4)$
(b) $(5x - 3)(x - 2)$
(c) $(-2x + 3)(x - 3)$
(d) $3(3x - 8)(x - 1)$

Chapter 4

Practise Now 1

1. (a) $30xy$
   (b) $-16xy$
   (c) $x^2y^3z^2$
   (d) $11x^4y^3$
2. $-\frac{4}{3}ab$

Practise Now 2

(a) $-5y + 2xy$
(b) $14x^2 + 6xy$

Practise Now 3

(a) $2xy - 5xz$
(b) $-7x^2 + 2xy$

Practise Now 4

(a) $2x^2 + 17xy - 9y^2$
(b) $6x^3 + 7x^2 - 18x - 21$

Practise Now 5

$5x^2 - 10xy + 3y^2$

Practise Now 6

(a) $x^2 - xy - x - 20y^2 + 5y$
(b) $x^3 - 4x^2 - 23x - 6$

Practise Now 7

1. (a) $(x + 3y)(x - 5y)$
   (b) $(6x + 5y)(x + y)$
2. $(3xy - 8)(xy - 2)$

Practise Now 8

1. (a) $x^2 + 4x + 4$
   (b) $25x^2 + 30x + 9$
2. $\frac{1}{4}x^2 + 3x + 9$

Practise Now 9

1. (a) $1 - 6x + 9x^2$
   (b) $4x^2 - 12xy + 9y^2$
2. $x^2 - \frac{2}{3}xy + \frac{1}{9}y^2$

Practise Now 10

1. (a) $25x^2 - 64$
   (b) $4x^2 - 49y^2$
2. $\frac{1}{16}x^2 - y^2$

Practise Now 11

(a) 1 002 001
(b) 635 209
(c) 89 975

Practise Now 12

533

Practise Now 13

1. (a) $(x + 6)^2$
   (b) $(2x + 5)^2$
2. $\left(2x + \frac{1}{2}\right)^2$

Practise Now 14

1. (a) $(2 - 9x)^2$
   (b) $(5x - y)^2$
2. $\left(6x - \frac{1}{3}y\right)^2$

Practise Now 15

1. (a) $(6x + 11y)(6x - 11y)$
   (b) $(9 + 2x)(9 - 2x)$
2. $\left(2x + \frac{3}{5}y\right)\left(2x - \frac{3}{5}y\right)$
3. $(2x + 9)(2x - 5)$

Practise Now 16

41 200

Practise Now 17

(a) $4x(2xy + 1)$
(b) $\pi r(r + l)$
(c) $a^2y(-ab + 1)$
(d) $3c^2(d + 2d^3 + c)$

Practise Now 18

(a) $(x + 1)(2 + a)$
(b) $(x + 2)(9 - b)$
(c) $3(2x - 3)(c - 2d)$
(d) $(4 - x)(7h + 1)$

Practise Now 19

(a) $(y + 4)(x + 3)$
(b) $(x + 3y)(4a + b)$
(c) $(x - 1)(x^2 + 1)$
(d) $(3y - 2)(2x + z)$

Chapter 5

Practise Now 1

(a) 0 or -2
(b) 0 or 1
(c) -5 or 7
(d) $-\frac{2}{3}$ or $1\frac{1}{4}$

Practise Now 2

1. (a) 0 or 5
   (b) −4
   (c) −1 or −4
   (d) $\frac{2}{3}$ or 5
2. (i) $1\frac{1}{3}$ or 2
   (ii) $\frac{1}{3}$ or 1
3. (i) 6
   (ii) −4

Practise Now 3

(a) 2 or −3
(b) $\frac{1}{3}$ or $\frac{2}{3}$

Practise Now 4

1. 8, 10
2. 4, 9

Practise Now 5

6 cm, 4 cm

Practise Now 6

1. (i) 576 m
   (ii) 24 s
2. (i) 18 m
   (ii) 3 s

Practise Now 7

$A\left(-1\frac{1}{2}, 0\right)$, $B(2, 0)$, $C(0, -6)$

Practise Now 8

(a) 5
(c) (i) 3.5
    (ii) 0.4 or 3.6
    (iii) 3, 2

Practise Now 9

(b) (i) ±1.5
    (ii) 5
(c) $x = 0$

Practise Now 10

1. (a) 28 m
   (c) (i) 65 m, 1.75 s
       (ii) 4.05 s
2. (b) (i) 17.25 m
       (ii) 3.85 s

Chapter 6

Practise Now 1

(a) $\frac{2x^2}{3y^3}$
(b) $\frac{x^2(x-y)^2}{3y^3}$

Practise Now 2

(a) $\frac{h + 7k}{5k}$
(b) $\frac{3}{2p - 1}$
(c) $\frac{z}{4}$

Practise Now 3

1. (a) $\frac{3v}{v + 3}$
   (b) $\frac{p - 3q}{5p}$
   (c) $\frac{x + 2z}{y - 2z}$
2. $\frac{n^2 - 2}{n^2 + 3}$

Practise Now 4

1. (a) $\frac{3}{4a^2c^2}$
   (b) $\frac{7p^4}{10r^5}$
   (c) $\frac{14(x - 3)}{15}$
   (d) $\frac{h - 3}{h}$
2. $-\frac{1}{2}$

Practise Now 5

1. (a) $\frac{63}{40a}$
   (b) $\frac{5}{3(2b + 3c)}$
   (c) $\frac{4h}{2 - 3k}$
2. (a) $\frac{6n^2 + 7mn - 3m^2}{6mn}$
   (b) $\frac{8q - 11p}{12(p - q)}$
   (c) $\frac{10x + 7y}{2(4x - 3y)}$

Practise Now 6

(a) $\frac{x - 13}{(x + 1)(2x - 5)}$
(b) $\frac{2 - y^2 - 3y}{(y + 3)(y - 3)}$
(c) $\frac{2}{z + 5}$

Practise Now 7

1. (i) $a = \frac{v - u}{t}$
   (ii) 10
2. (i) $T = \frac{100I}{PR}$
   (ii) 4 years

Practise Now 8

1. (i) $x = \frac{7y + 5}{3y - 2}$
   (ii) $1\frac{5}{11}$
2. (i) $k = \frac{bx^2}{3(p - a)}$
   (ii) −27

Practise Now 9

1. (i) $x = \frac{b^2 - 9y^2}{4a}$
   (ii) 1
2. (i) $x = \pm\sqrt{\frac{3k(p - a)}{b}}$
   (ii) ±3

Practise Now 10

1. (a) 2
   (b) $2\frac{3}{5}$
2. 13
3. $4\frac{4}{13}$

Practise Now 11

(a) $5\frac{9}{13}$
(b) −27

Chapter 7

Practise Now 1

(a) Yes
(b) No

Practise Now 2

(i) 44
(ii) −66
(iii) $-2\frac{2}{3}$
(iv) 2
(v) 36
(vi) $-2\frac{1}{2}$
(vii) 7
(viii) $-1\frac{2}{3}$
(ix) 3

Practise Now 3

(i) $2b - 5$
(ii) $7b + 5$
(iii) $18b - 28$

Chapter 8

Practise Now 1

$A, H; B, E; C, F; D, G, I$

Practise Now 2

(i) 5
(ii) DC
(iii) AD, 2
(iv) BC, 5.3
(v) $\angle ABC$, 90

Practise Now 3

(a) Not congruent
(b) $\triangle DEF \equiv \triangle TSU$
(c) Not congruent

Practise Now 4

(a) (i) 38°
(ii) 28°
(iii) 28°
(iv) 27 cm
(v) 9 cm
(b) AC and ED are parallel lines.

Practise Now 5

(a) No
(b) Yes

Practise Now 6

1. $x = 30, y = 4.2$
2. $w = 60, x = 100, y = 7.2, z = 4.5$

Practise Now 7

5 m

Practise Now 8

1. $a = 30, b = 7.5$
2. $x = 72, y = 9.125$

Practise Now 9

1. 18 cm, 30 cm
2. 7.5 cm, 8 cm
3. 4.5 m

Practise Now 10

(a) 1 : 150
(b) 1050 cm

Practise Now 12

(ii) 1 cm to 8 km
(v) (63 ± 0.5) km

Practise Now 13

1. (i) 3.125 m
(ii) 1.36 cm
(i) 268 m
(ii) 26.8 cm
(ii) 58.5 m

Practise Now 14

1. (i) 32.5 km
(ii) 5 cm
(iii) $\frac{1}{500\ 000}$
2. (i) 1 km
(ii) 29 cm

Practise Now 15

1. (i) 12 km^2
(ii) 4.5 cm^2
2. 126 km^2

Chapter 9

Practise Now 1

$A'(-3, -1), B'(-1, -5), A''(7, -1), B''(5, -5)$

Practise Now 2

(a) $3y + 4x = -12$
(b) $3y + 4x = 12$
(c) $3y + 4x = 36$

Practise Now 3

(a) $L(-1, 6), M(0, 9), N(1, 6)$
(b) (4, 3), 90° clockwise

Practise Now 4

$A''(8, 2), B''(9, 5), C''(11, 5)$ and $D''(12, 2)$

Practise Now 5

(a) $\begin{pmatrix} 6 \\ 9 \end{pmatrix}$

(b) $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$

(c) $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$

Chapter 10

Practise Now (Page 259)

(a) AB
(b) DE
(c) PQ

Practise Now 1

1. 10 cm
2. 25 cm

Practise Now 2

1. 9 m
2. 21 m

Practise Now 3

1. (i) 4 cm
(ii) 8.54 cm
2. (i) 60 cm
(ii) 40.5 cm
3. (i) 13.9 cm
(ii) 13.4 cm

Practise Now 4

1. 27.8 m
2. 2 m

Practise Now 5

11.6 m

Practise Now 6

30

Practise Now 7

(i) 20.8 km
(ii) 36.6 km

Practise Now 8

1. (a) No
(b) Yes; $\angle R$
2. (ii) 46.1 m

Chapter 11

Practise Now (Page 285)

1. (i) AB
(ii) BC
(iii) AC
2. (i) $\frac{3}{5}$
(ii) $\frac{4}{5}$
(iii) $\frac{3}{4}$
(iv) $\frac{4}{5}$
(v) $\frac{3}{5}$
(vi) $\frac{4}{3}$
3. (i) $\frac{a}{c}$
(ii) $\frac{b}{c}$
(iii) $\frac{a}{b}$
(iv) $\frac{b}{c}$
(v) $\frac{a}{c}$
(vi) $\frac{b}{a}$

Practise Now 1

(a) 0.914
(b) 3.63
(c) 0.952
(d) 3.78
(e) 15.1
(f) 0.360

Practise Now 2

(a) $x = 5.5$
(b) $y = 9.45$

Practise Now 3
1. 9.25 m
2. 12.3 m

Practise Now 4
1. 26.5 cm
2. 17.1 cm

Practise Now 5
1. (i) 20.5 m
   (ii) 10.4 m
2. (i) 6.39 m
   (ii) 6.62 m
   (iii) 12.4 m
   (iv) 8.28 m
   (v) 33.7 m
   (vi) 59.3 m^2

Practise Now (Page 297)
(a) 51.3°
(b) 69.5°
(c) 50.9°

Practise Now 6
(a) $x = 35.4$
(b) $y = 64.3$
(c) $z = 57.2$

Practise Now 7
1. (i) 23.3°
   (ii) 8.29 m
2. (i) 36.9°
   (ii) 3.75 cm

Practise Now 8
1. 38.5 m
2. 53.1 m
3. 20.0 m

Practise Now 9
28.4°

Practise Now 10
75.6 m

Practise Now 11
1.88 m

Chapter 12

Practise Now 1
1. 84 cm^3
2. 2 550 000 m^3

Practise Now 2
9 m

Practise Now 3
504 m^2

Practise Now 4
(i) 8 cm
(ii) 117 cm^3

Practise Now 5
1. 1140 cm^3
2. 7 m

Practise Now 6
6600 cm^3

Practise Now 7
1. 396 cm^2
2. 5.92 m

Practise Now 8
1. 427 m^2
2. 500 cm^3

Practise Now 9
1. 1890 g
2. 11.0 cm

Practise Now 10
1960 cm^2

Practise Now 11
5.64 cm

Practise Now 12
50 l

Practise Now 13
(a) (i) 15 300 cm^3
    (ii) 3210 cm^2
(b) (i) 31.2 cm
    (ii) 1092.5π cm^2

Chapter 13

Practise Now 2
(a) $x = 4$
(b) $y = 6$

Practise Now 4
(i) Infinite
(ii) 1
(iii) Infinite

Chapter 14

Practise Now (Page 393)
1. (a) {2, 4, 6, 8}
   (b) (i) True (ii) True
       (iii) False (iv) True
   (c) (i) $2 \in A$ (ii) $5 \notin A$
       (iii) $9 \notin A$ (iv) $6 \in A$
2. 10

Practise Now 1
(i) $C = \{11, 12, 13, 14, 15, 16, 17\}$
    $D = \{10, 11, 12, 13, 14, 15, 16, 17\}$
(ii) No

Practise Now 2
(a) No
(b) Yes
(c) Yes

Practise Now 3
(i) $P = \{\ \}$ (ii) No

Practise Now 4
(i) $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$
    $B = \{2, 3, 5, 7, 11, 13\}$
(iii) $B' = \{1, 4, 6, 8, 9, 10, 12\}$

Practise Now 5
1. (ii) Yes
2. (i) $P = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$
       $Q = \{2, 3, 5, 7, 11\}$
   (ii) $Q \subset P$
   (iii) $R = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$
   (iv) $R = P$

Practise Now (Page 403)
(i) (a) {7}, {8}, {7, 8}
    (b) {7}, {8}
(ii) (a) {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}
     (b) { }, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}

Practise Now 6
1. (i) $C = \{6, 12, 18\}$
       $D = \{3, 6, 9, 12, 15, 18\}$
   (ii) {6, 12, 18}
   (iv) Yes
2. (i) $E = \{1, 2, 3, 4, 6, 12\}$
       $F = \{5, 7, 11, 13\}$
   (ii) { }

Practise Now 7

1. (i) $C = \{1, 2, 4, 8\}$
   $D = \{1, 2, 4, 8, 16\}$
   (iii) {1, 2, 4, 8, 16}
   (iv) Yes
2. (i) $E = \{7, 14, 21, 28, 35, 42, 49, 56\}$
   $F = \{9, 18, 27, 36, 45, 54\}$
   (iii) {7, 9, 14, 18, 21, 27, 28, 35, 36, 42, 45, 49, 54, 56}

Practise Now 8

1. (a) $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
   $A = \{1, 3, 5, 7, 9\}$
   $B = \{3, 6, 9\}$
   (c) (i) {2, 4, 8}
   (ii) {1, 5, 7}

Chapter 15

Practise Now 1

Red, Orange, Purple, Yellow, Green; 5

Practise Now 2

(a) $B_1, B_2, B_3, B_4, B_5, R_1, R_2, R_3, R_4$; 9
(b) $N_1, A_1, T, I, O, N_2, A_2, L$; 8
(c) 357, 358, 359, ..., 389; 33

Practise Now 3

(i) $\frac{1}{15}$
(ii) $\frac{7}{15}$
(iii) $\frac{2}{3}$
(iv) 0

Practise Now 4

(i) $\frac{1}{2}$
(ii) $\frac{1}{13}$
(iii) $\frac{1}{52}$
(iv) $\frac{51}{52}$

Practise Now 5

. (i) $\frac{1}{8}$
(ii) $\frac{3}{4}$
(iii) $\frac{1}{4}$
. (i) $\frac{1}{6}$
(ii) $\frac{7}{12}$
(iii) 0
(iv) 1
12

Practise Now 6

(i) $\frac{2}{5}$
(ii) $\frac{7}{10}$

Practise Now 7

(i) $\frac{3}{8}$
(ii) $\frac{1}{6}$
(iii) 0
(iv) $\frac{5}{24}$

Practise Now 8

1. (i) $14 + x$
   (ii) $\frac{x + 2}{14 + x}$
   (iii) 6
2. 4

Chapter 16

Practise Now 1

(ii) 14

Practise Now 2

(i) 11
(ii) 20
(iii) 20

Practise Now 4

(ii) 90 kg
(iii) 25%

Practise Now 5

(ii) Michael
(iii) Khairul
(iv) Michael

Practise Now 6

(ii) Strong, positive correlation
(iv) 12.8 litres per 100 km
(v)
(vi) No

Practise Now 7

(ii) Strong, negative correlation
(iv) 12.7 s
(v)
(vi) No

Practise Now 8

(iii) 3
(iv) $\frac{19}{30}$

Practise Now 11

(a) 45

Chapter 17

Practise Now 1

70.5

Practise Now 2

56

Practise Now 3

1. (i) 77
   (ii) 9
2. 168 cm
3. 12

Practise Now 4

(i) 200
(ii) \$19 600
(iii) \$98

Practise Now 5

1. 1.4
2. 3.245
3. 35 minutes

Practise Now 6

47.3 years

Practise Now 7

(a) 15
(b) 17.3

Practise Now 8

(a) 17.5
(b) 8.4

Practise Now 9

1. 1
2. 3

Practise Now 10

4.55 km

Practise Now 11

7 minutes

Practise Now 12

(i) 60 cm, 110 cm
(ii) 60 cm

Practise Now 13

(a) 0.4
(b) 32, 37
(c) 1
(d) \$3000

Practise Now 14

(a) 5
(b) 1, 2, 3, 4, 5
(c) 3

Answers

Exercise 1A

1. (i) $41\frac{2}{3}$ kg
   (ii) 72
2. (i) 1.25 m
   (ii) 32
3. (i) $x = 1.5y$
   (ii) 9
   (iii) 8
4. (i) $Q = 7P$
   (ii) 35
   (iii) 6
5. (a) \$60
   (b) $\$\frac{ac}{b}$
6. $3\frac{3}{5}$ kg
7. 4.5
8. 4
9. (a) 36, 44; 1.5
   (b) 8, 9.5; 2.4, 6.6
10. (i) $y = 4x$
11. (i) $z = 8y$
12. (i) $F = 9.8m$
    (ii) 137.2
    (iii) 22
13. (i) $P = 2.5T$
    (ii) 60
    (iii) 4.8
14. (i) $V = 1.5R$
    (ii) 22.5
    (iii) 10
15. (i) \$1360
    (ii) 135
    (iii) $D = 8n + 600$
    (iv) No
16. 95 tonnes

Exercise 1B

1. (i) $x = 4y^3$
   (ii) 864
   (iii) 3
2. (i) $z^2 = 2w$
   (ii) ±6
   (iii) 12.5
3. (i) 2
   (ii) 3
4. (a) y and x^2
   (b) y and $\sqrt{x}$
   (c) y^2 and x^3
   (d) p^3 and q^2
5. ±27
6. −3 or 5
7. 5, 7; 81, 192
8. 0.5, 1.8; 0.016, 0.686

9. (i) $L = 2.5\sqrt{N}$
   (ii) 5 cm
   (iii) 36 hours
10. 16
11. $4a$
12. 800%

Exercise 1C

1. (b), (c) and (e)
2. 16 days
3. (i) 8
   (ii) $x = \frac{200}{y}$
   (iii) 0.5
4. (i) $Q = \frac{1}{2P}$
   (ii) 0.1
   (iii) 2.5
5. 5
6. (i) 840
   (ii) 40 days
7. 2 days
8. 0.5
9. 5
10. (a) 0.5, 8; 6, 4.8
    (b) 4.5, 14.4; 12, 1.44
11. (i) 600 kHz
    (ii) 375 m
12. (i) $t = \frac{24}{N}$
    (ii) 4 hours
    (iii) 32
13. 36 minutes
14. 11
15. $4\frac{14}{19}$ minutes
16. 70

Exercise 1D

1. (i) 6.25
   (ii) $x = \frac{400}{y^3}$
   (iii) 5
2. (i) $z = \frac{27}{\sqrt{w}}$
   (ii) 6.75
   (iii) 81
3. (a) y and x^2
   (b) y and $\sqrt{x}$
   (c) y^2 and x^3
   (d) n and $m - 1$
   (e) q and $(p + 1)^2$
4. $3\frac{1}{3}$
5. ±2.5
6. 10, 20; 10, 1.25
7. (i) $F = \frac{k}{d^2}$, where k is a constant
   (ii) 80 N
8. (i) 20 cm
   (ii) 12 cm
9. 6
10. $\frac{1}{9}b$
11. 0.04

Review Exercise 1

1. (i) $y = 3x$
   (ii) 33
   (iii) 4
2. (i) $\frac{2}{3}$
   (ii) $3\frac{3}{4}$
3. (i) $y = 4x^3$
   (ii) 1372
   (iii) 10
4. (i) 13.5
   (ii) ±11
5. (i) 5
   (ii) 8
6. (i) $y = \frac{12}{x}$
   (ii) 2
   (iii) 0.5
7. (i) $q = \frac{75}{p^2}$
   (ii) 0.75
   (iii) −15
8. (i) 2
   (ii) 7
9. 1, 1.25; 37.5, 6
10. (i) \$35.61
    (ii) 125 minutes
    (iii) $C = 0.086n + 9.81$; $C - 9.81$ and n
11. (i) $G = 55h$
    (ii) 1210 J
    (iii) 55 m
12. \$45, \$80
13. (i) 200 Pa
    (ii) 8000 dm^3
14. 7

Challenge Yourself – Chapter 1

2. (i) $z = \frac{kx^2}{\sqrt{y}}$
   (ii) 150
3. (i) $T = \frac{120B}{P}$
   (ii) 30 days
   (iii) 50

Exercise 2A

1. 0, undefined
2. (a) 1, 0
   (b) 1, −1
   (c) −2, 8
   (d) −3, −3
   (e) $1\frac{1}{2}$, 3
   (f) $-1\frac{1}{2}$, 3
   (g) $-1\frac{1}{3}$, 4
   (h) $\frac{4}{5}$, 4
   (i) $1\frac{1}{2}$, −6
   (j) $-\frac{1}{2}$, 4
3. −1, 0, 1, $-2\frac{1}{2}$, undefined
4. 0, −3, undefined, $\frac{1}{2}$

Exercise 2B

1. (a) 1000 hours
   (b) 50 km
   (c) (i) 50
   (ii) 0
   (iii) 60
2. (a) 40 km
   (b) $1\frac{1}{2}$ hours
   (c) (i) 20
   (ii) $-13\frac{1}{3}$
   (iii) −20

Exercise 2C

1. (a) $y = 6, y = -2$
2. (a) $x = 0.5, x = -2$
3. (a) $p = -\frac{1}{2}, q = 4\frac{1}{2}$
   (c) −3
   (d) (ii) $3\frac{1}{2}$
4. (a) −5, −3, 1
   (c) (ii) $1\frac{1}{4}$ units^2

Exercise 2D

1. (a) $x = -1, y = -3$
   (b) $x = -5, y = -2$
   (c) $x = 3, y = 1$
   (d) $x = 0, y = 2$
   (e) $x = 5, y = 3$
   (f) $x = 3, y = -4$
2. (a) $x = 4, y = 2$
   (b) $x = 1, y = -1$
   (c) $x = 2.6, y = -2.6$
   (d) $x = -1.5, y = -0.5$
3. (a) (i) −7, 9, 17
   (b) (i) 0, 2, 3
   (c) $x = -4, y = 1$
4. (a) Infinite number of solutions
   (b) No solution
   (c) Infinite number of solutions
   (d) No solution
5. (a) No solution
   (b) Infinite number of solutions

Exercise 2E

1. (a) $x = 8, y = 8$
   (b) $x = 12, y = 7$
   (c) $x = 1, y = \frac{1}{4}$
   (d) $x = 3, y = 2$
   (e) $x = 1, y = 2$
   (f) $x = 1, y = 1$
   (g) $x = 1\frac{1}{2}, y = -1\frac{1}{2}$
   (h) $x = -1\frac{1}{2}, y = 2$
   (i) $a = -7, b = -13$
   (j) $c = 2, d = \frac{1}{2}$
   (k) $f = 1, h = -\frac{1}{2}$
   (l) $j = 3\frac{1}{3}, k = -3$
2. (a) $x = 3, y = 2$
   (b) $x = 4, y = -5$
   (c) $x = -1, y = 2$
   (d) $x = -8, y = 19$
   (e) $x = 4, y = 5$
   (f) $x = -2, y = -4$
3. (a) $x = 3, y = 1$
   (b) $x = 7, y = -11$
   (c) $x = 1, y = 2$
   (d) $x = 3, y = -1$
   (e) $x = 2, y = 3$
   (f) $x = 4\frac{1}{3}, y = -\frac{1}{3}$
4. (a) $x = 6, y = 1$
   (b) $x = 1, y = 3$
   (c) $x = -1, y = -1$
   (d) $x = 3, y = 10$
   (e) $x = 1, y = 2$
   (f) $x = 5, y = -1$
   (g) $x = 3\frac{1}{7}, y = 5\frac{6}{7}$
   (h) $x = 0, y = 1\frac{1}{2}$
5. (a) $x = 0.75, y = -0.25$
   (b) $x = 3, y = 5$
   (c) $x = 2, y = -1.5$
   (d) $x = 0.5, y = 1.5$
6. (a) $x = 2, y = 1$
   (b) $x = 1\frac{6}{7}, y = 10$
   (c) $x = 1, y = 1$
   (d) $x = 2, y = -1$
   (e) $x = 2, y = -2$
   (f) $x = -6, y = 2$
7. (a) $x = 5, y = 6$
   (b) $x = 1, y = -1$
   (c) $x = 6, y = -4$
   (d) $x = 13, y = 11$
8. (a) $x = -4, y = 4$
   (b) $x = 4, y = -3$
   (c) $x = \frac{2}{3}, y = 0$
   (d) $x = 1, y = -2$
   (e) $x = -3, y = 5$
   (f) $x = -\frac{6}{11}, y = -1\frac{8}{11}$
9. (a) $x = 15, y = -5$
   (b) $x = -2, y = 11$
   (c) $x = 9, y = 4$
   (d) $x = 3, y = 6$
10. (a) $x = -1, y = 3$
    (b) $x = 2, y = 1$
    (c) $x = 1, y = 4$
    (d) $x = -2, y = \frac{1}{2}$
11. $p = 1, q = -2$
12. $p = 2, q = 3$
13. 19 m, 6 s

Exercise 2F

1. 25, 113
2. 5, 15
3. $15, $27
4. $2, $2.40
5. $6\frac{5}{7}, 6\frac{6}{7}$
6. 8, 40
7. 46°, 74°
8. 21 cm
9. 875 cm^2
10. 32 cm
11. $\frac{3}{5}$
12. 7, 6
13. $69
14. 7, 20
15. 10
16. $32, $48
17. $13 000, $12 000
18. 6, 14
19. 72
20. $0.45, $1.50
21. (i) 4000
    (ii) $5

Review Exercise 2

1. (a) 5, 7
   (b) −7, −4
2. $m = 2, n = 1.2$
3. (i) \$0.80
   (ii) \$3.80
   (iii) Company *B*
   (iv) Company *B*
   (v) Company *A*
4. (a) 10, 2, −6
   (c) 2
   (d) (ii) (−0.5, 3)
5. (a) (i) $p = -9, q = 11$
   (b) (i) 5.5, −0.5, −3.5
   (c) $x = 1, y = 1$
6. (a) $x = 2, y = -2$
   (b) $x = 2, y = 2\frac{1}{2}$
   (c) $x = 1, y = -4$
   (d) $x = 1\frac{1}{3}, y = -8$
   (e) $x = 3, y = -4$
   (f) $x = 2, y = 1.5$
7. 17, 14
8. 44 cm
9. $\frac{7}{10}$
10. 48
11. (i) 11
    (ii) 30
12. \$11, \$13
13. 21, 15
14. \$0.50
15. 14 kg, 6 kg
16. \$21.20
17. 38, 37

Challenge Yourself – Chapter 2

1. (i) $p = 4, q = 12$
   (ii) $p = 4, q \neq 12$
   (iii) $p \neq 4$, q is any real number
2. $x = \frac{1}{3}, y = 5$
3. 2, 3
4. (i) 8
   (ii) 7
   (iii) 5
5. $x = 4, y = 18, z = 78$ or $x = 8, y = 11, z = 81$ or $x = 12, y = 4, z = 84$

Exercise 3A

1. (a) $15x^2 + 11$
   (b) $12x^2 - 3x - 8$
   (c) $-y^2 - 11y$
   (d) $4x^2 + 9x$
   (e) $-x^2 - 16x$
   (f) $10y^2 + y - 3$
2. (a) $60x$
   (b) $6x^2$
   (c) $-16x^2$
   (d) $30x^2$
3. (a) $12x + 16$
   (b) $42x + 18$
   (c) $-8x - 24$
   (d) $-10x + 2$
   (e) $15x^2 - 20x$
   (f) $-24x^2 - 40x$
   (g) $-10x + 15x^2$
   (h) $x^2 + x$
4. (a) $13a + 27$
   (b) $63 - 33b$
   (c) $5c^2 + 7c$
   (d) $36d^2 - 28d$
5. (a) $x^2 + 10x + 21$
   (b) $12x^2 + 23x + 5$
6. (a) $-18a - 5$
   (b) $-4b + 3$
   (c) $29c - 3c^2$
   (d) $7d^2 - 12d$
   (e) $6f^2 - 41f$
   (f) $-13h^2 - h$
7. (a) $a^2 - 8a - 9$
   (b) $b^2 + 5b - 14$
   (c) $c^2 - 11c + 30$
   (d) $-6d^2 + 13d + 5$
   (e) $-7f^2 + f + 6$
   (f) $40 - 66h + 27h^2$
8. (a) $x^2 + 4x + 8$
   (b) $2x^2 + 16x - 7$
   (c) $11x^2 - 23x - 18$
   (d) $3x^2 - 10x - 12$
9. (a) $-2x^2 + 5x + 4$
   (b) $4x^2 - 21x - 35$
   (c) $7x^2 + 27x - 51$
   (d) $7x$

Exercise 3B

1. (a) $(a + 1)(a + 8)$
   (b) $(b + 3)(b + 5)$
   (c) $(c - 4)(c - 5)$
   (d) $(d - 2)(d - 14)$
   (e) $(f - 2)(f + 8)$
   (f) $(h - 10)(h + 12)$
   (g) $(k + 2)(k - 6)$
   (h) $(m + 1)(m - 21)$
2. (a) $(3n + 7)(n + 1)$
   (b) $(2p + 1)(2p + 3)$
   (c) $(3q - 4)(2q - 3)$
   (d) $(4r - 3)(r - 1)$
   (e) $(4s - 5)(2s + 3)$
   (f) $(6t - 5)(t + 4)$
   (g) $(2u + 3)(2u - 7)$
   (h) $(9w + 13)(2w - 3)$
3. (a) $(-a + 7)(a + 5)$
   (b) $(-3b + 1)(b - 25)$
   (c) $2(2c + 1)(c + 2)$
   (d) $5(d - 5)(d - 24)$
   (e) $4(2f - 5)(f + 3)$
   (f) $3(8h + 3)(h - 1)$
   (g) $2(-k + 5)(2k + 3)$
   (h) $5n(7m - 6)(m + 1)$
5. (a) $\frac{1}{9}(4p - 3)(p + 3)$
   (b) $-0.2r(16q - 3)(4q + 1)$

Review Exercise 3

1. (a) $20a^2 - 70a$
   (b) $-21b + 12b^2$
   (c) $c^2 - 15c + 44$
   (d) $-3d^2 + 17d - 20$
2. (a) $13f^2 - 16f$
   (b) $8h^2 + h - 3$
   (c) $-k^2 + 12k + 4$
   (d) $16m^2 + m - 43$
3. (a) $(a + 4)(a + 9)$
   (b) $(b - 7)(b - 8)$
   (c) $(c - 3)(c + 17)$
   (d) $(d + 3)(d - 15)$
4. (a) $(3f - 2)(3f + 8)$
   (b) $(3h + 2)(h - 7)$
   (c) $7(2k + 1)(k + 3)$
   (d) $3(3m - 2)(2m - 3)$
5. $\frac{1}{2}(3x + 2)(2x - 5)$

Challenge Yourself – Chapter 3

2 or 16

Exercise 4A

1. (a) $-12xy$
   (b) $7xy$
2. (a) $8xy - 8x$
   (b) $-27xy + 18xz$
   (c) $6x^2 + 21xy$
   (d) $3xy - 33y^2$
   (e) $-6a^2 - 9ab$
   (f) $-8c^2 + 20cd$
   (g) $-42hk + 18h^2$
   (h) $96m^2 + 56mn$
   (i) $6p^2 + 2pq + 14pr$
   (j) $-7s^2 + 28st + 21su$
3. (a) $13ab - 16ac$
   (b) $4d^2 - 14df + 14f^2$
4. (a) $x^2 + 7xy + 6y^2$
   (b) $x^3 + 5x^2 + 2x + 10$
5. (a) $-\frac{2}{3}xy$
   (b) $27x^5y^3$
   (c) $-26x^3y^3$
   (d) $8x^4y^4z^5$

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6. (a) $-3x^2y+6xy^2$
(b) $-27x^3y-63x^2z$
(c) $-39x^3y+13x^2y^2$
(d) $30x^2+20x^3y+15xy$

7. (a) $7ab-5ac$
(b) $-11df+10dh-7fh$
(c) $6k^2+19km$
(d) $-8n^2+10np$

8. (a) $a^2+2ab-3b^2$
(b) $3c^2+cd-14d^2$
(c) $-21k^2+32kh+5h^2$
(d) $7m^3-28m^2+2m-8$

9. (a) $8x^2-25xy-12y^2$
(b) $-3x^2-8xy+3y^2$

10. (a) $x^2+x+12xy+9y+27y^2$
(b) x^3+3x^2+3x+2

11. (a) $(a-b)(a+4b)$
(b) $(c+3d)(c-7d)$
(c) $(2h-3k)(h+5k)$
(d) $(3m+2n)(m-6n)$
(e) $3(p+2q)(p+3q)$
(f) $t(2r-5s)(r-2s)$

12. $\frac{4}{5}x^2y^2z^3$

13. (a) $16x^2-11xy-12y^2$
(b) $35x^2+49xy-22y^2$

14. (a) $2x^2-4x+7xy+6y-15y^2$
(b) $x^3-x^2-13x+28$
(c) x^3+x^2-3x+1
(d) $-3x^3+12x^2-13x+12$

15. (a) $(xy-3)(xy+5)$
(b) $(4xy+5)(3xy-8)$
(c) $2z(2xy-3)(xy-4)$
(d) $\frac{1}{3}(3x-2y)(2x+3y)$

Exercise 4B

1. (a) $a^2+8a+16$
(b) $9b^2+12b+4$
(c) $c^2+8cd+16d^2$
(d) $81h^2+36hk+4k^2$

2. (a) $m^2-18m+81$
(b) $25n^2-40n+16$
(c) $81-90p+25p^2$
(d) $9q^2-48qr+64r^2$

3. (a) s^2-25
(b) $4t^2-121$
(c) $49-4u^2$
(d) w^2-100x^2

4. (a) 1 447 209
(b) 795 664
(c) 3 999 996

5. 56

6. 40

7. (a) $\frac{1}{25}a^2+\frac{6}{5}ab+9b^2$
(b) $\frac{1}{4}c^2+\frac{2}{3}cd+\frac{4}{9}d^2$

8. (a) $\frac{9}{4}h^2-15hk+25k^2$
(b) $\frac{36}{25}m^2+\frac{36}{5}mn+9n^2$

9. (a) $-36p^2+25$
(b) $81r^2-\frac{16}{25}q^2$
(c) $-\frac{s^2}{4}+\frac{t^2}{9}$
(d) u^4-16

10. (a) $x^2+24x+84$
(b) $23x^2+8xy-57y^2$

11. 6

12. 25

13. $\frac{1}{256}x^4-\frac{1}{625}y^4$

14. (i) $4q^2$
(ii) 400

15. (i) a^2
(ii) 9

Exercise 4C

1. (a) $(a+7)^2$ (b) $(2b+1)^2$
(c) $(c+d)^2$ (d) $(2h+5k)^2$

2. (a) $(m-5)^2$ (b) $(13n-2)^2$
(c) $(9-10p)^2$ (d) $(7q-3r)^2$

3. (a) $(s+12)(s-12)$
(b) $(6t+5)(6t-5)$
(c) $(15+7u)(15-7u)$
(d) $(7w+9x)(7w-9x)$

4. (a) 1800 (b) 54

5. (a) $3(a+2)^2$ (b) $\left(5b+\frac{1}{2}c\right)^2$
(c) $\left(\frac{4}{7}d+\frac{1}{5}f\right)^2$ (d) $(h^2+k)^2$

6. (a) $4(3m-2n)^2$ (b) $\frac{1}{3}(p-q)^2$
(c) $(4r-\frac{1}{8}s)^2$ (d) $(5-tu)^2$

7. (a) $2(4a+7b)(4a-7b)$
(b) $\left(c+\frac{1}{2}d\right)\left(c-\frac{1}{2}d\right)$
(c) $\left(\frac{3h}{10}+4k\right)\left(\frac{3h}{10}-4k\right)$
(d) $(m+8n^2)(m-8n^2)$

8. (a) $a(a+6)$
(b) $-(5b+19)(5b+11)$
(c) $(c+d+2)(c-d-2)$
(d) $(2h-1+2k)(2h-1-2k)$
(e) $(5m+n-1)(5m-n+1)$
(f) $4p$

9. (i) $(x+2)$ cm
(ii) $(x^3+6x^2+12x+8)$ cm^3

10. (a) $-(11x+7)(7x+11)$
(b) $(4x+1+3y)(4x+1-3y)$
(c) $(2x+y-2)(2x-y+2)$
(d) $13(x+y+1)(x+y-1)$

Exercise 4D

1. (a) $9x(5x-9y)$ (b) $3x(13y-5xz)$
(c) $xy^2(z^2-xy)$ (d) $-5\pi x^3(3y+2)$

2. (a) $(x-2y)(6a+5)$
(b) $(x+3y)(2b-c)$
(c) $(5x-y)(3d-4f)$
(d) $5(x+3y)(h+2k)$

3. (a) $(x-5)(a+4)$
(b) $(a+b)(x+y)$
(c) $(1+y)(x+2y)$
(d) $(x-3)(x+2y)$

4. (a) $(a+b)(x-z)$
(b) $(c+2d)(-2c+9d)$
(c) $(2h-k)(x-3y)$
(d) $2(4m-n)(3x+y)$

5. (a) $(3x+4y)(a+7b)$
(b) $(3y+5)(4c-3)$
(c) $(d+f)(y-z)$
(d) $(x+2y)(3x-4z)$
(e) $(y-4)(2x-3)$
(f) $5(y-5x)(x-2)$
(g) $xy(y-5)(x+y)$
(h) $(k-h)(x-y)$

6. (a) $24(y-5x^2)(6p+q)$
(b) $2(x+2y)(x-2y)(5x-10y-6)$

7. (i) $\frac{1}{3}p^2(q+4r)$
(ii) 48

8. (i) $(x^2+3)(x-1)$
(ii) $(x^4-6x^2+10)(x^2-4)$

Review Exercise 4

1. (a) $-2a^2+10ab-14a$
(b) $6c^2+17cd+12d^2$
(c) $-4k^2-7hk+15h^2$
(d) $2m^3+7m^2+m-1$

2. (a) $6p^2-7pq-2q^2$
(b) $-12s^2-6sr-4r^2$
(c) $15t^2+69tu-9u^2$
(d) $-w^2+7wx+34x^2$

3. (a) $(x-7y)(x+9y)$
(b) $(2x+3y)(x+y)$
(c) $(3xy-4)(2xy+1)$
(d) $z(3-2xy)(1-2xy)$

4. (a) $x^2-10xy+25y^2$
(b) x^4-y^2
(c) $9x^2+\frac{24}{5}xy+\frac{16}{25}y^2$
(d) $\frac{1}{16}x^2+\frac{1}{12}xy+\frac{1}{36}y^2$
(e) $25x^2-\frac{49}{16}y^2$
(f) $\frac{9}{16}x^2y^2-\frac{1}{9}z^2$

5. (a) $(1+11x)(1-11x)$
(b) $(x+3y)^2$
(c) $25(x-2y)^2$
(d) $(6y+7x+7)(6y-7x-7)$

6. (a) $-7y(2x+3y)$
(b) $9xy(y-4x)$
(c) $(a+b)(3x-4y)$
(d) $(x-2y)(5-x+2y)$
(e) $(x+3y)(x+2)$
(f) $(3x-2)(x^2+1)$
(g) $2(2x-3y)(c-2d)$
(h) $(y-2)(5x+6y)$
7. $(x+1)(x+2)(x-2)$
8. (a) 808 201
(b) 318 000
9. 106
10. (i) f^2+6f+9
(ii) $4h^2+4hk+k^2+12h+6k+9$

Challenge Yourself – Chapter 4

1. 0
2. 16

Revision Exercise A1

1. $k=\frac{1}{4}$, $p=1.5$, $q=124$
2. 2 mm
3. $x=1$, $y=2$
4. 26, 8
5. (a) $3a^2+23ab+43b^2$
(b) $12c^2+\frac{4}{5}cd-\frac{26}{25}d^2$
6. (a) $2(2f-3)(f-1)$
(b) $(1-6hk)^2$
(c) $5mn(m-3n-5)$
(d) $(x-y)(2p-3q)$

Revision Exercise A2

1. 43.75%
2. (i) $y=\frac{91}{2x^2+5}$
(ii) $\frac{13}{19}$
(iii) ±2.5
3. $p=3$, $q=-4$
4. $x+y=70$, $2x+4y=196$; 42 chickens, 28 rabbits
5. (a) $a^3-23a^2+39a-12$
(b) $b^4-\frac{1}{81}$
6. (a) $(2cd-3)(cd+4)$
(b) $(5hk+1)^2$
(c) $-4m(4+m)$
(d) $(3r-s)(p+2q)$
7. (a) 648 025
(b) 806 000

Exercise 5A

1. (a) 0 or 9 (b) 0 or −7
(c) 0 or −1 (d) 0 or 6
(e) 0 or $2\frac{1}{3}$ (f) 0 or $-1\frac{1}{2}$
2. (a) 4 or 9 (b) 3 or −5
(c) −4 or 11 (d) −1 or −2
(e) $\frac{6}{7}$ or $1\frac{1}{4}$ (f) $1\frac{2}{3}$ or $-\frac{1}{2}$
(g) $-\frac{3}{5}$ or 2 (h) $-2\frac{1}{2}$ or $\frac{5}{8}$
3. (a) 0 or −9 (b) 0 or 7
(c) 0 or −5 (d) 0 or $1\frac{1}{3}$
(e) 0 or $\frac{1}{27}$ (f) 0 or −4
4. (a) −6 (b) 8
(c) ±4 (d) $-1\frac{2}{5}$
(e) $1\frac{1}{2}$ (f) ±5
5. (a) −3 or −7 (b) 7 or 9
(c) 3 or −9 (d) −3 or 8
(e) $-1\frac{1}{3}$ or −15 (f) $\frac{5}{6}$ or 4
(g) $\frac{2}{3}$ or −3 (h) $3\frac{1}{2}$ or −2
6. 0 or −3
7. (a) ±11 (b) ±8
(c) $\pm\frac{1}{2}$ (d) $\pm3\frac{1}{3}$
8. (a) 5 or −12 (b) $-1\frac{1}{4}$ or $1\frac{1}{2}$
9. (a) $\frac{1}{2}$ or −3 (b) $1\frac{1}{2}$ or 6
(c) 5 or −7 (d) −8 or 15
(e) $-\frac{1}{2}$ or 6 (f) −3 or 7
(g) −1 or 3 (h) −2 or 4
10. (i) $1\frac{2}{3}$ or $-1\frac{1}{2}$ (ii) $4\frac{2}{3}$ or $1\frac{1}{2}$
11. (a) $\frac{1}{2}$ or 5 (b) $\frac{4}{13}$ or $-\frac{2}{3}$
12. $y=\frac{2}{3x}$
13. $-10\frac{1}{2}$ or 1
14. (i) 7 (ii) 2

Exercise 5B

1. 2
2. 3
3. 7, 8
4. 5, 12
5. 9, 18 or −18, −9
6. 23 m, 9 m
7. 4 m
8. 5
9. 16 cm, 28 cm
10. (ii) ±2.51 (iii) 10.0 cm
11. (ii) $-6\frac{1}{3}$ or 5 (iii) 9 hours
12. (i) 4 m (ii) 4 s
13. 3 or 12
14. (i) 11.25 m (ii) 3.4 s
(iii) 0.75 m
15. (i) 110.25 m (ii) 8 s
(iii) 1 s or 7 s

Exercise 5C

1. (a) $a=0$, $b=-5$
(c) (i) 2.45 or −4.45
(ii) −9
(d) $x=-1$
2. (a) $p=-7$, $q=-3$
(c) (i) −12
(ii) 3.1, −0.75
3. (i) $A(-5, 0)$, $B(4, 0)$, $C(0, 20)$
(ii) 8
4. (b) (i) 9.3
(ii) 10.3, −0.5
(c) −3.45 or 2.45
5. (i) $y=x(64-8x)$
(iii) $4
6. (iii) 4.5 cm, 4.5 cm
7. (b) (i) −0.4 or 2.4
(ii) −1
(c) $x=1$
(d) 0 or 3
8. (ii) $5000, 1 year

Review Exercise 5

1. (a) 0 or $-2\frac{1}{2}$ (b) $-2\frac{3}{4}$ or $2\frac{1}{3}$
(c) 0 or $1\frac{3}{4}$ (d) −3
(e) $1\frac{3}{7}$ (f) ±7
2. (a) $\frac{2}{3}$ or $-2\frac{1}{2}$ (b) $1\frac{1}{3}$
(c) $-\frac{1}{3}$ or $2\frac{1}{3}$ (d) $\frac{3}{5}$ or −2
3. (i) $-1\frac{1}{2}$ or 7 (ii) $-3\frac{1}{2}$ or 5
4. (i) −3 (ii) $-\frac{1}{2}$
5. −2, 1 or 6, 9
6. (ii) 0 or 6 (iii) 30
7. (ii) 2 or −4 (iii) 24 cm
8. 30
9. (i) 20 m (ii) 1 s or 3 s
10. $A(-5, 0)$, $B(1, 0)$, $C(0, 5)$
11. (a) 6
(c) (i) −4.80
(ii) −4.85 or 1.85
(iii) −6.3
(d) $x=-1.5$
12. (ii) 64 m, 4 s

Challenge Yourself – Chapter 5

1. 8
2. (ii) 3.32

Exercise 6A

1. (a) $\frac{1}{3xy}$ (b) $\frac{2b^2}{3a^2}$
(c) $\frac{q^2}{3r^2s}$ (d) $\frac{n}{6m^2p^3}$
(e) $\frac{c^2}{5ab^4}$ (f) $-\frac{1}{4xyz^2}$

2. (a) $\frac{y}{4}$ (b) $\frac{4}{c}$
(c) $\frac{a+2b}{6}$ (d) $\frac{c}{c-d}$
(e) $\frac{m-n}{m}$ (f) $\frac{q}{3-2q}$

3. (a) $\frac{1}{2a-b}$ (b) $\frac{c-3d}{4c}$
(c) $\frac{3}{a+3}$ (d) $\frac{x+7}{x}$
(e) $\frac{k}{m-4}$ (f) $\frac{k}{m-4}$

4. (a) $\frac{3}{2b^4}$ (b) $\frac{3}{4}$
(c) $\frac{3}{8}$ (d) $2c$

5. (a) $\frac{1}{3x^2(a-b)}$ (b) $\frac{a(a-3b)^2}{3b}$
(c) $\frac{b^2(2a+3b)}{4a}$ (d) $\frac{n^2}{12a(b+c)}$
(e) $\frac{y+3}{y+2}$ (f) $\frac{-m-4}{2m+1}$
(g) $\frac{-y-3x}{y+x}$ (h) $\frac{3x-y}{4x-y}$
(i) $\frac{-a-b}{2a+3b}$ (j) $\frac{y+1}{2y-3}$
(k) $\frac{3}{a-1}$ (l) $\frac{a-b}{a+b}$
(m) $\frac{a-n}{a+n}$

(a) $\frac{5a^2c^5}{144b^6}$ (b) $\frac{27}{64d^3f^3}$
(c) $\frac{8}{5xy^2}$ (d) $\frac{6p^3s^4}{r^2q^3}$
(e) $-\frac{9}{5w^2}$ (f) $-\frac{9x}{4y}$
(g) $\frac{h}{h+3}$ (h) $\frac{d(c+d)^2}{c-d}$
(i) $\frac{m+2}{m}$ (j) $-\frac{z}{z+2}$
(k) $b(a-2b)$ (l) $\frac{y-2}{3(1-3y)}$

$\frac{x+y-z}{x-y-z}$

Exercise 6B

1. (a) $\frac{29}{18a}$
(b) $\frac{1}{b}$
(c) $\frac{d-c}{3cd}$
(d) $\frac{2f-17h}{24k}$
(e) $\frac{5a}{x-3y}$
(f) $\frac{1}{z}$

2. (a) $\frac{8a+20}{a(a+4)}$
(b) $\frac{c-5b}{2b(b+c)}$
(c) $\frac{10d+2}{(d-5)(2d+3)}$
(d) $\frac{-f-17}{(f+5)(f-1)}$
(e) $\frac{52-49h}{(3h-7)(6-5h)}$
(f) $\frac{2k+5}{(k+1)(k-1)}$
(g) $\frac{8-10m}{(2m+1)(2m-1)}$
(h) $\frac{2n-1}{(n-2)^2}$

3. (a) $\frac{7}{6(a-b)}$
(b) $\frac{2c-1}{2(3c-7)}$
(c) $\frac{12f+10d}{15(2f-d)}$
(d) $\frac{5u+7}{6(u-4)}$
(e) $\frac{7m-1}{6(3n-2)}$
(f) $\frac{5h+7k}{8(p-q)}$
(g) $\frac{3x^2}{2(x-y)}$
(h) $\frac{10-21x}{6(z-2y)}$

4. (a) $\frac{24a^2-23a}{(3a-5)(4a-1)}$
(b) $\frac{8b+5}{(2b+1)^2}$
(c) $\frac{5-2h}{h(h-6)}$
(d) $\frac{6m^2-25m+12}{m(m-4)(m-3)}$
(e) $\frac{x^2+6xy-6y^2}{(x-y)(x+y)}$
(f) $\frac{3}{2z+3}$

5. (a) $\frac{2a+5}{(a+3)(a+1)}$
(b) $\frac{-b^2-b+1}{(b+1)(b-6)}$
(c) $\frac{4p^2+4p+1}{2(p+1)(p-5)}$
(d) $\frac{-2x^2-xy+4}{(x+y)(x+2y)}$

6. $\frac{y+6x}{6y}$

Exercise 6C

1. (a) $y=\frac{k-ax}{b}$
(b) $n=\frac{PV}{RT}$
(c) $d=\frac{5b-3c}{2}$
(d) $a=\frac{R}{m}-g$

2. (a) $a=m(b+c)$
(b) $p=\frac{3}{2}(5q-r)$
(c) $k=\frac{a}{14}$
(d) $b=\frac{2A}{h}-a$

3. (a) $h=m^3+k$
(b) $D=b^2-4ac$
(c) $V=\pm\sqrt{PR}$
(d) $\theta=\frac{360A}{\pi r^2}$

4. ± 4

5. (a) $\sqrt{\frac{23}{5}}$ (b) $6\frac{15}{17}$

6. (a) 3 (b) 3
(c) 1 (d) 6
(e) $1\frac{1}{3}$ (f) $\frac{11}{42}$
(g) $-\frac{1}{2}$

7. (a) $C=\frac{5}{9}(F-32)$
(b) $l=\frac{A-2\pi r^2}{\pi r}$
(c) $u=\frac{s}{t}-\frac{1}{2}at$
(d) $d=\frac{2S-2an}{n(n-1)}$

8. (a) $h=\frac{3-k}{k-2}$
(b) $z=\frac{y^2}{y-x}$
(c) $p=\frac{q^2}{x-q}$
(d) $b=\frac{a}{a-1}$

9. (a) $r = \sqrt[3]{\frac{3V}{4\pi}}$
   (b) $u = \pm\sqrt{v^2 - 2as}$
   (c) $x = p \pm \sqrt{y - q}$
   (d) $z = \frac{t^2(m-3)}{4}$

10. (i) $h = \frac{V}{\pi r^2} - \frac{2}{3}r$
    (ii) -3.08

11. (i) $b = \frac{c + a^2c}{a^2 - 3}$
    (ii) 25

12. (a) $5\frac{5}{7}$ (b) 126
    (c) ± 11.5

13. (a) 2210 (b) 19.0

14. (a) $10\frac{1}{2}$ (b) -1
    (c) 3 (d) $1\frac{1}{2}$
    (e) $\frac{3}{8}$

15. (i) $v = \frac{fu}{u - f}$ (ii) 60 cm

16. (i) $l = g\left(\frac{T}{2\pi}\right)^2$ (ii) 0.0912 m

17. (i) $v = \sqrt{\frac{2(E - mgh)}{m}}$
    (ii) 19.0 m s^{-1}

18. $\frac{3}{5}$

19. (a) 17.6
    (b) 688
    (c) ± 27.5

20. (a) $R = \frac{kl}{r^2}$
    (b) $r = \sqrt{\frac{kl}{R}}$
    (c) (i) 0.0005
        (ii) 0.005 77 m

Review Exercise 6

1. (a) $\frac{4y}{x}$ (b) $\frac{x(c + 3d)}{5(c - d)}$
   (c) $\frac{1}{2c}$ (d) $\frac{1}{4(3f + 4d)}$
   (e) $\frac{p - 2}{p}$ (f) $\frac{-r - s}{q - 3}$
   (g) $\frac{3w + 2}{3(w - 2)}$ (h) $\frac{z + 6}{3(z + 2)}$

2. (a) $\frac{3ae^2}{4}$ (b) $-13\frac{1}{2}$
   (c) 6 (d) $\frac{9(4x - 1)}{4}$
   (e) $-n$ (f) $\frac{14p^2q(p - q)^2}{(p - q^2)^2}$
   (g) $\frac{6}{x - 1}$ (h) $\frac{y + 4}{4}$

3. (a) $\frac{7}{8a}$
   (b) $\frac{16b}{3(3b + 2c)}$
   (c) $\frac{11}{4(d - 2f)}$
   (d) $\frac{16h + 25}{6(2h + 5)}$
   (e) $\frac{10k + 17}{(k - 1)(4k + 5)}$
   (f) $\frac{3m^2 + 16m - 20}{(2m - 5)(8 + 3m)}$
   (g) $\frac{-7n + 5}{(2n - 1)^2}$
   (h) $\frac{2 - p}{(p + 4)(p - 1)}$

4. (a) $a = \frac{c(8 - b)}{2}$
   (b) $r = \frac{100(A - P)}{P}$
   (c) $t = \frac{1 - a}{1 + a}$
   (d) $h = \frac{5k}{5 + 2k}$
   (e) $b = \frac{a}{3a - 2}$
   (f) $y = \pm\sqrt{(k - x)^2 - z}$

5. (a) $2\frac{2}{5}$
   (b) $-140\frac{1}{16}$

6. (a) 4 (b) ± 10

7. (a) 7860 (b) 16.6
   (c) ± 6.44 (d) 9.81

8. (a) $4\frac{3}{5}$ (b) $2\frac{2}{9}$

9. (i) $R = \frac{R_1R_2}{R_1 + R_2}$
   (ii) 1.2 ohms

10. (i) $y = b \pm \sqrt{r^2 - (x - a)^2}$
    (ii) 7 or -1

Challenge Yourself – Chapter 6

1. $\frac{a}{c(a + b)^3}$
2. $n = 6$, $k = -5$

Exercise 7A

1. (a) Yes
   (b) No, 2 has two images.
   (c) Yes
   (d) Yes
   (e) No, 4 has no image.
   (f) Yes
2. 8, −28, −2, −7
3. (i) 3 (ii) 9 (iii) 5 (iv) 10
4. (i) 18 (ii) −17 (iii) 8 (iv) 1 (v) 2
5. (a) (i) $3\frac{1}{2}$ (ii) $5\frac{1}{4}$
       (iii) $2\frac{1}{2}$ (iv) 45
   (b) 20, 28
6. (i) 5 (ii) −2 (iii) $\frac{2}{7}$
   (iv) 3 (v) 1 (vi) $\frac{6}{7}$
7. 13, 17, 21
   (i) No (ii) No
   (iv) No (iv) No
8. 2, $\frac{1}{8}$, $-\frac{3}{4}$, $5\frac{1}{4}$
   (i) No (ii) No (iii) $\frac{9}{17}$
   (iv) $\frac{3}{4}a + \frac{1}{2}$, $1\frac{1}{2}a + \frac{1}{2}$, $1\frac{1}{4} - 2a$
   (v) 30 (vi) $\frac{3}{22}$

Review Exercise 7

1. (a) No, 1 has two images.
   (b) Yes
   (c) No, 3 has no image.
   (d) Yes
2. −8, 5, $2\frac{1}{16}$, $1\frac{2}{3}$
3. (i) 2 (ii) 27
   (iii) 12 (iv) 34
4. (i) 1 (ii) −34
   (iii) −6 (iv) −33
   (v) 5
5. (a) (i) 8 (ii) $9\frac{3}{4}$
       (iii) $8\frac{1}{2}$ (iv) 24
   (b) 40, 4
6. (i) 4 (ii) −4
   (iii) 2 (iv) $\frac{1}{2}$
   (v) $1\frac{4}{7}$
7. (i) $10a - 3$ (ii) $6a + 2\frac{3}{4}$
   (iii) $6\frac{1}{2}a - 4$

Challenge Yourself

$10\frac{1}{2}$

Exercise 8A

1. A, F; B, J; C, E; D, G; I, K
2. (i) 3.5 cm
   (ii) VZ
   (iii) WX, 3.5 cm
   (iv) ZY, 2.1 cm
   (v) YX, 2 cm
   (vi) ∠VWX, 90
3. EF = 3.4 cm, GH = 2.4 cm,
   ∠FEH = 100°, ∠FGH = 75°,
   MN = 5 cm, OL = 3 cm, ∠LMN = 65°,
   ∠NOL = 120°
4. (a) △ABC ≡ △PQR
   (b) △DEF ≡ △TSU
   (c) Not congruent
5. (i) 56°
   (ii) 16 cm

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6. (i) 75°
(ii) 10.9 cm
7. (i) 6 cm
(ii) 96°

Exercise 8B

1. (a) $x = 90, y = 35, z = 55$
(b) $x = 28, y = 34$
(c) $x = 7.2, y = 10.8$
(d) $x = 9.6, y = 5\frac{5}{6}$
2. (a) No
(b) No
3. (a) $x = 95, y = 52, z = 4.8$
(b) $x = 80, y = 10.5$
4. $x = 16, y = 1.875$
5. $x = 100, y = 270, z = 100$
6. 8 m
7. $x = 56, y = 6$
8. $x = 68, y = 10.5$
9. (i) $4y$ m
(ii) 1.5 m

Exercise 8C

1. 10 cm, 3.5 cm
2. (i) $k = 2$ (ii) 8 cm, 7 cm
3. (i) 1 cm to 8 km (ii) 56 km
5. (b) (ii) (125 ± 5) m
6. (i) 1920 m (ii) 0.1 cm
(iii) $\frac{1}{25\ 000}$
7. (i) 1.1 km (ii) 0.5 cm
8. (i) 320 km² (ii) 2 cm²
9. 4 cm, 9 cm
10. 8 cm
11. (i) 4.5 m by 3.75 m
(ii) 6.75 m²
(iii) 78.75 m²
12. (i) 1 cm to 0.25 m
(ii) 17 cm
13. (i) 636 m (ii) 53 cm
14. (i) 1.4 km (ii) 0.8 cm
15. (i) 1 : 12 500 (ii) 312.5 m
16. (i) 1 : 50 000 (ii) 56 cm
(iii) 3 km²
17. (i) 21.9 km² (ii) 0.505 cm²
18. 7140 m²
19. (i) $\frac{1}{6\ 000\ 000}$ (ii) 312 km
(iii) $82.80
(iv) 2 hours 42 minutes
(v) 42 km/h
20. 2800 hectares
21. (ii) 49 m

Review Exercise 8

1. (a) Not congruent
(b) $\triangle DEF \equiv \triangle TUS$
2. (i) 6 cm (ii) 95°
3. (i) 92° (ii) 4.2 cm
4. (i) 90° (ii) 6 cm; 8 cm
(iii) 53°
5. (a) Yes (b) Yes
6. $a = 53, b = 15$
7. $x = 85, y = 9$
8. (i) 60° (ii) 7.5 cm
9. (i) 20 m (ii) 50 cm
10. (i) 156 m (ii) 65 cm
11. (i) $\frac{1}{25\ 000}$ (ii) 0.75 km
(iii) 32 cm
12. (i) 19.83 cm (ii) 24.3 km
(iii) 1.54 cm²
13. (i) 1 : 150 000 (ii) 10.5 km
(iii) 36 cm²
14. (i) 14 cm (ii) 1 km²
15. (i) 98 m (ii) 25 cm²
(iii) 4 hectares

Challenge Yourself – Chapter 8

1. (i) $\triangle DPC, \triangle DAB; \triangle BPC, \triangle BQD; \triangle ABP, \triangle DQP$
(ii) 2 : 3
2. 4 : 1

Exercise 9A

1. (a) (i) (3, –4) (ii) (–1, –3)
(iii) (3, –3) (iv) (–3, 4)
(v) (3, 2) (vi) $(p, -q)$
(b) (i) (–3, 4) (ii) (1, 3)
(iii) (–3, 3) (iv) (3, –4)
(v) (–3, –2) (vi) $(-p, q)$
(c) (i) (4, 3) (ii) (3, –1)
(iii) (3, 3) (iv) (–4, –3)
(v) (–2, 3) (vi) (q, p)
2. (a) (i) (–1, 11)
(ii) (–1, –5)
(b) No
3. (1, 2)
4. (ii) (2, –2)
5. (i) (–2, 1) (ii) (6, 1)
(iii) (–6, 1) (iv) (10, 1)
6. (i) $y = x$ (ii) $x = 0$
7. (a) $x = 2$ (b) $y = 4$
(c) $y = x + 1$ (d) $x + y = 2$
(e) $x + y = 1$ (f) $y = 2x - 2$
8. (i) (1, –2) (ii) (2, 1)
9. (3, 10)
10. (ii) (–2, 2)
11. (a) $y = -3x - 2$ (b) $y = -3x + 2$
(c) $y = -3x + 14$

12. (a) (i) (5, 2) (ii) (5, 2)
(b) Yes
(c) (3, 3)
13. (a) (i) (4, 5) (ii) (4, –1)
(b) No
(c) (4, 2)

Exercise 9B

2. (a) (3, –3) (b) (7, 6)
(c) (9, –2)
3. (a) (6, 5) (b) (7, 0)
4. 120° anticlockwise rotation about O,
240° anticlockwise rotation about O
5. (ii) (2, 0), 180°
6. (a) (i) (6, 3) (ii) 180°
(b) (i) (5, 2) (ii) 90°
(c) (2, 2), (0, 0), (2, –1)
7. $x + y = 2$
8. (i) 6 (ii) (4.5, 1)
(iii) (1, 3.5)
9. (a) 77.5° (b) 31.5°

Exercise 9C

1. $A''(5, -2)$, $B''(6, 0)$, $C''(8, 0)$ and $D''(8, -1)$
2. $P'(4, 1)$, $Q'(10, 3)$, $R'(5, -2)$
3. (–5, 0), $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$
4. (i) (4, 8) (ii) $p = q = 0$
(iii) $h = 6, k = 12$
(iv) (–2, – 4)
5. (a) (4, 10) (b) (13, – 4)
(c) (10, 0) (d) (10, 0)
(e) (1, 14)

Review Exercise 9

1. (a) (–2, 1) (b) (–7, 4)
2. (0, –7)
3. (a) $2y = 5x - 28$
(b) (1, –1)
4. (a) Translation $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$
(b) A reflection in $x = 6$
(c) 180° rotation about (6, 3)
(d) 90° clockwise rotation about (6, 0)
(e) 90° clockwise rotation about (6, 6)
5. (a) (–2, –5) (b) (5, 2)
(c) (–2, –5)
6. $y + x = 0$
7. (a) $x + y = 8$ (b) $y = x + 5$
(c) $x + y = 4$
8. (a) (3, 2) (b) (–12, 5)
(c) (–3, 4) (d) (–1, 4)
9. $m = 2, c = -5$
10. (a) (3, 1) (b) (5, 2)
(c) (7, 3)

11. (a) (i) (–5, 3) (ii) (4, 7)
(b) (i) (4, –3) (ii) (–3, 2)
(c) (i) (–5, 2) (ii) (2, 5)

Challenge Yourself – Chapter 9

A translation of 9 cm along AC.

Revision Exercise B1

1. (a) 8 (b) $\pm\frac{4}{5}$
(c) $\frac{1}{2}$ or $-\frac{1}{5}$ (d) 0, 11 or –12
2. $x(x+5) = 66$; 34 m
3. (a) $p = -14$, $q = -5$
(c) (i) –2
(ii) 1.3 or 0.2
(iii) 1.1
(d) $x = 0.75$
4. (a) $\frac{-x+7}{6(2x-1)}$ (b) $\frac{-19x-4}{(3x+1)(3x-1)}$
5. (i) $x = \frac{10ay+4}{15a+3}$
(ii) 3
6. (i) $7a-2$
(ii) $10a+1$
(iii) $3\frac{1}{2}a + 9\frac{1}{8}$
7. (i) 8 cm
(ii) 50°
8. (i) $\frac{1}{800\ 000}$
(ii) 9 cm
(iii) 7.75 cm^2
9. (0, 5)
10. 14° clockwise rotation about O, 187° anticlockwise rotation about O

Revision Exercise B2

1. $1\frac{1}{4}$ or $\frac{1}{2}$
2. (i) –8 (ii) 5
3. $A(-4, 0)$, $B(-1, 0)$, $C(0, 4)$
4. (a) $\frac{2ab - 4a^2 - 7b^2}{2}$
(b) $-2cd^2 - 4c^2 + 3d$
5. (i) $v = \sqrt{\frac{Fr}{m}}$
(ii) 10 m s^{-1}
6. (i) –5 (ii) 17
(iii) 30 (iv) –25
7. (i) $9\frac{1}{6}$ cm (ii) 7.8 cm
8. (i) 183 m (ii) 15.25 cm
9. $y = \frac{1}{2}x + \frac{5}{2}$
10. (i) $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$
(ii) $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$
(iii) $\begin{pmatrix} 7 \\ -4 \end{pmatrix}$

Exercise 10A

1. (a) $a = 29$ (b) $b = 37$
(c) $c = 15.6$ (d) $d = 37.0$
2. (a) $a = 36$ (b) $b = 12.8$
(c) $c = 7.33$ (d) $d = 20.0$
3. 17 cm
4. 8.67 m
5. 56 cm
6. 8.98 m
7. (i) 28 cm (ii) 10.8 cm
8. 5.73 m
9. (a) $a = 22.6$ (b) $b = 21.9$
(c) $c = 15.1$ (d) $d = 28$
(e) $e = 44.6$
10. (a) $a = 15$, $b = 61.8$
(b) $c = 4.28$, $d = 30.3$
(c) $e = 4.29$, $f = 14.7$
(d) $g = 32.1$
11. (i) 11.4 m (ii) 111 m^2
12. 9.85 cm
13. 669 m^2

Exercise 10B

1. 50.3 m
2. 70.7 m
3. 58.3 m
4. 4.66 m
5. 24 inches
6. 17.5 m
7. 31.1 cm
8. 13 cm
9. (i) 12.0 m (ii) 12.9 m
10. (i) 4.34 m (ii) 1.72 m
11. 168 m^2
12. 3
13. (i) 9.22 cm (ii) 5.86 cm
14. 10.2 km
15. (a) (i) 33 cm
(ii) 21 cm
(b) 1089 cm^2, 1386 cm^2
(c) (i) 44 cm
(ii) 838 cm^2
(d) Circle

Exercise 10C

1. (a) Yes; $\angle B$ (b) No
(c) No (d) Yes; $\angle O$
3. No

Review Exercise 10

1. (a) $a = 11.5$ (b) $b = 10.3$
(c) $c = 10$ (d) $d = 18.5$
2. (i) 120 cm (ii) 903 cm^2
3. 21.7 cm
4. 1.73 cm
5. 108 cm^2

6. (i) 92.3 cm (ii) 2210 cm^2
7. (i) 60 cm
8. (i) 93 m^2 (ii) 26.6 m
(iii) 12.4 m
9. (i) 13.0 m (ii) 13.9 m
10. $13\frac{1}{3}$ s

Challenge Yourself – Chapter 10

1. (b) (i) 12, 16, 20
(ii) 7, 24, 25; 15, 20, 25
(c) (i) $25n^2$
(ii) 21, 28, 35
(d) (i) 7, 24, 25
(ii) No
(iii) 9, 40, 41
2. 11 units^2
3. Yes
4. (i) 4.56 cm (ii) No

Exercise 11A

1. (a) (i) PQ (ii) PR
(iii) QR
(b) (i) XY (ii) XZ
(iii) YZ
2. (a) (i) $\frac{5}{13}$ (ii) $\frac{12}{13}$
(iii) $\frac{5}{12}$ (iv) $\frac{12}{13}$
(v) $\frac{5}{13}$ (vi) $\frac{12}{5}$
(b) (i) $\frac{24}{25}$ (ii) $\frac{7}{25}$
(iii) $\frac{24}{7}$ (iv) $\frac{7}{25}$
(v) $\frac{24}{25}$ (vi) $\frac{7}{24}$
3. (a) (i) $\frac{y}{z}$ (ii) $\frac{x}{z}$
(iii) $\frac{y}{x}$ (iv) $\frac{x}{z}$
(v) $\frac{y}{z}$ (vi) $\frac{x}{y}$
(b) (i) $\frac{z}{x}$ (ii) $\frac{y}{x}$
(iii) $\frac{z}{y}$ (iv) $\frac{y}{x}$
(v) $\frac{z}{x}$ (vi) $\frac{y}{z}$
4. (a) 1.07 (b) 0.967
(c) 0.864 (d) 1.23
(e) 2.35 (f) 0.285
5. (a) 4.66 (b) 1.53
(c) 1.13 (d) –1.41
(e) 0.723 (f) 0.657
(g) 2.01 (h) 0.910

Exercise 11B

1. (a) $a = 13.8$ (b) $b = 37.5$
2. (a) $a = 10.9$ (b) $b = 35.1$
3. (a) $a = 7.44$ (b) $b = 6.77$
4. (a) $a = 6.71$, $b = 9.95$
   (b) $c = 11.7$, $d = 10.9$
   (c) $e = 6.81$, $f = 9.76$
   (d) $g = 22.6$, $h = 24.3$
5. (i) 7.38 m (ii) 10.9 m
6. (i) 21.7 cm (ii) 37.5 cm
   (iii) 68.3 cm
7. (i) 537 m (ii) 13 200 m^2
8. $\frac{2}{9}$

Exercise 11C

1. (a) 31.8° (b) 43.5°
   (c) 68.7°
2. (a) $a = 27.5$ (b) $b = 54.0$
   (c) $c = 67.8$ (d) $d = 28.4$
   (e) $e = 41.4$ (f) $f = 41.8$
   (g) $g = 48.7$ (h) $h = 41.8$
   (i) $i = 51.9$
3. (i) 64.6° (ii) 6.32 m
4. (i) 27.9° (ii) 8.04 cm
5. (i) 17.1° (ii) 7.50 m
6. 76.1°
7. (i) 71.8° (ii) 20.4 m
8. 19.3°

Exercise 11D

1. 21.2 m
2. 15.1 m
3. 72.2 m
4. 52.9°
5. (i) 4.33 m (ii) 2.5 m
6. 17.0°
7. 7.90 m
8. 20.6 m
9. 50.0°
10. 1.53 cm
11. 50.9 cm
12. (i) 38.6 cm (ii) 34.7 m
    (iii) 39.7°
13. 13.2 m

Review Exercise 11

1. (a) $a = 17.6$ (b) $b = 12.5$
   (c) $c = 57.9$
2. (i) 40 m (ii) 32 m
   (iii) $2\frac{2}{15}$
3. (i) 4 units (ii) $h = 3.46$
4. $\frac{4}{3}$
5. 57.1°
- (i) 14.5° (ii) 4.59 m
- (i) 25 cm (ii) 69.8 cm
- (i) 32.9° (ii) 43.7 m
  (iii) 28.3 m

9. 3.10 m
10. 130 m
11. 35.6 m
12. (i) 5.69 m (ii) 3.15 m

Challenge Yourself – Chapter 11

1. 136 units
2. $0° < x < 45°$

Exercise 12A

1. 20 cm^3
2. 46 cm^3
3. $23\frac{1}{3}$ m^3
4. 5 cm
5. 7.5 cm
6. 5 m
7. 7824 cm^2
8. 1041.6 g
9. $8\frac{1}{3}$ cm
10. (i) 14.1 cm (ii) 636 cm^3
11. (i) 6.75 cm (ii) 226 cm^2
12. (i) 9.375 m (ii) 584 m^2
13. $28\frac{8}{9}$ cm
14. *VA* is shorter than *VB*.
15. (i) 6.93 cm (ii) 60.3 cm^3

Exercise 12B

1. (a) 528 cm^3 (b) $256\frac{2}{3}$ cm^3
   (c) 180 cm^3 (d) 12 900 mm^3
2. 15 cm
3. 24 m
4. 3 cm
5. (a) 138 cm^2 (b) 1940 mm^2
   (c) 3040 cm^2
6. 14 mm
7. 6.22 cm
8. 16.0 m
9. 28.7 cm
10. 8192
11.
12. (i) 5 cm (ii) 39.3 cm^2
13. 204 cm^2
14. 1230 cm^3
15. (i) 2710 mm^3 (ii) 1290 mm^2
16. 6620 cm^3
17. (i) 660 cm^2 (ii) 1360 cm^3
18. 1540 m^3

Exercise 12C

1. (a) 2140 cm^3 (b) 11 500 mm^3
   (c) 268 m^3
2. (a) 6.97 cm (b) 14.3 mm
   (c) 5.71 m (d) 9 cm
   (e) 7.20 mm (f) 2.25 m
3. (a) 1810 cm^2 (b) 1020 mm^2
   (c) 113 m^2
4. 462 cm^2
5. (a) 4.09 cm (b) 24.0 mm
   (c) 15.9 m (d) 4 cm
   (e) 15.1 mm (f) 3.5 m
6. 13.5 cm
7. 709
8. 215 kg
9. 3 cm
10. 24.0 cm
11. 14.86 cm
12. 434 m^2
13. 7240 cm^3
14. 2 : 1
15. (i) 182 cm^2 (ii) $2\frac{4}{15}$ cm

Exercise 12D

1. 1980 m^2
2. 1560 cm^3
3. (i) 2260 cm^3 (ii) 902 cm^2
4. (i) 32 300 cm^3 (ii) 5080 cm^2
5. 42 *l*
6. (i) 352 m^2 (ii) 552 m^3
7. 273 m^3
8. 6.75 cm
9. (i) 84 cm (ii) 5635π cm^2
10. (i) 44 400 cm^3 (ii) 7610 cm^2

Review Exercise 12

1. (a) (i) 23 200 cm^3
       (ii) 5440 cm^2
   (b) (i) 2.59 m^3
       (ii) 11.9 m^2
   (c) (i) 134 000 cm^3
       (ii) 15 100 cm^2
   (d) (i) 334 m^3
       (ii) 242 m^2
2. (i) 658 cm^3 (ii) 553 cm^2
3. 10 115 kg/m^3
4. 1020 cm^2
5. (i) 3.44 kg (ii) 5.34 cm
6. (i) 400 (ii) 1.44 mm
8. $235.62
9. (i) 10 500 kg/m^3
   (ii) 794 g
10. (i) 0.9 cm
    (ii) 0.792 cm^3, 1.02 cm^3
11. 200 cm
12. 3 : 1 : 2
13. $8\frac{8}{9}$ cm
14. 7.93 cm

Challenge Yourself – Chapter 12

1. 454 cm^2

Exercise 13A

1. (a), (c), (f), (g)
2. (a) 1 (b) 1 (c) 1
   (d) 3 (e) 2 (f) 2
   (g) 2 (h) 2 (i) 4
   (j) 2 (k) 1
4. (a) 2 (b) 2 (c) 2
   (d) 1 (e) 3 (f) 3
   (g) 1 (h) 1 (i) 2
   (j) 1 (k) 1 (l) 2
8. (a) $x = 2$ (b) $y = 4$
   (c) $y = 3$ (d) $x = 4$
11. (a) W, M, A, T, H
   (b) E, H
   (c) H
   (d) N, S

Exercise 13B

2. 5
3. (a) (i) 1 (ii) 1
   (b) (i) 2 (ii) 2
   (c) (i) 0 (ii) 2
   (d) (i) 8 (ii) 8
   (e) (i) 2 (ii) 2
   (f) (i) 2 (ii) 2
   (g) (i) 2 (ii) 2
   (h) (i) 2 (ii) 2
   (i) (i) 0 (ii) 4

Exercise 13C

1. (a) True (b) False
   (c) True (d) True
   (e) True (f) True
   (g) False (h) True
   (i) False (j) False
2. (a) 4 (b) 6
   (c) Infinite (d) Infinite
4. 4
5. (a) (i) 3 (ii) 3
   (b) (i) 9 (ii) 13
   (c) (i) 4 (ii) 4
   (d) (i) 6 (ii) 6
   (e) (i) 6 (ii) 4
   (f) (i) Infinite (ii) 1
   (g) (i) Infinite (ii) Infinite
   (h) (i) Infinite (ii) 1
   (i) (i) 4 (ii) 1
   (j) (i) Infinite (ii) Infinite
6. 9, Yes
7. (i) 2
   (ii) 1
   (iii) 2

Review Exercise 13

2. (a) (i) 2 (ii) 2
   (b) (i) 0 (ii) 4
   (c) (i) 8 (ii) 8
   (d) (i) 3 (ii) 3
   (e) (i) 1 (ii) 1
   (f) (i) 0 (ii) 2
   (g) (i) 2 (ii) 4
3. (a) 1 (b) 2
   (c) 1 (d) 2
   (e) 2 (f) 1
   (g) 2 (h) 2

Challenge Yourself – Chapter 13

1. (i) Infinite
2. (i) Infinite

Revision Exercise C1

1. (i) $(2x + 1)^2 = (2x)^2 + (x - 5)^2$
   (ii) 12
   (iii) 84 cm^2
2. 53.5°
3. 12.2 cm
4. 0.75 cm
5. (i) 2.43 cm (ii) 37.1 cm^2
6. $x = 6$
7. 6

Revision Exercise C2

1. (i) 4.08 cm (ii) 12.5°
2. 121 m
3. 4.55 cm
4. (a) (i) 51.3 cm^2
   (ii) 96.9 cm^2
   (b) 16
5. 43.0 cm
6. $x = 4\frac{1}{2}$, $y = 4$
7. Infinite

Exercise 14A

1. (a) {1, 3, 5, 7, 9}
   (b) (i) True (ii) True
   (iii) False (iv) True
2. (a) 12 (b) 8
   (e) 7 (f) 12
3. (a) {2, 3, 4, 5, 6, 7, 8, 9}
   (b) {–10, –9, –8, –7, –6, –5, –4, –3, –2, –1}
   (c) {2, 4, 6, 8, 10, 12}
   (d) {A}
4. (a) {A, E, I}
   (b) {red, orange, yellow, green, blue, indigo, violet}
   (c) {9, 18, 27, 36, 45}
   (e) {12, 14, 16, 18, 20, 22}
5. (a) Yes (b) No
   (c) No (d) No
   (e) Yes (f) No
6. (a) Yes (b) Yes
   (c) No (d) Yes
   (e) Yes (f) Yes
   (g) No (h) No
   (i) Yes (j) Yes
7. (a) { }: Empty set
   (b) { }: Empty set
   (c) { }: Empty set
   (d) {2}: Not an empty set
8. (a) {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}
   (b) (i) Tuesday $\in D$
   (ii) Sunday $\in D$
   (iii) March $\in D$
   (iv) Holiday $\in D$
9. (i) No
   (ii) {4, 9, 16, 25, 36, 49}
10. (a) {red, orange, yellow, green, blue, indigo, violet}
   (b) {S, Y, M, T, R}
   (c) {January, June, July}
   (d) {11, 13, 15, 17}
   (e) {b, c, d, f, g}
   (f) {Tuesday, Thursday}
   (g) {2, 4, 6, 8, 10, 12}
   (h) {February}
11. (e) The set of first 5 letters in the alphabet
12. (a) China; the set of ASEAN countries
   (b) Rubber; the set of edible fruits
   (c) 20; the set of perfect squares
   (d) 75; the set of perfect cubes
   (e) Pie chart; the set of statistical averages
13. (i) $Q = \{\ \}$; $R = \{1\}$
   (ii) $Q = \emptyset$; $R \neq \emptyset$
14. (i) False (ii) False
   (iii) False (iv) False
   (v) True (vi) False
   (vii) False (viii) True
15. (a) True (b) True
   (c) False (d) False
   (e) False (f) False
16. (a) $S = \{x : x$ is a girl in my current class wearing spectacles$\}$
   (b) $T = \{x : x$ is a prime number$\}$
   (c) $U = \{x : x$ is a multiple of 4$\}$
   (d) $V = \{x : x$ is a multiple of 4 between –8 and 12 inclusive$\}$
17. (i) False (ii) True
   (iii) False (iv) True

Exercise 14B

1. (a) {2, 4, 6, 8, 10, ..., 20}
   (b) {4, 8, 12, 16, 20}
   (c) {3, 6, 9, 12, 15, 18}
   (d) {1, 3, 5, 7, 9, 11, 13, 15, 17, 19}
   (e) {1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19}
   (f) {1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20}
2. (a) {30, 31, 32, 34, 35, 36, 38, 40, 41, 43, 45}
   (b) {35, 43, 44}
   (c) {31, 37, 41, 43}
   (d) {30, 31, 33, 34, 35, 37, 38, 39, 41, 42, 43, 45}
   (e) {31, 32, 34, 35, 37, 38, 40, 41, 43, 44}
3. (a) {2, 3, 5, 7}
   (b) {2, 4, 6, 8, 10}
   (c) {5, 10}
   (d) {1, 4, 6, 8, 9, 10}
   (e) {1, 3, 5, 7, 9}
   (f) {1, 2, 3, 4, 6, 7, 8, 9}
4. (i) ξ = {tiger, cat, dog, mouse, lion}
   A = {cat, dog, mouse}
   (ii) {tiger, lion}
6. (i) ξ = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
   B = {2, 4, 6, 8, 10}
   (iii) {1, 3, 5, 7, 9}
7. (a) {20, 40, 60, 80}
   (b) {60}
   (c) {40, 80}
   (d) ∅
8. (i) A = {s, t, u}
   B = {s, t, u, v, w, x, y, z}
   (ii) Yes
9. (ii) Yes
10. (i) ξ = {1, 2, 3, 4, 5, 6, 7, 8, 9}
   C = {1, 4, 6, 8, 9}
   C' = {2, 3, 5, 7}
11. (i) ξ = {a, b, c, d, e, f, g, h, i, j}
   D = {b, c, d, f, g, h, j}
   D' = {a, e, i}
12. (i) E = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20}
   F = {4, 8, 12, 16}
   (ii) F, E
   (iii) G = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20}
   (iv) $E = G$
13. $I \subset H$
14. (a) True (b) True
   (c) True (d) False
   (e) True
15. (a) { }, {1}, {2}, {1, 2}
   (b) { }, {pen}, {ink}, {ruler}, {pen, ink}, {pen, ruler}, {ink, ruler}, {pen, ink, ruler}
   (c) { }, {Thailand}, {Vietnam}, {Thailand, Vietnam}
   (d) { }, {a}, {e}, {i}, {o}, {a, e}, {a, i}, {a, o}, {e, i} {e, o}, {i, o}, {a, e, i}, {e, i, o}, {a, e, o}, {a, i, o}, {a, e, i, o}
16. (a) { }, {x}, {y}
   (b) { }, {Singapore}, {Malaysia}
   (c) { }, {3}, {4}, {5}, {3, 4}, {3, 5}, {4, 5}
   (d) { }, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d} {b, c, d}
17. (i) O' = {1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20}
   (ii) O' = {x : x is a positive integer less than 21 that is not divisible by 3}
18. (a) ≠ (b) ⊃
   (c) = (d) ≠
   (e) ≠ (f) =
   (g) ⊂ (h) =
   (i) ≠ (j) =
   (k) ⊂ (l) ≠
19. (a) True (b) True
   (c) False (d) False
   (e) False (f) True
   (g) True (h) False
   (i) True (j) True
   (k) False (l) True
   (m) False

Exercise 14C

1. (a) {b, c} (b) ∅
   (c) {m, n, y} (d) {t, s, c}
   (e) {i, r, a, l} (f) {a, i, o}
2. (i) {2, 4}
3. (i) {blue, yellow, pink}
4. (a) {1, 2, 3, 4, 5, 7, 9}
   (b) {1, 2, 3, 4, 5, 6, 7, 8, 9}
   (c) {2, 4, 6, 8}
   (d) {2, 3, 5, 6, 7, 9}
   (e) {2, 4, 6, 8}
5. (ii) {11, 12, 13, 14, 15, 16, 17, 18, 20}
6. (ii) {apple, orange, banana, grape, durian, pear, strawberry}
7. (i) {durian, mango, pineapple, rambutan, soursop}
   (ii) {durian, mango}
8. (a) {3, 6, 8, 9, 12}, {6, 9}
   (b) {a, b, x, y, m, n, o, p}, ∅
   (c) {monkey, goat, lion, tiger}, {goat}
   (d) {a, m, k, y}, ∅
9. (a) ξ = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}
   I = {4, 8, 12}
   J = {1, 2, 4, 8}
   (c) (i) {3, 5, 6, 7, 9, 10, 11, 13, 14, 15}
   (ii) {12}
11. (i) M = {1, 4, 9, 16, 25, 36, 49, 64}
   P = {1, 8, 27, 64}
   (iii) {1, 64}
12. (i) N = {8, 16, 24, 32}
   Q = {4, 8, 12, 16, 20, 24, 28, 32}
   (ii) {8, 16, 24, 32}
   (iv) Yes
13. (i) R = {1, 2, 3, 6, 9, 18}
   S = {10, 12, 14, 15, 16}
   (ii) { }
14. (i) T = {4, 8, 12}
   U = {1, 2, 3, 4, 6, 8, 12, 24}
   (iii) {1, 2, 3, 4, 6, 8, 12, 24}
   (iv) Yes
15. (i) V = {1, 5, 25}
   W = {6, 12, 18, 24}
   (iii) {1, 5, 6, 12, 18, 24, 25}
16. (a) ξ = {4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18}
   Y = {6, 9, 12, 15, 18}
   Z = {9, 18}
   (c) (i) {4, 5, 7, 8, 10, 11, 13, 14, 16, 17}
   (ii) {6, 12, 15}
17. (a) ξ = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}
   P = {2, 3, 5, 7, 11}
   Q = {0, 1, 4, 6, 8, 9, 10}
   (c) (i) {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}
   (ii) { } or ∅
   (iii) {0, 1, 4, 6, 8, 9, 10}
18. (b) (i) They are the same.
   (ii) They are the same.
21. (i) A (ii) ξ
   (iii) ∅ (iv) A
22. (i) A (ii) B
   (iii) Not possible to simplify further.
   (iv) Not possible to simplify further.
   (v) A (vi) ∅
   (vii) Not possible to simplify further.
   (viii) B

Review Exercise 14

1. (a) A = {1, 3, 5, 7, 9}
   (b) (i) True (ii) True
   (iii) False (iv) False
   (c) (i) $-3 \notin A$ (ii) $3 \notin A$
   (iii) $0 \notin A$ (iv) $9 \notin A$
2. (a) B = {2} is not an empty set.
   (b) C = {Saturday, Sunday} is not an empty set.
   (c) D = { } = ∅ (d) E = { } = ∅

3. (a) $\{-4, -2, 0, 1, 3\}$
(b) $\{\ \}$
(c) $\{-5, -4, -3, -2, -1, 0, 1\}$
(d) $\{-5, -4, -3, -2, -1, 0, 1, 2\}$

4. (a) True (b) False
(c) True (d) True

5. (ii) $\{4, 5, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23\}$

6. (i) $\{-7, 7\}$
(ii) $\{1, 2, 3, 4, 5, 6\}$
(iii) $\{-7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$

7. (ii) $\{3, 5, 6, 7, 8, 10, 11, 12, 14, 15\}$

8. (ii) $\{5\}$

9. (i) $\{\ \}, \{s\}, \{i\}, \{t\}, \{s, i\}, \{s, t\}, \{i, t\}, \{s, i, t\}$
(ii) $\{\ \}, \{s\}, \{i\}, \{t\}, \{s, i\}, \{s, t\}, \{i, t\}$

Challenge Yourself – Chapter 14

2. $2^n - 1$

Exercise 15A

1. 1, 2, 3, 4, 5, 6; 6

2. (a) 2, 3, 4, 5; 4
(b) A, B, C, D, E, F, G, H, I, J; 10
(c) $R_1, R_2, R_3, R_4, R_5, B_1, B_2, B_3, G_1, G_2$; 10
(d) T, E_1, A, C, H, E_2, R; 7
(e) 100, 101, 102, …, 999; 900

3. (i) $\frac{3}{8}$ (ii) $\frac{3}{8}$
(iii) 1 (iv) $\frac{7}{8}$

4. (i) $\frac{7}{13}$ (ii) $\frac{7}{13}$
(iii) $\frac{3}{13}$ (iv) 0
(v) $\frac{3}{13}$

5. (i) $\frac{1}{52}$ (ii) $\frac{1}{2}$
(iii) $\frac{3}{13}$ (iv) $\frac{10}{13}$

6. (i) $\frac{1}{11}$ (ii) $\frac{2}{11}$
(iii) $\frac{4}{11}$ (iv) $\frac{7}{11}$

7. (i) $\frac{1}{5}$ (ii) $\frac{3}{5}$
(iii) $\frac{1}{5}$ (iv) $\frac{2}{5}$

8. (i) $\frac{1}{4}$ (ii) $\frac{1}{2}$
(iii) $\frac{3}{4}$

9. $\frac{3}{8}$

10. (i) $\frac{7}{10}$ (ii) $\frac{7}{10}$

11. (i) $\frac{1}{9}$ (ii) $\frac{1}{15}$

12. (i) $\frac{13}{27}$ (ii) $\frac{2}{27}$
(iii) $\frac{1}{27}$ (iv) $\frac{4}{27}$

13. (i) $\frac{1}{3}$ (ii) $\frac{1}{3}$
(iii) $\frac{3}{13}$ (iv) $\frac{12}{13}$

14. (i) $\frac{1}{6}$ (ii) $\frac{1}{10}$
(iii) 1 (iv) $\frac{2}{5}$

15. $\frac{2}{3}$

16. (a) (i) $\frac{62}{117}$ (ii) $\frac{2}{9}$
(iii) $\frac{7}{39}$
(b) (i) $\frac{4}{57}$ (ii) $\frac{53}{57}$

17. (i) 26
(ii) 64

18. (i) $\frac{9}{80}$ (ii) $\frac{13}{80}$
(iii) $\frac{1}{5}$ (iv) $\frac{1}{10}$

19. (i) $\frac{1}{4}$ (ii) $\frac{1}{3}$

20. $\frac{8}{15}$

Exercise 15B

1. (i) $\frac{2}{5}$ (ii) $\frac{11}{30}$
(iii) $\frac{9}{10}$ (iv) 0

2. (i) $\frac{3}{10}$ (ii) $\frac{2}{5}$

3. (i) $\frac{5}{12}$ (ii) $\frac{5}{24}$
(iii) $\frac{3}{8}$

4. (i) $\frac{1}{8}$ (ii) $\frac{3}{8}$
(iii) $\frac{1}{2}$

5. (i) $15 + x$ (ii) $\frac{15}{15 + x}$
(iii) 60

6. (i) $\frac{4}{19}$ (ii) $\frac{9}{19}$

7. (a) (i) $\frac{2}{5}$ (ii) $\frac{1}{8}$
(b) (i) $\frac{1}{13}$ (ii) $\frac{7}{13}$

8. 9

9. $3\frac{1}{4}$

10. 10

11. 7

12. (i) $\frac{7}{20}$ (ii) $2x + 46$
(iii) 30

13. $x = 16, y = 18$

Review Exercise 15

1. (a) (i) 5, 6, 8, 56, 58, 65, 68, 85, 86, 568, 586, 658, 685, 856, 865
(ii) 15
(b) (i) $\frac{2}{5}$ (ii) $\frac{1}{3}$

2. (i) $\frac{1}{2}$ (ii) $\frac{1}{3}$
(iii) $\frac{1}{6}$

3. (i) $\frac{1}{26}$ (ii) 0
(iii) $\frac{1}{13}$ (iv) $\frac{12}{13}$

4. (i) $\frac{1}{6}$ (ii) $\frac{1}{2}$
(iii) 0

5. (i) $\frac{7}{22}$ (ii) $\frac{1}{11}$
(iii) 0 (iv) $\frac{13}{22}$

6. (i) $\frac{1}{5}$ (ii) $\frac{1}{2}$
(iii) $\frac{13}{20}$

7. (a) (i) 0 (ii) 1
(b) (i) $\frac{2}{5}$ (ii) $\frac{19}{30}$

8. (i) $\frac{5}{18}$ (ii) $\frac{17}{36}$
(iii) 0

9. (i) 55 (ii) $\frac{11}{18}$

10. (i) $37p - 2q - 145 = 0$
(ii) $3p + 2q - 55 = 0$
(iii) $p = 5, q = 20$

Challenge Yourself – Chapter 15

1. 22, 30

2. (i) $\frac{1}{2}$ (ii) 0
(iii) $\frac{1}{6}$

Exercise 16A

1. (ii) 15 kg (iii) 36 kg
(iv) 9

2. (i) 1 minute (ii) 13 minutes
(iii) 31 (iv) 11 : 31

3. (ii) 15% (iii) $\frac{7}{20}$

4. (a) Strong, negative
(b) Moderate, positive
(c) Strong, positive
(d) none

5. \$7000

6. (ii) 60%

7. (i) 5.4 minutes (ii) $\frac{6}{11}$

8. (a) 2
(b) (i) $41\frac{2}{3}$% (ii) $16\frac{2}{3}$%
(c) Girls

9. (ii) 26
(iii) 35%
10. (ii) 7 minutes, 9.8 minutes
(iii) 30%
(iv) $\frac{3}{10}$
11. (a) 2007, 2008, 2010, 2011, 2012
12. (i) School Q (ii) School P
(iii) School P
13. (ii) 25%, 60% (iii) Class B
14. (ii) Strong, positive correlation
(iv) 137 mm Hg
(v) No
15. (ii) Strong, negative correlation
(iv) 68 years
(v) $y = \frac{41}{50}x + 88$
(vi) No
16. (ii) No correlation
17. (ii) 3 (iii) $56\frac{1}{4}$%
(iv) 8
18. Stem-and-leaf diagram

Exercise 16B

1. (iii) 2 hours (iv) 26
(v) $\frac{3}{13}$
2. (ii) 4 : 1 (iii) No
3. (ii) 3
4. (ii) 40 (iii) 150
(iv) $46\frac{2}{3}$%
5. (a) 55.5, 60.5, 7; 60.5, 65.5, 6; 65.5, 70.5, 5; 70.5, 75.5, 10; 75.5, 80.5, 5; 80.5, 85.5, 5; 85.5, 90.5, 2; 90.5, 95.5, 3; 95.5, 100.5, 3
(b) 91-95; 66-70; 81-85
6. (a) 84
7. (a) 10, 2 × standard, 4;
5, 1 × standard, 6;
5, 1 × standard, 11;
5, 1 × standard, 13;
15, 3 × standard, 1
8. (a) 10, 1.2; 10, 1.1; 20, 0.4; 30, 0.2
11. (i) 8, 4, 3, 4, 3 (iii) $114\frac{2}{7}$%
15. 32, 37, 42, 47, 52, 57
16. (ii) 9, 8 (iii) 349, 99
(iv) 3

Review Exercise 16

1. (ii) 1 minute and 2 minutes
(iii) $73\frac{1}{3}$%
2. (i) 4 (ii) 6
(iii) $\frac{4}{25}$
3. (ii) 165 cm (iii) 145 cm
(iv) 2 : 1
4. (i) 82 km/h (ii) 5%
(iii) 84
5. (ii) \$96, \$80 (iii) Group A
6. (c) (i) 9
(ii) 14
(d) $y = \frac{1}{2}x + \frac{2}{5}$
(e) Strong, positive
7. No correlation
8. (ii) $\frac{37}{60}$ (iii) \$1278
9. (iii) $\frac{6}{35}$
10. (a) $20 < x \leqslant 25$, 12;
$30 < x \leqslant 35$, 7;
$35 < x \leqslant 40$, 4;
$40 < x \leqslant 45$, 3
(b) 39

Challenge Yourself – Chapter 16

2 students obtain 3 marks, 5 students obtain 4 marks, 5 students obtain 9 marks.

Exercise 17A

1. 38.1
2. \$35.86
3. 8
4. 54 kg
5. (i) 96 (ii) 13
6. (i) 30 (ii) 60
(iii) 2
7. 1.35 days
8. (a) 9 (b) \$8.90
(c) 4.85 years
9. (i) 128 (ii) 21
10. \$1110
11. (i) 27 cm (ii) 42 cm
12. (a) 4, 5
(b) (i) $37\frac{1}{2}$% (ii) 40%
13. (i) 28 cm (ii) $\frac{3}{4}$
14. (i) 123.52 minutes
(ii) $\frac{29}{50}$
15. (i) 52.7 km/h (ii) 2 : 3
16. (i) 7; 0; 1; 1; 8; 3
(ii) 22.55 million km
17. 11
18. (i) 172.1 hours (iii) 172.7 hours

Exercise 17B

1. (a) 5 (b) 29.5
(c) 2.8 (d) 12.75
2. (a) 39.5 (b) 70
(c) 5.7 (d) 42.5
3. (a) 3 (b) 7.7, 9.3
4. (a) Red (b) 78, 79
(c) 60 (d) 30
(e) No mode
5. (i) 27 °C (ii) 22 °C, 27 °C
6. (b) (i) 2.05 (ii) 2
(iii) 2
8. 5
9. (a) (i) 32.3 km
(ii) 32 km
(iii) 32 km
(b) $\frac{1}{5}$
10. (a) (i) \$32.17 (ii) \$32
(iii) \$32
(b) $\frac{3}{5}$
11. (a) (iii) $x = 12, y = 6$
(b) (i) 7 (ii) 10
12. (a) 7 (b) 3, 4, 5
13. (i) 5, 4.67 (ii) Jun Wei; No
(iii) 3, 4 (iv) 2, 6
(v) Mode
14. (ii) 7, 7 (iii) 6, 8
(iv) Mode
15. (ii) Mean
16. (a) $x = 10, y = 4$
(b) (i) 2 (ii) 2
(c) 1
17. (a) 7 (b) 7
(c) 7
18. (i) 4 (ii) 2

Review Exercise 17

1. (a) 12, 13, 14 (b) 94, 95.5, 98
2. 10
3. $a = 26, b = 29$
4. (a) (i) $h + k = 1000$
(ii) $h = 100, k = 900$
(b) (ii) O
5. (i) 6.55% (ii) 6.5%
(iii) 8%
6. (a) (i) 74.68 A (ii) 73 A
(iii) 72 A
7. (a) (iii) $x = 20, y = 24$
(b) (i) 3 (ii) 3
8. (a) 14 (b) 9
(c) 11
9. (b) (ii) $130\frac{1}{3}$ cm (iii) 90%
10. (a) (i) Jaguar (ii) Cheetah
(b) Median

Challenge Yourself – Chapter 17

1. $x, x, x + 2y, x + 2y + 20$
2. 3.5
3. 172 cm

Revision Exercise D1

1. (a) {21, 22, 23, 25, 26, 27, 29, 30, 31, 32, 33, 34}
2. (i) $\frac{1}{2}$ (ii) $\frac{1}{30}$
   (iii) $\frac{1}{10}$ (iv) $\frac{4}{45}$
3. (i) 0 (ii) $\frac{1}{3}$
   (iii) $\frac{2}{13}$ (iv) $\frac{12}{13}$
4. (ii) 5 (iii) $2\frac{6}{7}$ %
5. (c) (i) 71.5 (ii) 3
   (d) $y = -11.4x + 89$
   (e) Strong, negative correlation
6. (i) 175 (ii) 3 : 5
   (iii) $\frac{11}{35}$
7. (i) 12.675 (ii) 13
   (iii) 13
8. (i) 32 days (ii) $\frac{17}{25}$

Revision Exercise D2

1. (ii) 0 (iii) {3, 4, 6, 8, 9}
2. (i) $\frac{3}{14}$ (ii) $\frac{5}{7}$
3. 280
4. (ii) 70%
5. (ii) 2
6. 6.8
7. (a) $x = 9, y = 4$
   (b) (i) 6 (ii) 5

Problems in Real-World Contexts

2. (i) 144, 16 (ii) $8411\frac{2}{3}$ m^3
   (iii) 473 m^2
3. (a) 50.6%, 61.4%, 55.4%
   (b) $22 000, $35 000, $41 000
5. (i) Ethan's method
   (iii) Rui Feng's method
6. (a) No
   (b) (i) 16 m, 10 m
   (ii) 2 m, 0.5 m
   (c) Quadratic trendline
   (d) 0 m; 5.56 m; 11.1 m; 16.7 m; 22.2 m; 27.8 m; 33.3 m; 38.9 m; 44.4 m; 50 m; 55.6 m; Yes

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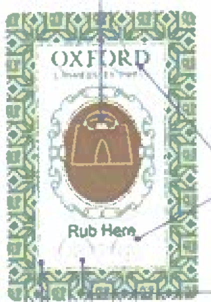

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